Turk J Phy
24 (2000) , 311 – 318.

Gauge invariant reduction and integrability of SL(2,R)/U(1) WZNW theory*

Chris FORD
*DESY Zeuthen, Platanenallee 6, 15738
Zeuthen-GERMANY*
George JORJADZE
*Razmadze Mathematical Institute, M. Aleksidze 1, 380093,
Tbilisi-GEORGIA*
Gerhard WEIGT
*DESY Zeuthen, Platanenallee 6, 15738
Zeuthen-GERMANY*

Received 20.04.2000

Abstract

The general solution of $SL(2,\mathbf{R})/U(1)$ WZNW theory is constructed by a gauge invariant reduction. This is done within both Lagrangian and Hamiltonian frameworks.

1. Introduction

About ten years ago the Dublin group [1] showed that Toda theories can be obtained via nilpotent gauging (or Hamiltonian reduction) of Wess-Zumino- Novikov-Witten (WZNW) theory. Due to their relationship with Lie algebras, the Toda systems are among the models of the theory of integrable non-linear equations [2]. More recently U. Muller and G. Weigt found a Lax pair representation for *non-nilpotent* gaugings of WZNW theory [3]. Without integrating the Lax pair the authors gave the general solution for the $SL(2,\mathbf{R})/U(1)$ case [3]. In this note we show how one can obtain the solution given in [4] in systematic way directly from the gauge invariant reduction procedure.

*Talk presented by G. Jorjadze in Regional Conference on Mathematical Physics IX held at Feza Gürsey Institute, Istanbul, August 1999.

2. Classical Dynamics and Reduction

2.1. Lagrangian for $SL(2,\mathbf{R})$ WZNW Model

Let us introduce a 2-form

$$F_{uv} = \frac{2}{a + \langle u\ g\ v\ g^{-1}\rangle}\ L_u \wedge R_v, \tag{1}$$

given on $SL(2,\mathbf{R})$ group manifold. Here $g \in SL(2,\mathbf{R})$, a is a parameter, u and v are some fixed non-zero elements of $sl(2.\mathbf{R})$ algebra, $\langle u\ g\ v\ g^{-1}\rangle := -1/2\ \mathrm{tr}\ (ugvg^{-1})$ and the 'left', 'right' 1-forms are defined by

$$L_u = \langle u\ dg\ g^{-1}\rangle, \qquad R_v = \langle v\ g^{-1}\ dg\rangle. \tag{2}$$

One can verify (see Appendix) that, for $a^2 = \langle uu\rangle\langle vv\rangle$, (1) provides

$$dF_{uv} = \frac{2}{3}\langle g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\rangle. \tag{3}$$

The function $\Lambda(u,v;g) = \langle u\ g\ v\ g^{-1}\rangle$ is positive (see Appendix) for 'time-like' u and v ($\langle uu\rangle > 0$, $\langle vv\rangle > 0$). Choosing $a = \sqrt{\langle uu\rangle\langle vv\rangle}$ one gets the globally well defined 2-form

$$F_{\hat{u}\hat{v}} = \frac{2}{1 + \langle \hat{u}\ g\ \hat{v}\ g^{-1}\rangle}\ L_{\hat{u}} \wedge R_{\hat{v}}, \tag{4}$$

with normalized vectors $\hat{u}, \hat{v}$ ($\langle\hat{u}\hat{u}\rangle = 1$, $\langle\hat{v}\hat{v}\rangle = 1$), which satisfies (3).

