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ON GENERALIZED SHIODA - INOSE STRUCTURES

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To Professor G. Ikeda with admiration.

This note concerns algebraic K3 surfaces X admitting a finite group of symplectic automorphisms G such that the quotient X/G is birational to a generalized Kummer surface A_G. To extend the classical case of $G = \mathbf{Z}_2$ which has been extensively studied ([9], [3], [5]), to more general groups one needs to determine finite groups with suitable actions both on K3 surfaces and on abelian surfaces. To this end, finite groups with symplectic actions on K3 surfaces were completely determined in ([7]) and ([11]) and in the latter article the configurations of singularities on the quotients were also listed. On the complementary side, Katsura's article ([4]) contains the classification of all finite groups acting on abelian surfaces so as to yield generalized Kummer surfaces (cf. ([1]) for related lattice theoretic discussion).

In this note, we mainly have two results :

1) a K3 surface X admitting a Shioda-Inose structure with $G \neq \mathbf{Z}_2$ has $\rho(X) \geq 19$ in general and $\rho(X) = 20$ if G is noncyclic,

2) on a singular K3 surface X, all Shioda-Inose structures are induced by a unique abelian surface.

Throughout the paper we will consider only algebraic K3 surfaces over $\mathbf{C}$.

Our notation will be as follows :

A (resp. X) denotes an abelian (resp. an algebraic K3) surface.

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A_G is the Kummer surface constructed from A/G for a suitable finite group G.

$K_?$ denotes the canonical class of ?.

$T_?$ = the transcendental lattice of ?.

$\rho(?)$ is the Picard number of ?.

We use the standard notation A_k, D_k, E_k to denote the rational singularities on surfaces.

We recall that a generalized Kummer surface A_G is a K3 surface which is the minimal resolution of the quotient A/G of an abelian surface A by some finite group G ([4], Definition 2.1).

Definition 1 : *A K3 surface X admits a Shioda-Inose structure with group G if G acts on X symplectically and the quotient X/G is birational to a generalized Kummer surface A_G.*

Generalized Kummer surfaces (in characteristic 0) arise only if G is isomorphic to one of the following groups ([4], Corollary 3.17) :

$\mathbf{Z}_k, k = 2, 3, 4, 6,$

binary dihedral groups Q_8, Q_{12} and

binary tetrahedral group T_{24}.

All of these possibilities occur ([4], Examples).

Comparing this list with the list of finite groups acting symplectically on K3 surfaces ([11], Table 2), we see that all such G appear as a group of symplectic automorphisms of some K3 surface.

In the classical case of $G = \mathbf{Z}_2$, we have the following lattice theoretic characterization of K3 surfaces admitting Shioda-Inose structure ([5], Corollary 6.4).

Theorem [5] : *An algebraic K3 surface X admits a Shioda-Inose structure if and only if X satisfies one of the following conditions :*

(i) $\rho(X) = 19$ *or 20,*

(ii) $\rho(X) = 18$ *and* $T_X = U \oplus T'$,

(iii) $\rho(X) = 17$ *and* $T_X = U^2 \oplus T'$.

In this case, one can also prove the following elementary result.

Lemma 2 : *Given an abelian surface A, there exists a K3 surface X with* $\rho(X) =$

$16 + \rho(A)$ *admitting a Shioda-Inose structure induced by* A.

It follows from ([4], p. 17) that a K3 surface X admitting a Shioda-Inose structure with cyclic group $G \neq \mathbf{Z}_2$ has Picard number $\rho(X) \geq 19$. It follows from Proposition 3 below that if G is noncyclic, then X is a singular K3 surface.

Proposition 3 : *If* G *is a non-cyclic group acting on an abelian surface* A *to yield a generalized Kummer surface, then the singularities of* A/G *are given as follows :*

$3A_1 + 4D_4$ *for* $G = Q_8$,

$A_1 + 2A_2 + 3A_3 + D_5$ *for* $G = Q_{12}$ *and*

$4A_2 + 2A_3 + A_5$ *or* $A_1 + 4A_2 + D_4 + E_6$ *for* $G = T_{24}$.

Remark :

An important application of classical Shioda-Inose structures is the proof of Tate's conjecture for zeta functions of singular K3 surfaces ([9], Theorem 6) which can be trivially extended to give

Lemma 4 : *Let* X *be a K3 surface defined over a number field* $\mathbf{K}$*. If* X *admits a classical Shioda-Inose structure induced from* A *with complex multiplication over* $\mathbf{K}$*, then Tate's conjecture for* $\zeta_X(s)$ *holds true.*

By the remark preceeding Proposition 3, we see that a K3 surface X admitting a generalized Shioda-Inose structure already admits a classical Shioda-Inose structure and hence these generalized structures do not lead to new interesting arithmetic results.

As to the variation of Shioda-Inose structures with respect to the isogenies of abelian surfaces, we have

Lemma 5 : *If* X *is a singular K3 surface, then each and every Shioda-Inose structure on* X *is induced only by* A.

Remark :

Let A_1 and A_2 are two abelian surfaces which are isogeneous via an isogeny of degree n. If X_1, X_2 are two K3 surfaces which admit Shioda-Inose structures (not necessarily with the same group) induced from A_1, A_2 respectively, then X_1, X_2 are isogeneous in the sense of ([6], Definition 1.8). In case of singular K3 surfaces, it follows from Lemma 5 by using ([2]) that stronger form of isogeny holds, that is we have rational maps $X_1 \to X_2$, $X_2 \to X_1$ of degree n.

References

[1] J. Bertin, Resaux de Kummer et surfaces K3, Invent. Math. 93, 267-284 (1988).

[2] H. Inose, On defining equations of singular K3 surfaces and a notion of isogeny, in Proc. Int. Symp. Alg. Geo., Kyoto 1977, Kinokuniya Books (1978), 495-502.

[3] H. Inose, On certain Kummer surfaces which can be realized as nonsingular quartic surfaces in $\mathbf{P^3}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 23 No 3, 545-560 (1976).

[4] T. Katsura, Generalized Kummer surfaces and their unirationality in characteristic p, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. 34, 1-41 (1987).

[5] D. Morrison, On K3 surfaces with large Picard number, Inv. Math. 75, 105-121 (1984).

[6] S. Mukai, On the moduli spaces of bundles on K3 surfaces I, in Vector Bundles in Algebraic Geometry, Tata Institute of Fundamental Research Studies 11, Oxford Uni. Press 1987, 341-413.

[7] S. Mukai, Finite groups of automorphisms of K3 surfaces and the Mathieu group, Inv. Math. 94, 183-221,(1988).

[8] T. Shioda, The period map of abelian surfaces, J. Fac. Sci. Uni. Tokyo, 25, 47-59 (1978).

[9] T. Shioda, H. Inose, On singular K3 surfaces, in Complex Analysis and Algebraic Geometry, Iwanami Shoten (1977), 119-136.

[10] T. Shioda, N. Mitani, Singular abelian surfaces and binary quadratic forms, in CLassification of Algebraic Varieties and Compact Complex Manifolds, LNM No. 412, 259-287 (1974).

[11] G. Xiao, Galois covers between K3 surfaces, Ann. Inst. Fourier, Grenoble 46, 73-88 (1996).

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