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UNIVALENT HARMONIC MAPPINGS ONTO HALF PLANES

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Abstract

We consider the class $S_H(D,\Omega)$ of complex functions f which are univalent, harmonic, sense preserving on a simple connected domain $D \neq \mathbb{C}$ containing the origin, satisfy $f(0) = a_0$, $f_{\bar{z}}(0) = 0 < f_z(0)$, and have the fixed range $f(D) = \Omega$, where $\Omega = \{w : \mathrm{Re}w > a,\ a \in \mathbb{R}\}$. In particularly, we describe the closure $\overline{S_H(D,\Omega)}$ of $S_H(D,\Omega)$ and characterize its extreme points, as well as sharp estimates for coefficients and distortion theorems.

Key Words: Harmonic mappings, Extremal problems.

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1. Introduction.

Recently, there has been interest [1,2,4,5,6] in studying the class S_H of all complex-valued, harmonic, sense preserving univalent mappings f defined on the open unit disk U, which are normalized at the origin by

$$f(0) = 0 \qquad \text{and} \qquad f_z(0) > 0. \tag{1}$$

Such functions can be represented as $f = h + \bar{g}$ where $h(z) = z + a_2z^2 + \ldots$ and $g(z) = b_1z + b_2z^2 + \ldots$ are analytic in U. Since f is sense preserving, $J_f(z) = |h'(z)|^2 - |g'(z)| > 0$ and then $|g'(z)| < |h'(z)|$ for $z \in U$. It follows that $|b_1| < 1$ and hence $f_0 = (f - \overline{b_1 f})/(1 - |b_1|^2)$ also belongs to S_H. Thus we obtain restriction to the subclass

S_H^0 of S_H, consisting of those functions in S_H with $f_{\bar{z}}(0) = 0$. If we let F and G be analytic in U and satisfy Re $F=$ Re $f = u$ and Re $G =$Im $f = v$, then $h = (F + iG)/2$ and $g = (F - iG)/2$.

In contrast to conformal mappings, harmonic univalent functions f are not at all determined (up to normalization (1)) by their image domains. So, it is natural to study the class of harmonic, sense preserving univalent mappings of a simple connected domain $D \neq \mathbb{C}$ onto another domain Ω. We shall assume that D contain the origin and Ω contain any fixed real point a_0 and that functions $f \in S_H(D,\Omega)$ are normalized so that

$$f(0) = a_0, \qquad f_z(0) > 0 \qquad \text{and} \qquad f_{\bar{z}}(0) = 0.$$

Hengartner and Schober [5] and later Cima and Livingston [2] considered the case of Ω being a strip, Abu-Muhana and Schober [1] considered the case of Ω being a wedge or half- plane, Livingston [6] considered the case of $\Omega = \mathbb{C}\backslash(-\infty, a]$, $a < 0$, and Grigoriyan and Szapial [4] considered the case of $\Omega = \mathbb{C}\backslash\{(-\infty, a] \cup [b, +\infty)\}$, $a < 0 < b$.

Our purpose is to study the closure of the class $S_H(D,\Omega)$ where $\Omega = \{w :$Re $w > a,\ -\infty < a < a_0 < +\infty\}$. Also, we will give coefficient estimations and a sharp upper bound for the area of the image $f(\{z : |z| \leq r\})$ for these functions.

2. Harmonic mappings onto half plane

We shall use the half plane

$$\Omega = \{w : \text{Re} w > a,\ -\infty < a < a_0 < +\infty\}$$

and a simply connected domain $D \neq \mathbb{C}$ containing the origin. Then $S_H(D,\Omega)$ consists of harmonic, sense preserving univalent mappings $f = h + \bar{g}$ from D onto Ω normalized by

$$f(0) = a_0, \qquad f_z(0) > 0 \qquad \text{and} \qquad f_{\bar{z}}(0) = 0,$$

where h and g are analytic in D and have the expansion

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n$$

in a neighborhood of origin. Since f is sense preserving, the function $w(z) = -g'(z)/h'(z)$ satisfies $|w(z)| < 1$, and the normalizations implies $w(0) = 0$.

