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GEODESICS IN A TENSOR BUNDLE

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Abstract

The main purpose of the present paper is to study geodesics in a tensor bundle $T_q^p(M_n)$ with respect to the horizontal lift ${}^H\nabla$ of an affine connection ∇.

Key words and phrases: Tensor, Tensor Bundle, Connection, Horizontal Lift, Geodesic.*

1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $T_q^p(Q)$ the vector space of tensors type (p,q) at a point Q of M_n that is, the set of all tensors of type (p,q), of M_n at Q. Then the set

$$T_q^p(M_n) = \bigcup_{Q\in M_n} T_q^p(Q)$$

is, by definition, the tensor bundle over the manifold M_n. For any point $\tilde{Q}$ of $T_q^p(M_n)$ such that $\tilde{Q} \in T_q^p(Q)$, the correspondence $\tilde{Q} \to Q$ determines the bundle projection $\pi : T_q^p(M_n) \to M_n$.

Let x^i be local coordinates in a neighborhood U of $Q \in M_n$. Then a tensor t of type (p,q) at Q which is an element of $T_q^p(M_n)$ is expressible in the form $(x^i, t_{i_1\cdots i_q}^{j_1\cdots j_p}) = (x^i, x^{\overline{i}})$ $(x^{\overline{i}} = t_{i_1\cdots i_q}^{j_1\cdots j_p}, \overline{i} = h+1, \cdots, h+n^{p+q})$, where $t_{i_1\cdots i_q}^{j_1\cdots j_p}$ are components of t with respect to the natural frame $\frac{\partial}{\partial x^i}$. We may consider $(x^i, x^{\overline{i}})$ as local coordinates in a neighborhood $\pi^{-1}(U)$ of $T_q^p(M_n)$.

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To a transformation of local coordinates of M_n; $x^{i'} = x^{i'}(x^1, \cdots, x^n)$, there corresponds in $T^p_q(M_n)$ the coordinates transformation

$$x^{i'} = x^{i'}(x^1, \cdots, x^n)$$
$$x^{\bar{i}'} = t^{j'_1 \cdots j'_p}_{i'_1 \cdots i'_q} = A^{j'_1}_{j_1} \cdots A^{j'_p}_{j_p} A^{i_1}_{i'_1} \cdots A^{i_q}_{i'_q} t^{j_1 \cdots j_p}_{i_1 \cdots i_q} = A^{(j')}_{(j)} A^{(i)}_{(i')} x^{\bar{i}'} \tag{1}$$

where $A^i_{i'} = \frac{\partial x^i}{\partial x^{i'}}$, $A^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}$, $A^{(j')}_{(j)} = A^{j'_1}_{j_1} \cdots A^{j'_p}_{j_p}$, $A^{(i)}_{(i')} = A^{j'_1}_{j_1} \cdots A^{i_q}_{i'_q}$.

The Jacobian of (1) is given by the matrix

$$\left(\frac{\partial x^{I'}}{\partial x^I} \right) = \begin{pmatrix} A^{i'}_i & 0 \\ t^{(j)}_{(k)} \partial_i (A^{(k)}_{(i')} A^{(j')}_{(j)}) & A^{(i)}_{(i')} A^{(j')}_{(j)} \end{pmatrix} \tag{2}$$

where $I = (i, \bar{i})$, $I' = (i', \bar{i}')$, $t^{(j)}_{(k)} = t^{j_1 \cdots j_p}_{k_1 \cdots k_q}$

2. Horizontal Lifts of Affine Connection

We denote by $\mathcal{T}^p_q(M_n)$ the set of all tensor fields of class C^∞ and of type (p, q) in M_n.

We now assume that M_n is a manifold with an affine connection ∇. Let X^h and Γ^h_{ji} be components of $X \in \mathcal{T}^1_0$ and ∇, respectively, with respect to the local coordinates (x^h) in M_n. Then the horizontal lift of X have components

$${}^H X = \begin{pmatrix} {}^H X^i \\ {}^H X^{\bar{i}} \end{pmatrix} = \begin{pmatrix} X^i \\ \sum_{\mu=1}^q \Gamma^m_{hi_\mu} X^h t^{j_1 \cdots j_p}_{i_1 \cdots m \cdots i_q} - \sum_{\lambda=1}^p \Gamma^{j\lambda}_{hm} X^h t^{j_1 \cdots m \cdots j_p}_{i_1 \cdots i_q} \end{pmatrix}$$

with respect to the coordinates $(x^i, x^{\bar{i}})$ in $T^p_q(M_n)$ (see [1]).

