Turk J Math
23 (1999) , 519 – 530.

ON THE ARTIN CONDUCTOR $\mathfrak{f}_{\mathrm{Artin}}(\chi)$ OF A CHARACTER χ of Gal(E/K) I: BASIC DEFINITIONS

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Dedicated to Professor Masatoshi Ikeda on the occasion of his 70^{th} birthday

Abstract

Let K be a local field with finite residue class field and E a finite Galois extension over K. In this paper, we study the Artin conductor $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ of a character χ_ρ associated to a representation $\rho : \mathrm{Gal}(E/K) \to GL(V)$ of Gal(E/K) with metabelian kernel ker(ρ). In order to do so, we first review the Artin character $a_{\mathrm{Gal}(E/K)}$ of Gal(E/K) and review the metabelian local class field theory. We finally propose the definition of the conductor $\mathfrak{f}(E/K)$ of a metabelian extension E/K in the sense of Koch-de Shalit local class field theory, and compute $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ under a suitable assumption.

§1. Introduction

Let K be a local field with finite residue class field $O_K/\mathfrak{p}_K =: \kappa_K$ of q_K elements, where as usual, O_K stands for the ring of integers in K with the unique maximal ideal $\mathfrak{p}_K$. Let ν denote the corresponding normalized valuation on K (normalized by $\nu(K^\times) = \mathbb{Z}^\times$), and $\widetilde{\nu}$ the unique extension of ν to a fixed seperable closure K^{sep} of K.

Let E be a finite Galois extension over K, and let $\rho : \mathrm{Gal}(E/K) \to GL(V)$ be an irreducible (finite-dimensional) representation of the Galois group Gal(E/K) of E/K over $\mathbb{C}$. Furthermore, setting $E^{\ker(\rho)}$ to denote the fixed field of ker(ρ), suppose that $E^{\ker(\rho)}/K$

Key words and phrases. Local fields, higher-ramification groups, Artin conductor, metabelian local class field theory

*This work has been carried out under the frame of the Project "$GL(n)$" at Feza Gürsey Institute of Theoretical Physics, İstanbul-Turkey.

is an abelian extension. It is then well-known that

$$\mathfrak{f}(E^{\ker(\rho)}/K) = \mathfrak{f}_{\mathrm{Artin}}(\chi_\rho), \tag{1.1}$$

where $\mathfrak{f}(E^{\ker(\rho)}/K)$ is the conductor of the abelian extension $E^{\ker(\rho)}/K$, which is defined in the sense of abelian local class field theory (cf. chapter 5, paragraph 1.6 of [5]), and $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ is the Artin conductor of the character $\chi_\rho : \mathrm{Gal}(E/K) \to \mathbb{C}$ associated to the representation (ρ, V) of $\mathrm{Gal}(E/K)$ (which will be reviewed in the next section).

It is then natural to ask more generally the behaviour of $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ without any assumption on the kernel $\ker(\rho)$ of the representation (ρ, V). It seems that, this general problem is closely related with the 2-abelian local class field theory of Koch and de Shalit, and more generally with the n-abelian local class field theory.

In this paper, we start our investigation on this problem for the metabelian kernel $\ker(\rho)$ case. In particular, in this paper, we will collect the necessary basic definitions, propose the definition of the conductor $\mathfrak{f}(E/K)$ of a finite metabelian (=2-abelian) extension E/K in the sense of metabelian local class field theory (following the main theorem of higher ramification theory in metabelian local class field theory), and under a suitable assumption, compute $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$.

Our aim, which is the subject of our next paper is to study the following problem (which should be viewed as the metabelian generalization of eq. no. (1.1)):

Problem 1.1 *Let E/K be a finite Galois extension, $\rho : Gal(E/K) \to GL(V)$ an irreducible (finite-dimensional) representation of the Galois group of E/K over $\mathbb{C}$. Suppose that $E^{\ker(\rho)}/K$ is a 2-abelian extension. Then,*

$$\mathfrak{f}(E^{\ker(\rho)}/K) = \mathfrak{f}_{Artin}(\chi_\rho).$$

§2. Artin representation $A_{\mathbf{Gal}(E/K)}$ of Gal(E/K)