φ_D denotes the conformal mapping from D onto the unit disk U normalized by

$$\varphi_D(0)=0 \qquad \text{and} \qquad \varphi_D'(0)>0.$$

Since $S_H(D,\Omega)=S_H(U,\Omega)\circ\varphi_D$, it is sufficient for many problems to consider the class $S_H(U,\Omega)$. Particularly if $D=\Omega$, then $S_H(\Omega,\Omega)$ consists of automorphisms of Ω.

For $z\in U$ and $|\eta|=1$, define now the kernel

$$\begin{aligned} k(z,\eta) &= \int_0^z \frac{1+\eta\zeta}{1-\eta\zeta}\frac{d\zeta}{(1-\zeta)^2} \\ &= \begin{cases} \frac{z}{(1-z)^2} & \text{if} \quad \eta=1 \\ \left[\frac{2\eta}{(\eta-1)^2}\log\left(\frac{1-z}{1-\eta z}\right)+\frac{1+\eta}{1-\eta}\frac{z}{1-z}\right] & \text{if} \quad \eta\neq 1 \end{cases} \end{aligned}$$

Then we define the family

$$\mathcal{F}=\left\{f:f(z)=\operatorname{Re}\left(\frac{cz+a_0}{1-z}\right)+i\,\operatorname{Im}\left[2(a_0-a)\int_{|\eta|=1}k(z,\eta)d\mu,\ \mu\in\wp\right]\right\}$$

where $\wp$ is the set of probability measures on the Borel sets of $|\eta|=1$ and $c=a_0-2a$.

Theorem 1. $S_H(U,\Omega)\subset\mathcal{F}$.

Proof. Let $f=h+\bar{g}\in S_H(U,\Omega)$. $w(z)=-g'(z)/h'(z)$ satisfies the hypotesis of Schwarz's lema. Since Ω is convex in the direction of the imaginary axis, by a result of Clunie an Sheil-Small [3], $\psi=h+g$ is a conformal univalent mapping of U onto Ω. Since the function ψ satisfies normalizations $\psi(0)=h(0)=a_0$ and $\psi'(0)=h'(0)>0$, such a conformal mapping is determined uniquely by Riemmann mapping theorem. Hence

$$\psi(z)=h(z)+g(z)=\frac{cz+a_0}{1-z}.$$

Thus we obtain

$$u(z)=\operatorname{Re}\,f(z)=\operatorname{Re}\psi(z)=\operatorname{Re}\left(\frac{cz+a_0}{1-z}\right) \quad \text{and} \quad h'(z)+g'(z)=\frac{2(a_0-a)}{(1-z)^2}.$$

At the same time,

$$\begin{aligned} h'(z)-g'(z) &= [h'(z)+g'(z)]\frac{h'(z)-g'(z)}{h'(z)+g'(z)} \\ &= \frac{2(a_0-a)}{(1-z)^2}\cdot\frac{1+w(z)}{1-w(z)}=\frac{2(a_0-a)}{(1-z)^2}p(z), \end{aligned}$$

where, by the Herglotz formula $p(z) = \int_{|\eta|=1}(1+\eta z)/(1-\eta z)d\mu$ for some measure $\mu \in \wp$. Thus

$$\begin{aligned} v(z) &= \text{Im}f(z) = \text{Im}[h(z) - g(z)] = 2(a_0 - a)\text{Im}\left[\int_0^z \frac{p(\zeta)}{(1-\zeta)^2}d\zeta\right] \\ &= 2(a_0 - a)\text{Im}\left\{\int\limits_{|\eta|=1}\left[\int_0^z \frac{1+\eta\zeta}{1-\eta\zeta}\cdot\frac{d\zeta}{(1-\zeta)^2}\right]d\mu\right\} \\ &= 2(a_0 - a)\text{Im}\left[\int\limits_{|\eta|=1} k(z,\eta)d\mu\right] \end{aligned}$$