Integration of F over 2d closed surface gives the topological Wess-Zumino term of $SL(2,\mathbf{R})$ WZNW theory. As a result, we find the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{WZ}, \qquad \text{with} \qquad \mathcal{L}_0 = \langle g^{-1}\partial_+ g\ g^{-1}\partial_- g\rangle, \qquad (x_\pm := x \pm t)$$

$$\mathcal{L}_{WZ} = \frac{\langle \hat{u}\ \partial_+ g\ g^{-1}\rangle\langle \hat{v}\ g^{-1}\partial_- g\rangle - \langle \hat{u}\ \partial_- g\ g^{-1}\rangle\langle \hat{v}\ g^{-1}\ \partial_+ g\rangle}{1 + \langle \hat{u}\ g\ \hat{v}\ g^{-1}\rangle}, \tag{5}$$

which leads to the same dynamical equations as WZNW theory

$$\partial_-(\partial_+ g\ g^{-1}) = 0, \qquad\qquad \partial_+(g^{-1}\partial_- g\ ) = 0. \tag{6}$$

Lagrangian (5) is invariant under the global $U(1)$ transformations

$$g \mapsto h_{\hat{u}}(\varepsilon) g h_{\hat{v}}(\varepsilon), \quad \text{with} \quad h_{\hat{u}}(\varepsilon) = e^{\varepsilon\hat{u}} \quad \text{and} \quad h_{\hat{v}}(\varepsilon) = e^{\varepsilon\hat{v}}, \tag{7}$$

We construct the coset $SL(2,\mathbf{R})/U(1)$ model by gauging of (7) symmetry.

2.2. Coset Model

The gauging procedure amounts to the introduction of auxiliary gauge fields $A_\pm$ and construction of a new Lagrangian

$$\mathcal{L}_G(g, A_\pm, \partial_\pm g) = \mathcal{L}(g, \partial_\pm g - A_\pm(\hat{u}g + g\hat{v})), \tag{8}$$

which is invariant under the gauge transformations

$$A_\pm \mapsto A_\pm + \partial_\pm \varepsilon, \qquad g \mapsto h_{\hat{u}}(\varepsilon) g h_{\hat{v}}(\varepsilon) \quad (\varepsilon = \varepsilon(x_+, x_-)).$$

The gauge fields $A_\pm$ can be easily eliminated from (8) using the corresponding variational equations $\partial\mathcal{L}_G/\partial A_\pm = 0$. These equations define

$$A_+ = \frac{\langle \hat{u}\ \partial_+ g\ g^{-1}\rangle}{1 + \langle \hat{u}\ g\ \hat{v}\ g^{-1}\rangle}, \qquad A_- = \frac{\langle \hat{v}\ g^{-1}\ \partial_- g\rangle}{1 + \langle \hat{u}\ g\ \hat{v}\ g^{-1}\rangle} \tag{9}$$

and after elimination of $A_\pm$ we obtain the gauged Lagrangian

$$\mathcal{L}_G| = \langle g^{-1}\partial_+ g\ g^{-1}\partial_- g\rangle - \frac{\langle \hat{u}\ \partial_+ g\ g^{-1}\rangle\langle \hat{v}\ g^{-1}\partial_- g\rangle + \langle \hat{u}\ \partial_- g\ g^{-1}\rangle\langle \hat{v}\ g^{-1}\ \partial_+ g\rangle}{1 + \langle \hat{u}\ g\ \hat{v}\ g^{-1}\rangle}. \tag{10}$$

The gauged Lagrangian (10) can be rewritten in terms of gauge invariant variables. Let us consider the case $\hat{u} = \hat{v}$. To analyze (10) it is convenient to introduce the basis of $sl(2, \mathbf{R})$ algebra T_n $(n = 0, 1, 2)$

$$T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

Without loss of generality, we can assume $\hat{u} = T_0$. Then, the gauge invariant fields are (see Appendix)

$$q_1 = \langle T_1 g\rangle, \qquad q_2 = \langle T_2 g\rangle. \tag{12}$$

Introducing $q_0 = \langle T_0 g\rangle$, one can parameterize $g \in SL(2, \mathbf{R})$ by

$$g = cI + q^n\ T_n = \begin{pmatrix} c - q_2 & -q_1 - q_0 \\ -q_1 + q_0 & c + q_2 \end{pmatrix}, \quad \text{with} \quad c^2 + q^n q_n = 1. \tag{13}$$