For a finite seperable extension L/K, and for any $\sigma \in \mathrm{Hom}_K(L, K^{sep})$, introduce

$$i_{L/K}(\sigma) := \min_{x \in O_L} \{\widetilde{\nu}(\sigma(x) - x)\},$$

put

$$\gamma_t := \# \left\{\sigma \in \mathrm{Hom}_K(L, K^{sep}) : \ i_{L/K}(\sigma) \geq t\right\}$$

for $-1 \leq t \in \mathbb{R}$, and define the function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ (the Hasse-Herbrand transition function of the extension L/K) by

$$\varphi_{L/K}(u) := \begin{cases} t_0^u \frac{\gamma_t}{\gamma_0}dt, & 0 \leq u \in \mathbb{R}, \\ u, & -1 \leq u \leq 0. \end{cases}$$

It is well-known that, $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is a monotone-increasing (piecewise linear) function, and induces a homeomorphism $\mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$. Let $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ be the mapping inverse to the function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$.

Now, assume that E is a finite Galois extension over K with Galois group $\mathrm{Gal}(E/K) =: G$. The subgroup G_u of G defined by

$$G_u = \{\sigma \in G : \ i_{E/K}(\sigma) \geq u + 1\}$$

for $-1 \leq u \in \mathbb{R}$ is called the u^{th} ramification group of G (in the lower numbering), and $\{G_u\}_{u \in \mathbb{R}_{\geq -1}}$ induces a filtration on G, called the lower ramification filtration of G. A break in the filtration $\{G_u\}_{u \in \mathbb{R}_{\geq -1}}$ of G is defined to be any number $u \in \mathbb{R}_{\geq -1}$ satisfying $G_u \neq G_{u+\varepsilon}$ for every $0 < \varepsilon \in \mathbb{R}$. The function $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ induces the upper ramification filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ on G by setting

$$G^v := G_{\psi_{L/K}(v)}$$

for $-1 \leq v \in \mathbb{R}$, where G^v is called the v^{th} upper ramification group of G. A break in the upper filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of G is defined to be any number $v \in \mathbb{R}_{\geq -1}$ satisfying $G^v \neq G^{v+\varepsilon}$ for every $0 < \varepsilon \in \mathbb{R}$.

The basic properties of lower and upper ramification filtrations on G are as follows: Suppose that $K \subseteq F \subseteq E$ is a sub-extension of E/K, let $\mathrm{Gal}(E/F) = H$.

(i) The lower numbering on G passess well to the subgroup H of G in the sense that:

$$H_u = H \cap G_u$$

for $-1 \leq u \in \mathbb{R}$;

(ii) and if furthermore, $H \lhd G$, the upper numbering on G passess well to the quotient G/H as:

$$(G/H)^v = G^v H/H$$

for $-1 \leq v \in \mathbb{R}$;

(iii) (transitivity of the Hasse-Herbrand function) $\varphi_{E/K} = \varphi_{F/K} \circ \varphi_{E/F}$.

Let $\mathcal{B}_{\mathcal{E}/\mathcal{K}}$ be the set of all numbers $v \in \mathbb{R}_{\geq -1}$ which occur as breaks in the upper ramification filtration of G. Then,

(iv) (Hasse-Arf theorem) If E/K is an abelian extension, then $\mathcal{B}_{\mathcal{E}/\mathcal{K}}$ is a finite subset of $\mathbb{Z} \cap \mathbb{R}_{\geq -1}$.

Remark 2.1 If E/K is an infinite Galois extension with Galois group $\mathrm{Gal}(E/K) = G$ (which is a topological group under the Krull topology), define the upper ramification filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ on G by the projective limit

$$G^v := \varprojlim_{K \subseteq F \subset E} \mathrm{Gal}(F/K)^v$$

defined over the transition morphisms

$$\begin{array}{ccc} (G/\mathrm{Gal}(E/F))^v & \xleftarrow{t_F^{F'}} & (G/\mathrm{Gal}(E/F'))^v \\ \| & & \| \\ G^v\mathrm{Gal}(E/F)/\mathrm{Gal}(E/F) & \xleftarrow{\text{can.}} & G^v\mathrm{Gal}(E/F')/\mathrm{Gal}(E/F') \end{array}$$

induced from (ii), as $K \subseteq F \subseteq F' \subseteq E$ runs over all finite Galois extensions F and F' over K inside E. Observe that:

(v) $G^{-1} = G$ and G^0 is the inertia group of G;

(vi) $\bigcap_{v \in \mathbb{R}_{\geq -1}} G^v =< 1 >$;

(vii) G^v is a closed subgroup of G (with respect to the Krull topology) for $-1 \leq v \in \mathbb{R}$.