and the theorem is proved. □

Lemma 2. *If* $f \in S_H(U,\Omega)$ *and* $\tilde{f} \in S_H(D,\Omega)$, *then*

$$\tilde{a}_0(\tilde{f}) = a_0(f), \quad \tilde{a}_1 = \tilde{a}_1(\tilde{f}) = a_1 \cdot \varphi_D'(0) \quad \textit{and} \quad a_1 = a_1(f) = 2(a_0 - a).$$

Proof. For each $\tilde{f} \in S_H(D,\Omega)$ and $f \in S_H(U,\Omega)$, as we can write $\tilde{f} = f \circ \varphi_D$,

$$\tilde{a}_0(\tilde{f}) = \tilde{f}(0) = (f \circ \varphi_D)(0) = f(0) = a_0,$$

$$\tilde{a}_0(\tilde{f}) = \tilde{f}_z(0) = (f \circ \varphi_D)'(0) = a_1 \cdot \varphi_D'(0).$$

Also

$$a_1 = a_1(f) = h'(0) = \psi'(0) = 2(a_0 - a) > 0.$$

□

Theorem 3. $\mathcal{F}$ *is convex and compact.*

Proof. For $\mu \in \wp$ the transformation $\mathcal{L}(\mu) =\text{Im}\left[a_1\int_{|\eta|=1}k(z,\eta)\,d\mu\right]$ is a linear transformation of $\wp$. Then for $f_1, f_2 \in \mathcal{F}$, $\mu_1, \mu_2 \in \wp$ and for the constant C, we can write $f_1 = C + \mathcal{L}(\mu_1)$ and $f_2 = C + \mathcal{L}(\mu_2)$. From this, for $0 < t < 1$ we can obtain

$$tf_1 + (1-t)f_2 = C + f_1 = C + \mathcal{L}(t\mu_1 + (1-t)\mu_2).$$

Therefore the convexity of $\wp$ implies the convexity of $\mathcal{F}$.

Similarly it can be shown that the compactness of $\wp$ implies the compactness of $\mathcal{F}$. $\square$

Theorem 4. *If $f \in \mathcal{F}$, then f is normalized harmonic, sense preserving and univalent from U into Ω.*

Proof. Let $f = h + \bar{g} = \mathrm{Re}F \ + i \ \mathrm{Re}G$, then with

$$F(z) = \frac{cz + a_0}{1 - z} \qquad \text{and} \qquad G(z) = -i \ a_1 \int_0^z \frac{p(\zeta)}{(1-\zeta)^2} d\zeta.$$

Since

$$g'(z)/h'(z) = [F'(z) - iG'(z)]/[F'(z) + iG'(z)] = [1 - p(z)]/[1 + p(z)],$$

it follows that $|g'(z)| < |h'(z)|$ for $z \in U$. Thus f is locally univalent and sense preserving in U. Also

$$h(z) + g(z) = \frac{a_1 z}{1 - z} + a_0$$

is convex in the direction of the real axis. By the theorem of Clunie and Sheil-Small [3], f is univalent in U. Morever, since

$$\mathrm{Re} f(z) = \mathrm{Re} \frac{cz + a_0}{1 - z} > a$$

for all $f \in \mathcal{F}$ and $z \in U$, it follows that $f(U) \subset \Omega$. $\square$

Remark 1. $S_H(U, \Omega) \neq \mathcal{F}$. *For istance, if μ is a unit point mass at $\eta = -1$, then*

$$f(z) = Re \left(\frac{cz + a_0}{(1-z)^2} \right) + i \frac{a_1}{2} arg \left(\frac{1+z}{1-z} \right)$$

maps U onto the half-strip $\{w : Rew > a, \ |Imw| < a_1 \pi/4\}$. Therefore $f \in \mathcal{F} \backslash S_H(U, \Omega)$.