Let $A^{j_1 \cdots j_p}_{i_1 \cdots i_q}$ be components of $A \in T^p_q(M_n)$. We can easily verify by means of (2) that the $\tilde{A}^J$ defined by

$$\tilde{A}^i = 0, \tilde{A}^{\bar{i}} = A^{j_1 \cdots j_p}_{i_1 \cdots m \cdots i_q}$$

determine in $T^p_q(M_n)$ a vector field. This vector field is called the vertical lift of the tensor field $A \in \mathcal{T}^p_q(M_n)$ to $T^p_q(M_n)$ and denoted by ${}^V A$ (see [2]).

We shall now define the horizontal lift ${}^H\nabla$ of an affine connection ∇ in M_n to $T^p_q(M_n)$ by the conditions

$${}^H(\nabla_X Y) = {}^H \nabla_{{}^H X} {}^H Y, {}^V (\nabla_X A) = {}^H \nabla_{{}^H X} {}^V A, {}^H \nabla_{{}^V A} {}^H X = 0, {}^H \nabla_{{}^V A} {}^V B = 0$$

for any $X,Y \in \mathcal{T}_0^1(M_n)$, $A,B \in T_q^q(M_n)$, from wich we have (see [3])

$$^{H}\Gamma^{i}_{ms} = \Gamma^{i}_{ms}$$

$$^{H}\Gamma^{\overline{i}}_{\overline{m}s} = \sum_{b=1}^{q} \Gamma^{j_b}_{sl_b} \delta^{j_1}_{l_1} \cdots \delta^{j_{b-1}}_{l_{b-1}} \delta^{j_{b+1}}_{l_{b+1}} \cdots \delta^{j_p}_{l_p} \delta^{m_1}_{i_1} \delta^{m_2}_{i_2} \cdots \delta^{m_q}_{i_q}$$

$$- \sum_{c=1}^{q} \Gamma^{m_c}_{si_c} \delta^{m_1}_{i_1} \cdots \delta^{m_{c-1}}_{i_{c-1}} \delta^{m_{c+1}}_{i_{c+1}} \cdots \delta^{m_q}_{i_q} \delta^{j_1}_{l_1} \cdots \delta^{j_p}_{l_p},$$

$$^{H}\Gamma^{\overline{i}}_{m\overline{s}} = \sum_{b=1}^{p} \Gamma^{j_b}_{mk_b} \delta^{j_1}_{k_1} \cdots \delta^{j_{b-1}}_{k_{b-1}} \delta^{j_{b+1}}_{k_{b+1}} \cdots \delta^{j_p}_{k_p} \delta^{s_1}_{i_1} \cdots \delta^{s_q}_{i_q}$$

$$- \sum_{c=1}^{q} \Gamma^{s_c}_{mi_c} \delta^{s_1}_{i_1} \cdots \delta^{c-1}_{i_{c-1}} \delta^{s_{c+1}}_{i_{c+1}} \cdots \delta^{s_q}_{i_q} \delta^{j_1}_{k_1} \cdots \delta^{j_p}_{k_p}, \tag{3}$$

$$^{H}\Gamma^{\overline{i}}_{ms} = \sum_{b=1}^{p} (\partial m \Gamma^{jb}_{sq} + \Gamma^{jb}_{mr} \Gamma^{r}_{sq} - \Gamma^{r}_{ms} \Gamma^{jb}_{ra}) t^{j_1 \cdots j_{b-1} a j_{b+1} \cdots j_p}_{i_1 \cdots i_q}$$

$$+ \sum_{c=1}^{q} (-\partial_m \Gamma^{a}_{si_c} + \Gamma^{r}_{mi_c} \Gamma^{a}_{sr} + \Gamma^{r}_{ms} \Gamma^{a}_{ri_c}) t^{j_1 \cdots j_p}_{i_1 \cdots i_{c-1} a i_{c+1} \cdots i_q}$$

$$- \sum_{b=1}^{p} \sum_{c=1}^{q} t^{j_1 \cdots j_{b-1} r j_{b+1} \cdots j_q}_{i_1 \cdots i_{c-1} a i_{c+1} \cdots i_p} (\Gamma^{jb}_{mr} \Gamma^{a}_{si_c} + \Gamma^{a}_{mi_c} \Gamma^{jb}_{sr})$$