In this setting, a number $-1 \leq v \in \mathbb{R}$ is said to be a break in the upper ramification filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of G, if v is a break in the upper filtration of some finite quotient G/H for some $H \triangleleft G$. As introduced previously, let $\mathcal{B}_{\mathcal{E}/\mathcal{K}}$ be the set of all numbers $v \in \mathbb{R}_{\geq -1}$, which occur as breaks in the upper ramification filtration of G. Then,

(viii) (Hasse-Arf theorem.) $\mathcal{B}_{\mathcal{K}^{\dashv\lfloor}/\mathcal{K}} = \mathbb{Z} \cap \mathbb{R}_{\geq -\infty}$;

and the final important result, in the spirit of Koch-de Shalit local class field theory, is

(ix) $\mathcal{B}_{\mathcal{K}^{\int\rceil}\surd/\mathcal{K}} = \mathbb{Q} \cap \mathbb{R}_{\geq -\infty}$. $\square$

Again assume that E/K is a finite Galois extension. Introduce the class function $a_{\mathrm{Gal}(E/K)} : \mathrm{Gal}(E/K) \to \mathbb{C}$ on the Galois group $\mathrm{Gal}(E/K)$ of the extension E/K by

$$a_{\mathrm{Gal}(E/K)}(\sigma) = \begin{cases} -f(E/K) i_{E/K}(\sigma), & \sigma \neq 1; \\ f(E/K) \sum_{\omega \neq 1} i_{E/K}(\omega), & \sigma = 1. \end{cases}$$

As the set of irreducible representations χ on $\mathrm{Gal}(E/K)$ forms an orthonormal basis of the vector space $C(\mathrm{Gal}(E/K))$ of all class functions on $\mathrm{Gal}(E/K)$ over $\mathbb{C}$ equipped with the hermitian scalar-product

$$(\ ,\ ) : C(\mathrm{Gal}(E/K)) \times C(\mathrm{Gal}(E/K)) \to \mathbb{C}$$

defined by

$$(f, g) = \frac{1}{[E:K]} \sum_{\sigma} f(\sigma) \overline{g(\sigma)}$$

on the $\mathbb{C}$-vector space $C(\mathrm{Gal}(E/K))$, the function $a_{\mathrm{Gal}(E/K)} : \mathrm{Gal}(E/K) \to \mathbb{C}$ is a $\mathbb{C}$-linear combination

$$a_{\mathrm{Gal}(E/K)} = \sum_{\chi} f(\chi) \chi$$

of irreducible characters χ of $\mathrm{Gal}(E/K)$, where the uniquely defined χ-coordinate of $a_{\mathrm{Gal}(E/K)}$ is $f(\chi) = (a_{\mathrm{Gal}(E/K)}, \chi) \in \mathbb{C}$.

Now, applying Hasse-Arf theorem for the characters χ of $\mathrm{Gal}(E/K)$ of degree 1, and then using Brauer induction theory for an arbitrary character χ of $\mathrm{Gal}(E/K)$, it follows that $f(\chi) \in \mathbb{Z}_{\geq 0}$. Hence the class function $a_{\mathrm{Gal}(E/K)} : \mathrm{Gal}(E/K) \to \mathbb{C}$ is in fact a character (called the Artin character of $\mathrm{Gal}(E/K)$) of a representation $A_{\mathrm{Gal}(E/K)} : \mathrm{Gal}(E/K) \to GL(V)$, the Artin representation of $\mathrm{Gal}(E/K)$. The explicit construction of this representation is not known.