Although, functions in $\mathcal{F}$ do not necesserily map U onto Ω but they map U into subdomains.

Using an argument similar to that in [5, Lemma 2.5 and Theorem 2.7] we obtain the following results. We omit the proofs.

Theorem 5. *If* $f \in \mathcal{F}$, *then* $f(U)$ *is convex.*

Theorem 6. $\mathcal{F} = \overline{S_H(U,\Omega)}$ *and* $\overline{S_H(D,\Omega)} = \mathcal{F} \circ \varphi_D$, *where the* $\overline{S_H(U,\Omega)}$ *is closure of* $S_H(U,\Omega)$.

The set of extreme points of $\overline{S_H(U,\Omega)}$ is the set of functions

$$f_\eta(z) = \mathrm{Re}\left(\frac{cz + a_0}{1-z}\right) + i\ a_1 \mathrm{Im}\ k(z,\eta).$$

Proof. Let us show the set of extremum points of $\overline{S_H(U,\Omega)}$ by $E_{\overline{S_H(U,\Omega)}}$. If $f_\eta \in E_{\overline{S_H(U,\Omega)}}$, then the associated μ must be an extreme point of the set $\wp$ of probability measures on $|\eta| = 1$. Thus, we show that μ be a point mass. We suppose that $f_\eta = tf_1 + (1-t)f_2$ for some $f_1, f_2 \in \overline{S_H(U,\Omega)}$ and $0 < t < 1$. Then Re $f_\eta = t\mathrm{Re} f_1 + (1-t)\mathrm{Re} f_2$ and therefore $\mathrm{Re} f_1 = \mathrm{Re} f_2$. Also, the map $\mathcal{L}(\mu) = \mathrm{Im}\left[a_1 \int_{|\eta|=1} k(z,\eta) d\mu\right]$ is linear and one-to-one for $\mu \in \wp$. Since Im $f_\eta = t\,\mathrm{Im} f_1 + (1-t)\,\mathrm{Im} f_2$, and for this equation to be satisfied it must be the $\mathrm{Im} f_1 = \mathrm{Im} f_2$ and μ must be of unit point mass. The unit point masses are the extremum points of $\wp$. Therefore the relation $f_\eta = tf_1 + (1-t)f_2$ is only walid when $f_1 = f_2$ and $\mu \in E_\wp$. Therefore $f_\eta \in E_{\overline{S_H(U,\Omega)}}$. □

3. The Mapping Properties of Extreme Points.

In this section we obtain the image of U under the extreme points $f_\eta(z)$ of $\overline{S_H(U,\Omega)}$. If $\eta = 1$, then the extreme point is

$$f(z) = \mathrm{Re}\left(\frac{cz + a_0}{1-z}\right) + i\ a_1 \mathrm{Im}\left(\frac{z}{(1-z)^2}\right); \quad c = a_0 - 2a.$$

Its boundary values are all a except at the point 1. Also, f maps U onto the region $\Omega = \{w : \mathrm{Re}\ w > a\}$. If $\eta = e^{i\theta} \neq 1$, then the extreme point is

$$f(z) = \mathrm{Re}\left(\frac{cz + a_0}{1-z}\right) + i\ a_1 \left[\frac{1}{2\sin^2(\theta/2)} \arg\left(\frac{1 - e^{i\theta}z}{1-z}\right) - \cot(\theta/2)\ \mathrm{Re}\left(\frac{z}{1-z}\right)\right].$$

Suppose $\eta = e^{i\theta}$, $0 < \theta < \pi$. If ζ is on the open arc of the unit circle going from 1 to η to -1 to $\bar{\eta}$ in the counter-clockwise direction, then $\arg[(1-\eta\zeta)/(1-\zeta)] = \theta/2$ and $\lim\limits_{z\to\zeta} f(z) = a + iq_1$ where $q_1 = a_1(\theta + \sin\theta)/(4\sin^2(\theta/2)) > 0$. Since $dq_1/d\theta < 0$ for