Inserting this parameterization in (10) we find the gauged Lagrangian expressed only in terms of gauge invariant fields

$$\mathcal{L}_G| = -\frac{1}{1 + q_1^2 + q_2^2}\left(\partial_+ q_1 \partial_- q_1 + \partial_+ q_2 \partial_- q_2\right). \tag{14}$$

This Lagrangian has a natural complex structure and for the complex valued field $w = q_1 + iq_2$ we get the dynamical equation

$$\partial^2_{+,-} w = \bar{w}\ \frac{\partial_+ w\ \partial_- w}{1 + \bar{w} w}. \tag{15}$$

Described gauging procedure can be done for arbitrary time-like u and v. It can also be generalized for the cases when both u and v are space-like ($\langle uu\rangle < 0$, $\langle vv\rangle < 0$) or one of them is light-like ($\langle uu\rangle = 0$ or/and $\langle vv\rangle = 0$). For all these cases the Wess-Zumino term $\mathcal{L}_{WZ}$ has singularities at $\langle u\ g\ v\ g^{-1}\rangle + \sqrt{\langle uu\rangle\langle vv\rangle} = 0$.

Note that the space of gauge orbits (which defines the space of gauge invariant variables) essentially depends on the choice of u and v generators. For example, in the case $\hat{u} = -\hat{v} = T_0$ the gauge invariant fields are $c = -\langle g\rangle$ and $q_0 = \langle T_0 g\rangle$. According to (13) $c^2 + q_0^2 \geq 1$. Therefore, in this case the configuration space of the reduced system is the manifold with edge. Consistent quantization of coset models should take into account this peculiarities.

2.3. Integrability of the Coset Model

The dynamical equations (6) are invariant under the transformations

$$g(x_+, x_-) \mapsto g_+(x_+)\ g(x_+, x_-)\ g_-(x_-), \tag{16}$$

where $g_\pm(x_\pm)$ are arbitrary $SL(2, \mathbf{R})$ group valued functions. This symmetry provides integrability of WZNW theory and the general solution has the form

$$g(x_+, x_-) = g_+(x_+)\ g_-(x_-). \tag{17}$$

One can check that the Lagrangian (5) is invariant under (16) upto a total derivative.

Let $\tilde{g}(x_+, x_-)$ be a solution of (6), which satisfies the conditions

$$\langle \hat{T}_0\ \partial_+\tilde{g}\ \tilde{g}^{-1}\rangle = 0 \qquad \text{and} \qquad \langle \hat{T}_0\ \tilde{g}^{-1}\ \partial_-\tilde{g}\rangle = 0. \tag{18}$$

Then, the set $(\tilde{g}, A_+, A_-)$, with $A_\pm = 0$ (see (9)) is a solution of the dynamical equations for the system (8), and vice-versa, if the set $(\tilde{g}, A_+ = 0, A_- = 0)$ is a solution for (8), then $\tilde{g}$ satisfies (6) and provides (18). Since the dynamics of the gauge invariant fields q_1 and q_2 does not depend on the choice of gauge fields $A_\pm$, the solution to (15) can be written as

$$q_1 = \langle T_1\ g_+(x_+)\ g_-(x_-)\rangle, \qquad q_2 = \langle T_2\ g_+(x_+)\ g_-(x_-)\rangle, \tag{19}$$

where $g_\pm(x_\pm)$ satisfy the restrictions

$$\langle \hat{T}_0\ g'_+(x_+)\ g_+^{-1}(x_+)\rangle = 0 \quad \text{and} \quad \langle \hat{T}_0\ g_-^{-1}(x_-)\ g'_-(x_-)\rangle = 0. \tag{20}$$