Let $\phi \in C(\mathrm{Gal}(E/K))$ and put $f_{\mathrm{Artin}}(\phi) = (\phi, a_{\mathrm{Gal}(E/K)}) \in \mathbb{C}$. If χ is a character of $\mathrm{Gal}(E/K)$, define the Artin conductor $\mathfrak{f}_{\mathrm{Artin}}(\chi)$ of χ to be the ideal in O_K defined by

$$\mathfrak{f}_{\mathrm{Artin}}(\chi) = \mathfrak{p}_K^{f_{\mathrm{Artin}}(\chi)}.$$

§3. Metabelian local class field theory

Recall that, a group G is called n-abelian, if the n^{th}-commutator subgroup $G^{(n)}$ of G is $< 1_G >$. Now, an extension E over any field K is called n-abelian, if it is a Galois extension with an n-abelian Galois group $\mathrm{Gal}(E/K)$. Fix a seperable closure K^{sep} of K (when it is convenient, instead of K^{sep}, sometimes the notation $\overline{K}$ will be used to denote the seperable closure of K), and let $K^{(ab)^n}$ denote the maximal n-abelian extension of K inside K^{sep} (cf. part 2 of [1]).

In [4], Koch and de Shalit have constructed class field theory for metabelian (that is, 2-abelian) extensions of a local field K, which induces the abelian local class field theory when restricted to the abelian extensions of K. The main idea in [4] is the following: let K be a local field with finite residue field κ_K of q_K elements. Let $\overline{\kappa}_K$ be a fixed seperable closure of κ_K. Fix an extension $\phi \in \mathrm{Gal}(K^{sep}/K)$ of the Frobenius automorphism ϕ_K over K, that is fix a Lubin-Tate splitting ϕ over K. It is then well-known that, depending on the choice of ϕ, there exists a unique norm-compatible set of primes

$$\mathcal{L}_\phi^o = \{\pi_L \in L : \ K \subseteq L \subset K_\phi \text{ s.t } [L:K] < \infty\},$$

(where K_ϕ denotes the fixed field of ϕ in K^{sep}), which has a canonical extension to a norm-compatible set of primes

$$\mathcal{L}_\phi = \{\pi_L \in \widetilde{L^{nr}} : K \subseteq L \subset K^{sep} \text{ s.t } [L:K] < \infty\},$$

called the Lubin-Tate labelling over K attached to the Lubin-Tate splitting ϕ over K. For a finite extension L over K which is pointwise fixed by ϕ, there exists a unique Lubin-Tate formal power series $f_L \in O_L[[X]]$ (belonging to π_L for $\widetilde{L^{nr}}$, where π_L chosen from $\mathcal{L}_\phi^o$) satisfying certain properties (cf. 0.3 in [4]), and a unique Lubin-Tate formal group law $F_L \in O_{\widetilde{L^{nr}}}[[X,Y]]$ attached to f_L. Let $[u]_{f_L} : F_L \to F_L$ be the unique endomorphism of F_L over $O_{\widetilde{L^{nr}}}$ of the form $[u]_{f_L} = uX + (\text{higher-degree terms}) \in XO_{\widetilde{L^{nr}}}[[X]]$ for $u \in O_L$, and let

$$\{u\}_{f_L} = [u]_{f_L} \pmod{\pi_L}.$$

Let $\widehat{K^\times}$ denote the profinite completion of $K^\times$, and fix the isomorphism $\widehat{K^\times} \xrightarrow{\sim} U(K) \times \widehat{\mathbb{Z}}$ defined by $a = u_a \pi_K^{\nu_a} \mapsto (u_a, \nu_a)$ for $a \in \widehat{K^\times}$, where π_K is the prime element in K chosen uniquely from $\mathcal{L}_\phi^o$. For $1 \leq d \in \mathbb{Z}$, consider the topological group

$$\mathfrak{G}_d^{[2]}(K,\phi) := \left\{ (a,\xi) \in \widehat{K^\times} \times \overline{\kappa}_K[[X]]^\times : \frac{\xi^{\phi^d}}{\xi} = \frac{\{u_a\}_{f_K}}{X} \right\}$$

under the law of composition defined by

$$(a,\xi)(b,\psi) = \left(ab, \xi(\psi^{\phi^{-\nu_a}} \circ \{u_a\}_{f_K})\right)$$