$0 < \theta < \pi$, q_1 is decreasing in this interval and $q_1 \to a_1\pi/4$ as $\theta \to \pi$. If ζ is the open arc from $\bar{\eta}$ to 1, then $\arg[(1-\eta\zeta)/(1-\zeta)] = (\theta/2) - \pi$ and we obtain $\lim_{z\to\zeta} f(z) = a + iq_2$ where $q_2 = a_1(\theta + \sin\theta - 2\pi)/[4\sin^2(\theta/2)] < 0$. Since $dq_2/d\theta > 0$ for $0 < \theta < \pi$, q_2 is increasing in this interval and $q_2 \to -a_1\pi/4$ as $\theta \to \pi$. Thus, the cluster set of $f(z)$ at $\bar{\eta}$ is the closed segment of the imaginary line joining $a + iq_1$ and $a + iq_2$. The cluster set of f at 1 contains the rest of $\partial f(U)$. It consists of the half-lines

$$\{a + iy : y > q_1\} \qquad \textit{and} \qquad \{a + iy : y < q_2\}$$

4. Applications.

In this section we will use our knowledge of extreme points to solve some extremal problems on $\overline{S_H(U,\Omega)}$.

Theorem 7. *$f = h + \bar{g} \in S_H(U,\Omega)$ and*

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \textit{and} \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n$$

Then

$$|a_n| \le \frac{n+1}{2}\, a_1, \quad |b_n| \le \frac{n-1}{2}\, a_1 \quad \textit{and} \quad ||a_n| - |b_n|| \le a_1. \tag{2}$$

Equality occurs in all cases for the functions

$$f(z) = Re\left(\frac{cz + a_0}{1 - z}\right) + i\ a_1 Im\ \left(\frac{z}{(1-z)^2}\right).$$

Proof. We need only prove these inequalities for the extreme points of $\overline{S_H(U,\Omega)}$. Let

$$f_\eta(z) = \mathrm{Re}\left(\frac{cz + a_0}{1 - z}\right) + i\ a_1 \mathrm{Im}\ k(z,\eta).$$

In our notation

$$F(z) = \frac{cz + a_0}{1 - z} \quad \text{and} \quad iG(z) = a_1\left[\frac{2\eta}{(\eta - 1)^2}\log\left(\frac{1 - z}{1 - \eta z}\right) + \frac{1 + \eta}{1 - \eta}\frac{z}{1 - z}\right].$$

Thus

$$h(z) = \frac{1}{2}[F(z) + iG(z)] = \frac{1}{2}\left[\frac{cz + a_0}{1 - z} + a_1\ k(z,\eta)\right] = \sum_{n=0}^{\infty} a_n z^n,$$

$$g(z) = \frac{1}{2}[F(z) - iG(z)] = \frac{1}{2}\left[\frac{cz + a_0}{1 - z} - a_1\ k(z, \eta)\right] = \sum_{n=2}^{\infty} b_n z^n.$$

Thus, for $n \geq 1$

$$\begin{aligned} 2a_n &= a_1\left[1 + \frac{2\eta}{(1-\eta)^2}\frac{\eta^n - 1}{n} + \frac{1+\eta}{1-\eta}\right] \\ &= a_1\left[1 + \frac{1}{n(\eta - 1)}\right]\left[2\sum_{k=2}^{n}\eta^k + (2+n)\eta - n\right] \end{aligned}$$

and it follows that

$$a_n = a_1\left[1 + \frac{1}{n}\sum_{k=1}^{n-1}(n-k)\eta^k\right] = \left[\frac{a_1}{n}\sum_{k=0}^{n-1}(n-k)\eta^k\right].$$

Thus

$$|a_n| \leq \frac{n+1}{2}\ a_1 = (n+1)(a_0 - a).$$

Similarly, for all $n \geq 2$

$$b_n = -\frac{a_1}{n}\sum_{k=0}^{n-1}(n-k)\eta^k \quad \text{and} \quad |b_n| \leq \frac{n-1}{2}\ a_1 = (n-1)(a_0 - a)$$

with equality for $\eta = 1$.