$$+ \frac{1}{2} \sum_{b=1}^{q} \sum_{c=1}^{q} t^{j_1 \cdots j_p}_{i_1 \cdots i_{b-1} r i_{b+1} \cdots i_{c-1} l i_{c+1} \imath_q} (\Gamma^{l}_{mi_c} \Gamma^{r}_{mi_b} + \Gamma^{r}_{mi_b} \Gamma^{l}_{si_c})$$

$$+ \frac{1}{2} \sum_{b=1}^{p} \sum_{c=1}^{p} t^{j_1 \cdots j_{b-1} r j_{b+1} \cdots j_{c-1} l j_{c+1} \cdots j_p}_{i_1 \cdots i_q} (\Gamma^{jb}_{mr} \Gamma^{jc}_{sl} + \Gamma^{jc}_{ml} \Gamma^{jb}_{sr}),$$

where δ^i_j-Kronecker delta, $x^{\overline{m}} = t^{l \cdots l_p}_{m_1 \cdots m_q}$, $x^{\overline{s}} = t^{k \cdots k_p}_{s_1 \cdots s_q}$.

3. Geodesisc (paths) In A Tensor Bundle of The Horizontal Lift ${}^{H}\nabla$

Let $\tilde{C}:[0,1]\rightarrow T_q^p(M_n)$ be a curve in $T_q^p(M_n)$ and suppose that $\tilde{C}$ is expressed locally by $x^A=x^A(t)$, i.e.,

$$\begin{aligned} x^h &= x^h(t) \\ x^{\overline{h}} &= x^{\overline{h}}(t) \end{aligned}$$

with respect to the induced coordinates $(x^h,x^{\overline{h}})$ in $T_q^p(M_n)$, t being a parameter. Then the curve $C=\pi\circ\tilde{C}$ in M_n is called the projection of the curve $\tilde{C}$ and denoted by $\pi\tilde{C}$ which is expressed locally by $x^h=x^h(t)$.

A curve $\tilde{C}$ in $T_q^p(M_n)$ is a geodesic with respect to ${}^{H}\nabla$ (a path of ${}^{H}\nabla$), when it satisfies the differential equation

$$\frac{d^2x^I}{dt^2}+^H\Gamma^I_{MS}\frac{dx^M}{dt}\frac{dx^S}{dt}=0 \qquad (4).$$

Consider the case where $p=1$, $q=2$, for example. By means of (3), (4) reduces to

$$\frac{d^2x^i}{dt^2}+\Gamma^i_{ms}\frac{dx^m}{dt}\frac{dx^s}{dt}=0$$

$$\begin{aligned} \frac{d^2x^{\overline{i}}}{dt^2}+^H\Gamma^{\overline{i}}_{ms}\frac{dx^m}{dt}\frac{dx^s}{dt}+^H\Gamma^{\overline{i}}_{\overline{m}s}\frac{dx^{\overline{m}}}{dt}\frac{dx^s}{dt} \\ +^H\Gamma^{\overline{i}}_{m\overline{s}}\frac{dx^m}{dt}\frac{dx^{\overline{s}}}{dt}=\frac{d^2t^{j_1}_{i_1i_2}}{dt^2} \\ +[(\partial_m\Gamma^{j_1}_{sa}+\Gamma^{j_1}_{mr}\Gamma^2_{sa}-\Gamma^2_{ms}\Gamma^{j_1}_{ra})t^a_{i_1i_2} \\ +\sum_{c=1}^{2}(-\partial_m\Gamma^a_{si_c}+\Gamma^2_{mi_c}\Gamma^a_{sr}+\Gamma^2_{ms}\Gamma^a_{ri_c})t^{j_1}_{.a.} \qquad (5) \\ -\sum_{c=1}^{2}t^r_{.a.}(\Gamma^{j_1}_{mr}\Gamma^a_{si_c}+\Gamma^a_{mi_c}\Gamma^{j_1}_{sr}) \\ +\frac{1}{2}\sum_{b=1}^{2}\sum_{c=1_{c\neq b}}^{2}t^{j_1}_{.r.l}(\Gamma^l_{mi_c}\Gamma^r_{si_b}+\Gamma^r_{mi_b}\Gamma^l_{si_c})]\frac{dx^m}{dt}\frac{dx^s}{dt} \end{aligned}$$

$$+[\Gamma^{j_1}_{sl_1}\delta^{m_1}_{i_1}\delta^{m_2}_{i_2}-\sum_{c=1}^{2}\Gamma^{m_c}_{si_c}\delta^{m_{c-1}}_{i_{c-1}}\delta^{m_{c+1}}_{i_{c+1}}\delta^{j_1}_{l_1}]\frac{dt^{l_1}_{m_1m_2}}{dt}\frac{dx^s}{dt}$$