for $(a,\xi),(b,\psi) \in \mathfrak{G}_d^{[2]}(K,\phi)$, and with a basis of neighborhoods of the underlying topology

$$\mathfrak{G}_d^{[2]}(K,\phi)^{(i,j)} = \left\{(a,\xi) \in \mathfrak{G}_d^{[2]}(K,\phi) : a \in U^i(K), \quad \xi \equiv 1 \pmod{X^j}\right\}$$

for $0 \leq i,j \in \mathbb{Z}$. Let $\mathfrak{G}^{[2]}(K,\phi) := \varprojlim_d \mathfrak{G}_d^{[2]}(K,\phi)$, where the projective limit is defined over the transition morphisms $\mathfrak{G}_{d'}^{[2]}(K,\phi) \to \mathfrak{G}_d^{[2]}(K,\phi)$ for every $1 \leq d, d' \in \mathbb{Z}$ with $d \mid d'$, which is defined by

$$(a,\xi) \mapsto \left(a, \prod_{0 \leq i \lneq \frac{d'}{d}} \xi^{\phi^{di}}\right)$$

for $(a,\xi) \in \mathfrak{G}_{d'}^{[2]}(K,\phi)$.

Let L be a ϕ-compatible extension over K, $\phi' = \phi^{f(L/K)}$ (cf. 0.4 in [4]), and

$$\mathfrak{G}^{[2]}(L,\phi') \xrightarrow{M_{\phi,L/K}} \mathfrak{G}^{[2]}(K,\phi)$$

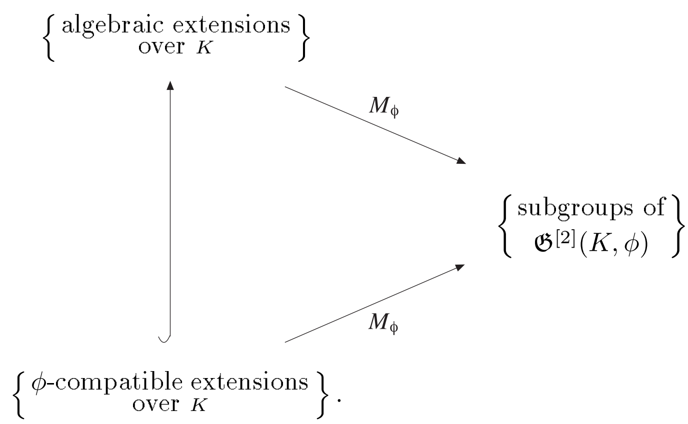
the "2-abelian" norm map which is a canonical morphism defined via Coleman theory (cf. 1.5 in [4]). As a notation, put $M_\phi(L/K) := M_{\phi,L/K}\left(\mathfrak{G}^{[2]}(L,\phi')\right)$, and for an infinite extension E/K which is a union of ϕ-compatible extensions L over K (such an E will be called as an infinite ϕ-compatible extension over K), define $M_\phi(E/K) := \bigcap_{K \subseteq L \subset E} M_\phi(L/K)$ where L runs over all ϕ-compatible sub-extensions in E/K. The map

$$\left\{\begin{matrix}\phi\text{-compatible} \\ \text{extensions over } K\end{matrix}\right\} \xrightarrow{M_\phi} \left\{\begin{matrix}\text{subgroups of} \\ \mathfrak{G}^{[2]}(K,\phi)\end{matrix}\right\}$$

defined by

$$L/K \mapsto M_\phi(L/K)$$

for every ϕ-compatible extension L/K, has a canonical extension



The metabelian local class field theory of Koch-de Shalit states the following.