Next we are concerned with estimates $|\tilde{a}_n|$ and $|\tilde{b}_n|$ that are valid for all domains D containing the origin. Let $f = h + \bar{g} \in \overline{S_H(U,\Omega)}$ and $\tilde{h}(z) = a_0 + \tilde{a}_1 z + \tilde{a}_2 z^2 + ...$ and $\tilde{g}(z) = \tilde{b}_2 z^2 + ...$ in a neighborhood of $z = 0$. By the Lemma 2, the leading coefficient a_0 is independent of D. We can write $\tilde{f} = f \circ \varphi_D \in S_H(D,\Omega)$ and so $\tilde{h} = h \circ \varphi_D$, $\tilde{g} = g \circ \varphi_D$ and $\tilde{f} = \tilde{h} + \overline{\tilde{g}}$. Since

$$\tilde{h}' + \tilde{g}' = \frac{a_1\varphi_D'}{(1-\varphi_D)^2} \quad \text{and} \quad \tilde{h}' - \tilde{g}' = \frac{a_1\varphi_D'}{(1-\varphi_D)^2}\ p(\varphi_D) \tag{3}$$

we have by Theorem 1

$$\tilde{a}_n + \tilde{b}_n = \left[\frac{a_1\varphi_D'(z)}{(1-\varphi_D(z))^2}\right]_{z=0}^{(n-1)}, \quad \tilde{b}_n = \frac{a_1}{n!}\left[\frac{\varphi_D'(z)}{(1-\varphi_D(z))^2}\frac{\varphi_D(z)}{\varphi_D(z) - 1}\right]_{z=0}^{(n-1)},\ n \geq 2$$

and

$$\tilde{a}_n = \frac{a_1}{n!}\left[\frac{\varphi_D'(z)}{(1-\varphi_D(z))^3}\right]_{z=0}^{(n-1)}, \quad n \geq 1.$$

For automorphism of Ω containing the $a_0 \in \mathbb{R}$ we have the following coefficient estimates. $\square$

Theorem 8. *Let* $\tilde{f} = \tilde{h} + \overline{\tilde{g}} \in \overline{S_H(\Omega,\Omega)}$ *and suppose* $\tilde{h}$ *and* $\tilde{g}$ *have expansions*

$$\tilde{h}(z) = \sum_{n=0}^{\infty} \tilde{a}_n (z-a_0)^n \qquad \textit{and} \qquad \tilde{g}(z) = \sum_{n=2}^{\infty} \tilde{b}_n (z-a_0)^n \tag{4}$$

in a neighborhood of $z = a_0$. *Then* $\tilde{a}_1 = 1$ *and*

$$|\tilde{a}_n| = |\tilde{b}_n| \leq \frac{2^{n-2}}{n\ a_1^{n-1}}\ ; \qquad n \geq 2.$$

Equality occurs for the functions

$$f(z) = Re(z) - i(a_1/2)arg(z-a)$$

which arise from unit point measures at $\eta = -1$.

Proof. Since $\varphi_\Omega(z) = (z-a_0)/(z+c)$, it follows from Lemma 2 that $\tilde{a}_1 = 1$

$$\tilde{h}'(z) + \tilde{g}'(z) = \frac{a_1 \varphi_\Omega'(z)}{(1-\varphi_\Omega(z))^2} = 1$$

and then $\tilde{a}_n = -\tilde{b}_n$ for all $n \geq 2$. Also, since

$$\tilde{h}'(z) = 1 + p(\varphi_\Omega(z)) = \int_{|\eta|=1} \frac{1}{1-\varphi_\Omega(z)} d\mu,$$