$$+([\Gamma^{j_1}_{mk_1}\delta^{s_1}_{i_1}\delta^{s_2}_{i_2}-\sum_{c=1}^{2}\Gamma^{s_c}_{mi_c}\delta^{s_{c-1}}_{i_{c-1}}\delta^{s_{c+1}}_{i_{c+1}}\delta^{j_1}_{k_1})\frac{dx^m}{dt}\frac{dt^{k_1}_{s_1s_2}}{dt}=0,$$

where $x^{\bar{i}}=t^{j_1}_{i_1i_2}$.

From first equation in (5), we have

$$\Gamma^{j_1}_{ma}t^{a}_{i_1i_2}\frac{d^2x^m}{dt^2}=-\Gamma^{j_1}_{ma}t^{a}_{i_1i_2}\Gamma^m_{rs}\frac{dx^2}{dt}\frac{dx^s}{dt},$$
$$-\Gamma^{a}_{mi_1}t^{j_1}_{ai_2}\frac{d^2x^m}{dt^2}=\Gamma^{a}_{mi_1}\Gamma^m_{rs}t^{j_1}_{ai_2}\frac{dx^2}{dt}\frac{dx^s}{dt},\tag{6}$$
$$-\Gamma^{a}_{mi_2}t^{j_1}_{i_1a}\frac{d^2x^m}{dt^2}=\Gamma^{a}_{mi_2}\Gamma^m_{rs}t^{j_1}_{i_1a}\frac{dx^2}{dt}\frac{dx^s}{dt}$$

. By means of (6), the second equation in (5) is reduced to

$$\frac{\delta^2t^{j_1}_{i_1i_2}}{dt^2}=0\tag{7}$$

where the left-hand side is defined by

$$\frac{\delta^2t^{j_1}_{i_1i_2}}{dt^2}=\frac{d}{dt}(\frac{dt^{j_1}_{i_1i_2}}{dt}+\Gamma^{j_1}_{ma}\frac{dx^m}{dt}t^a_{i_1i_2}$$
$$-\Gamma^a_{mi_1}\frac{dx^m}{dt}t^{j_1}_{ai_2}-\Gamma^a_{mi_2}\frac{dx^m}{dt}t^{j_1}_{i_1a})$$
$$+\Gamma^{j_1}_{sr}\frac{dx^s}{dt}(\frac{dt^r_{i_1i_2}}{dt}+\Gamma^r_{ma}\frac{dx^m}{dt}t^a_{i_1i_2}$$

$$-\Gamma^a_{mi_1}\frac{dx^m}{dt}t^r_{ai_2}-\Gamma^a_{mi_2}\frac{dx^m}{dt}t^r_{i_1a})-\Gamma^r_{si_1}\frac{dx^s}{dt}(\frac{dt^{j_1}_{ri_2}}{dt}$$

$$+\Gamma^{j_1}_{ma}\frac{dx^m}{dt}t^a_{ri_2}-\Gamma^a_{mr}\frac{dx^m}{dt}t^{j_1}_{ai_2}$$

$$-\Gamma^a_{mi_2}\frac{dx^m}{dt}t^{j_1}_{ra})-\Gamma^r_{si_2}\frac{dx^s}{dt}(\frac{dt^{j_1}_{i_1r}}{dt}+\Gamma^{j_1}_{ma}\frac{dx^m}{dt}t^a_{i_2}$$
$$-\Gamma^a_{mi_1}\frac{dx^m}{dt}t^{j_1}_{ai_2}-\Gamma^a_{ma_2}\frac{dx^m}{dt}t^{j_1}_{i_1a}).$$

By similar devices, we can prove the formula (7) for general case. Thus we have from (5) and (7).

Theorem. A curve $\tilde{C} : x^i = x^i(t)$, $x^{\bar{i}} = t_{i_1 \cdots i_p}^{j_1 \cdots j_q} = t_{i_1 \cdots i_p}^{j_1 \cdots j_q}(t) = x^{\bar{i}}(t)$, in $T_q^p(M_n)$ is a geodesic of the horizontal lift ${}^H\nabla$ of an affine connection ∇ given in M_n, if and only if the projection $\pi\tilde{C}$ is a geodesic of ∇ in M_n and the tensor field $t_{i_1,\cdots,i_p}^{j_1,\cdots,j_q}$ along $\pi\tilde{C}$ has vanishing second covariant derivative.

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