Theorem 3.1 (Metabelian local class field theory) *Let K be a local field with finite residue field κ_K of q_K elements. Fix an extension $\phi \in Gal(K^{sep}/K)$ of the Frobenius automorphism $\phi_K \in Gal(K^{nr}/K)$.*

(i) There exists an order-preserving bijection

$$\left\{ \begin{matrix} 2\text{-}abelian\ extensions \\ over\ K \end{matrix} \right\} \xrightarrow{M_\phi} \left\{ \begin{matrix} closed\ subgroups \\ of\ \mathfrak{G}^{[2]}(K,\phi) \end{matrix} \right\}$$

defined by

$$L/K \mapsto M_\phi(L/K)$$

for any 2-abelian extension L/K satisfying

$$[L:K] = \Big(\mathfrak{G}^{[2]}(K,\phi) : M_\phi(L/K) \Big) ;$$

(ii) there exists a topological isomorphism

$$\mathfrak{G}^{[2]}(K,\phi) \xrightarrow[\sim]{(?,K)_\phi} Gal(K^{(ab)^2}/K)$$

called the "2-abelian" local Artin map, which depends only on the choice of ϕ;

(iii) for any 2-abelian extension L over K, the surjective morphism

$$\begin{array}{c}\mathfrak{G}^{[2]}(K,\phi) \xrightarrow[\sim]{(?,K)_\phi} Gal(K^{(ab)^2}/K) \xrightarrow{res.^{K^{(ab)^2}}_{l}} Gal(L/K) \\ \text{(composite: } (?,K)_{\phi|L}\text{)}\end{array}$$

induces the canonical topological isomorphism

$$\mathfrak{G}^{[2]}(K,\phi)/M_\phi(L/K) \xrightarrow[\sim]{(?,L/K)_\phi} Gal(L/K),$$

where $M_\phi(L/K)$ is the closed normal subgroup $M_{\phi,L/K}(\mathfrak{G}^{[2]}(L,\phi'))$ of $\mathfrak{G}^{[2]}(K,\phi)$;

(iv) if K' is ϕ-compatible over K, then the square

$$\begin{array}{ccc}\mathfrak{G}^{[2]}(K',\phi') & \xrightarrow{(?,K')_{\phi'}} & Gal(K'^{(ab)^2}/K') \\ {\scriptstyle M_{\phi,K'/K}}\downarrow & & \downarrow{\scriptstyle res_K} \\ \mathfrak{G}^{[2]}(K,\phi) & \xrightarrow{(?,K)_\phi} & Gal(K^{(ab)^2}/K)\end{array}$$

is commutative;

(v) the breaks in the upper-numbering of the ramification groups for 2-abelian extensions over K occur at $r=0$ and at rational numbers of the form

$$u_{i,j} = i - \frac{q_K^i - 1 - j}{q_K^i - q_K^{i-1}}$$

with $1 \le i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ such that $q_K^{i-1} \le j < q_K^i$. If E/K is a finite 2-abelian extension, and

$$i-1 \le u_{i,j-1} < r \le u_{i,j} \le i,$$

then

$$\alpha \in M_\phi(E/K)\mathfrak{G}^{[2]}(K,\phi)^{(i,j)} \Longleftrightarrow (\alpha, E/K)_\phi \in Gal(E/K)^{(r)}.$$

The 2-abelian local class field theory is related with the abelian local class field theory as follows: if E/K is a 2-abelian extension, then

$$\mathrm{pr}_1(M_\phi(E/K)) = N(E/K).$$

The abelian extension over K which is class field to $N(E/K)$ is the maximal abelian sub-extension of E/K.

§4. Conductor of a metabelian extension E/K

In this section, let E be a finite 2-abelian extension over K.

Observe that, if $0 \leq r \in \mathbb{R}$, then there exists a unique $1 \leq i(r) = i \in \mathbb{Z}$ and a unique $j(r) = j \in \left\{q_K^{i-1}, \ldots, q_K^{i-1} + (q_K^i - q_K^{i-1} - 1)\right\}$ such that

$$i(r) - 1 \leq u_{i(r),j(r)-1} < r \leq u_{i(r),j(r)} \leq i(r).$$

In fact, $i(r) - 1 = [[r]]$, where $0 \leq [[r]] \in \mathbb{Z}$ is the "integer part" of the number r.