it follows that for $z = a_0$ we have,

$$n\tilde{a}_n = \frac{1}{a_1^{n-1}} \int_{|\eta|=1} \eta(1-\eta)^{n-2} d\mu\ , \quad |n\tilde{a}_n| \leq \frac{2^{n-2}}{a_1^{n-1}} \int_{|\eta|=1} d\mu = \frac{2^{n-2}}{a_1^{n-1}}$$

and so

$$|\tilde{a}_n| = |\tilde{b}_n| \leq \frac{2^{n-2}}{n\ a_1^{n-1}} = \frac{1}{2n(a_0-a)^{n-1}}.$$

The next theorem is concerned with the estimates of $|\tilde{a}_n|$ and $|\tilde{b}_n|$ that are valid for all domains D. $\square$

Theorem 9. *Let $\tilde{f}=\tilde{h}+\overline{\tilde{g}}\in\overline{S_H(D,\Omega)}$ and $\tilde{h}$, $\tilde{g}$ have expansions (4). Then $\tilde{a}_1=a_1\varphi'_D$ and for all $n\geq 2$*

$$|\tilde{a}_n|\leq a_1\left[\frac{(2n)!}{4(n!)^2}+2^{2n-3}\right]|\varphi'_D(0)|,\qquad |\tilde{b}_n|\leq a_1\left[\frac{(2n)!}{4(n!)^2}-2^{2n-3}\right]|\varphi'_D(0)|.$$

Equality occurs for

$$\tilde{f}_0=Re\left(\frac{a_1}{2\sqrt{1-4z}}+a\right)+i\ a_1 Im\left(\frac{z}{1-4z}\right).$$

Proof. It follows from Lemma 2 that $\tilde{a}_1=a_1\varphi'_D$. It is no loss of generality to assume $\tilde{a}_1=a_1\varphi'_D=1$. Let $f\in S_H(U,\Omega)$ have coefficients a_n, b_n and $\varphi_D(z)=z+A_2z^2+...$ near $z=0$.

By a theorem of Loewner [7] the coefficients A_n are dominated by the coefficients of the fuction

$$\varphi_{D_0}(z)=\frac{1-2z-\sqrt{1-4z}}{2z}$$

and its rotations. Since the estimates of Theorem 8 are sharp for the function

$$f_0(z)=\mathrm{Re}\left(\frac{cz+a_0}{1-z}\right)+i\ a_1\mathrm{Im}\left(\frac{z}{(1-z)^2}\right).$$

$\tilde{a}_n$ and $\tilde{b}_n$ are dominated by the corresponding coefficients of $f_0\circ\varphi_{D_0}=\tilde{h}_0+\overline{\tilde{g}}_0$.That is $\tilde{a}_n$ and $\tilde{b}_n$ are bounded by the corresponding coefficients of

$$\tilde{h}_0(z)=\frac{1}{2}[\tilde{F}_0(z)+i\ \tilde{G}_0(z)]\quad\text{ and }\quad \tilde{g}_0(z)=\frac{1}{2}[\tilde{F}_0(z)-i\ \tilde{G}_0(z)].$$

Thus,

$$\begin{aligned}\tilde{h}_0(z) &= \frac{1}{2}\left[\frac{c\varphi_{D_0}(z)+a_0}{1-\varphi_{D_0}(z)}+a_1\frac{\varphi_{D_0}(z)}{(1-\varphi_{D_0}(z))^2}\right]=\frac{1}{2}\left[\frac{a_1}{\sqrt{1-4z}}+\frac{a_1z}{1-4z}+a_0\right]\\ &= a_1\sum_{n=1}^{\infty}\frac{(2n)!}{4(n!)^2}\ z^n+\frac{a_1}{2}\sum_{n=1}^{\infty}4^nz^{n+1}+\frac{a}{2}=a_1\sum_{n=1}^{\infty}\left[\frac{(2n)!}{4(n!)^2}+2^{2n-3}\right]\ z^n+\frac{a}{2}.\end{aligned}$$

Similarly, we have

$$\tilde{g}_0(z)=a_1\sum_{n=1}^{\infty}\left[\frac{(2n)!}{4(n!)^2}-2^{2n-3}\right]\ z^n+\frac{a}{2}.$$

Therefore, the proof of the theorem is completed. □

Theorem 10. *If $f = h + \bar{g} \in S_H(U,\Omega)$, then*

$$|f_z(z)| = |h'(z)| \le \frac{a_1}{(1-|z|)^3} \qquad \text{and} \qquad |f_{\bar{z}}(z)| = |g'(z)| \le \frac{a_1|z|}{(1-|z|)^3}.$$

Equality occurs for the function

$$f(z) = Re\left(\frac{cz + a_0}{1-z}\right) + i\ a_1 Im\left(\frac{z}{(1-z)^2}\right).$$

Proof. We need only consider extreme points $f_\eta(z)$. In this case

$$h(z) = \frac{1}{2}\left\{\frac{cz+a_0}{1-z} + a_1\left[\frac{2\eta}{(\eta-1)^2}\log\left(\frac{1-z}{1-\eta z}\right) + \frac{1+\eta}{1-\eta}\frac{z}{1-z}\right]\right\},$$

$$\begin{aligned} h'(z) &= \frac{a_1}{2}\left[\frac{1}{(1-z)^2} - \frac{2\eta}{(1-\eta)(1-z)(1-\eta z)} + \frac{1+\eta}{(1-\eta)(1-z)^2}\right] \\ &= \frac{a_1}{(1-\eta z)(1-z)^2} \end{aligned}$$

and

$$|h'(z)| = \frac{a_1}{|1-\eta z||1-z|^2} \le \frac{a_1}{(1-|z|)^3}.$$

Similarly we get

$$|g'(z)| = \left|-\frac{a_1 \eta z}{(1-\eta z)(1-z)^2}\right| \le \frac{a_1|z|}{(1-|z|)^3}.$$

□

Theorem 11. *If $f = h + \bar{g} \in S_H(U,\Omega)$ and $U_r = \{z : |z| \le r < 1\}$, then the area of $f(U_r)$ is A_r and*

$$A_r \le \pi a_1^2\ \frac{r^2(1+r^2)}{(1-r^2)^3}.$$

The bound is sharp.

Proof. Let $f = u + iv$, $\partial U_r = C_r$ and $f(C_r) = \Gamma_r$. Then the area of the domain enclosed by Γ_r is by

$$A_r = \frac{1}{2}\int_{\Gamma_r}(udv - vdu) = \frac{1}{2}\int_0^{2\pi}[u(\theta)\frac{dv}{d\theta} - v(\theta)\frac{du}{d\theta}]d\theta.$$

Since $u = \mathrm{Re}(h+g)$ and $v = \mathrm{Im}(h-g)$, we have

$$u(\theta) = \frac{1}{2}\sum_{n=0}^{\infty} r^n[(a_n+b_n)e^{in\theta} + (\bar{a}_n+\bar{b}_n)e^{-in\theta}]$$
$$v(\theta) = \frac{1}{2}\sum_{n=0}^{\infty} r^n[(a_n-b_n)e^{in\theta} - (\bar{a}_n-\bar{b}_n)e^{-in\theta}].$$

By putting these values in (5) and taking integral of both sides, we obtain

$$A_r = \pi a_1^2 r^2 + \pi \sum_{n=2}^{\infty} n r^{2n}(|a_n|^2 - |b_n|^2).$$

By Theorem 8, we get

$$A_r \leq \pi a_1^2 \sum_{n=2}^{\infty} n^2 r^{2n} = \pi a_1^2 \, \frac{r^2(1+r^2)}{(1-r^2)^3} \;;\; 0 \leq r < 1.$$

□

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