Define the conductor $\mathfrak{f}(E/K)$ of the extension E/K (in the sense of metabelian local class field theory) to be the ideal

$$\mathfrak{f}(E/K) = \mathfrak{p}_K^i$$

in O_K, where $1 \leq i \in \mathbb{Z}$ is the smallest integer defined by the condition

$$\mathfrak{G}^{[2]}(K, \phi)^{(i, q_K^i - 1)} \subseteq M_\phi(E/K).$$

Observe that, by the metabelian local class field theory, the conductor $\mathfrak{f}(E/K)$ of the 2-abelian extension E/K is defined by $\mathfrak{f}(E/K) = \mathfrak{p}_K^i$, where $1 \leq i \in \mathbb{Z}$ is the smallest integer such that $\mathrm{Gal}(E/K)^{(r)} = 1$ for every $i - 1 \leq i - \frac{1}{q_K^i - q_K^{i-1}} < r \leq i$.

§5. Computation of $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$

Let $\rho : \mathrm{Gal}(E/K) \to GL(V)$ be an irreducible representation of the Galois group $\mathrm{Gal}(E/K)$ of a finite Galois extension E/K. Let $\ker(\rho)$ be the kernel of the representation (ρ, V) of $\mathrm{Gal}(E/K)$. Let $E^{\ker(\rho)}$ be the fixed field of $\ker(\rho)$. Note that, the representation $\widetilde{\rho} : \mathrm{Gal}(E^{\ker(\rho)}/K) \xrightarrow{\sim} \mathrm{Gal}(E/K)/\ker(\rho) \to GL(V)$ is an irreducible representation of $\mathrm{Gal}(E^{\ker(\rho)}/K)$.

Now, if $\phi : \mathrm{Gal}(E^{\ker(\rho)}/K) \to \mathbb{C}$ is any central function on $\mathrm{Gal}(E^{\ker(\rho)}/K)$, and ϕ' is the corresponding central function on $\mathrm{Gal}(E/K)$ defined by

$$\phi' : \mathrm{Gal}(E/K) \xrightarrow{\text{canonical map}} \mathrm{Gal}(E^{\ker(\rho)}/K) \xrightarrow{\phi} \mathbb{C},$$

then $f_{\mathrm{Artin}}(\phi) = f_{\mathrm{Artin}}(\phi')$. In particular, for the case $\phi' = \chi_\rho$ and $\phi = \chi_{\widetilde{\rho}}$, we get $f_{\mathrm{Artin}}(\chi_\rho) = f_{\mathrm{Artin}}(\chi_{\widetilde{\rho}})$.

We will now explicitly compute $(\chi_{\tilde{\rho}}, a_{\mathrm{Gal}(E^{\ker(\rho)}/K)})$ when $E^{\ker(\rho)}/K$ is a metabelian extension. For simplicity, fix the following notation: $G = \mathrm{Gal}(E^{\ker(\rho)}/K)$ and $H = \mathrm{Gal}(E^{\ker(\rho)}/E_1)$, where $E_1 = E \cap K^{ab}$ is the maximal abelian sub-extension in E/K. We assume that H is not contained in the center $Z(G)$ of G. Then it is well-known that there exists a subgroup $H \subseteq H_o \subsetneq G$ and an irreducible representation ψ of H_o such that

$$\rho = \mathrm{Ind}_{H_o}^{G}(\psi).$$

We further assume that $H = H_o$. Then, ψ is a 1-dimensional representation of H, since H is abelian and by the Frobenius reciprocity:

$$f_{\mathrm{Artin}}(\chi_\rho) = (\mathrm{Ind}_H^G(\psi), a_G) = (\psi, \mathrm{Res}_H(a_G)).$$

But $\mathrm{Res}_H(a_G) = \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{d}_{E_1/K}) r_H + [\kappa_{E_1} : \kappa_K] a_H$, where $\mathfrak{d}_{E_1/K}$ is the discriminant of E_1/K and r_H is the character of the regular representation of H. Hence,

$$f_{\mathrm{Artin}}(\chi_\rho) = \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{d}_{E_1/K}) + [\kappa_{E_1} : \kappa_K] f_{\mathrm{Artin}}(\psi),$$

which yields

$$\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho) = \mathfrak{d}_{E_1/K} \mathfrak{f}_{\mathrm{Artin}}(\psi)^{[\kappa_{E_1}:\kappa_K]} = \mathfrak{d}_{E_1/K} \mathfrak{f}(E_1^{\ker(\psi)}/E_1)^{[\kappa_{E_1}:\kappa_K]}.$$

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Received 17.07.2000