Using (19) and (20) one can derive the general solution of (15). To give the explicit form we use the representation (13) for the chiral and anti-chiral fields

$$g_\pm(x_\pm) = c_\pm(x_\pm)I + q_\pm^n(x_\pm)T_n \tag{21}$$

and introduce polar coordinates for the components $(c_\pm, q_\pm^n)$:

$$\begin{aligned} c_\pm &= R_\pm \cos\beta_\pm, \qquad & q_\pm^0 &= R_\pm \sin\beta_\pm, \\ q_\pm^1 &= -r_\pm \cos\alpha_\pm, \qquad & q_\pm^2 &= \pm r_\pm \sin\alpha_\pm. \end{aligned} \tag{22}$$

Conditions (20) lead to $R_\pm^2 \beta'_\pm - r_\pm^2 \alpha'_\pm = 0$. Since $R_\pm^2 - r_\pm^2 = 1$ (see (13)), we find

$$R_\pm = \sqrt{\frac{\alpha'_\pm}{\alpha'_\pm - \beta'_\pm}}, \qquad r_\pm = \sqrt{\frac{\beta'_\pm}{\alpha'_\pm - \beta'_\pm}}. \tag{23}$$

Inserting (21) and (22) in (19), we get

$$w = q_1 + iq_2 = R_+ r_- e^{i(\alpha_- - \beta_+)} + r_+ R_- e^{-i(\alpha_+ - \beta_-)}. \tag{24}$$

One can check that (24) indeed satisfies (15) if $R_\pm$ and $r_\pm$ are given by (23). Since the solution (24)–(23) depends on four arbitrary functions $\alpha_\pm(x_\pm), \beta_\pm(x_\pm)$ we get the general solution of (15).

As a conformal field theory (14) has a traceless energy momentum tensor ($T_{+-} = 0$) and for the chiral and anti-chiral parts we find

$$T_{\pm\pm} = \frac{1}{1 + \bar{w}w} \partial_\pm \bar{w} \partial_\pm w = \alpha'_\pm \beta'_\pm + \frac{(\alpha''_\pm \beta'_\pm - \beta''_\pm \alpha'_\pm)^2}{4\alpha'_\pm \beta'_\pm (\alpha'_\pm - \beta'_\pm)^2}. \tag{25}$$

2.4. Hamiltonian approach

Hamiltonian reduction of WZNW theory is an alternative method for the construction of coset models. Passing to the Hamiltonian approach we introduces the phase space as a set of functions $R(x), g(x)$ $(x \in [a, b])$, where $R(x)$ and $g(x)$ take values in the $sl(2, \mathbf{R})$ algebra and $SL(2, \mathbf{R})$ group respectively. The boundary behaviour of these fields should provide non-degeneracy of the symplectic form. The 1-form and the Hamiltonian obtained from (5) are

$$\theta = \int_a^b dx \left[-\langle R g^{-1} dg \rangle + \frac{\langle T_0\ g^{-1} g' \rangle \langle T_0\ dg\ g^{-1} \rangle - \langle T_0\ g' g^{-1} \rangle \langle T_0\ g^{-1} dg \rangle}{1 + \langle T_0\ g\ T_0\ g^{-1} \rangle} \right], \tag{26}$$

$$H = -\frac{1}{2} \int_a^b dx\ [\langle R\ R \rangle + \langle g^{-1}\ g'\ g^{-1}\ g' \rangle]. \tag{27}$$

The functions $R(x)$ and $g(x)$ are dynamically related by $g^{-1}\dot{g} = R$. Taking into account this relation and the form of the general solution (17) we introduce the 'chiral' and 'anti-chiral' fields $g_\pm(x)$, which parameterize the phase space

$$g(x) = g_+(x) g_-(x), \qquad R(x) = g_-^{-1}(x)[g_+^{-1}(x)\ g'_+(x) - g'_-(x) g_-^{-1}(x)] g_-(x). \tag{28}$$

The Hamiltonian (27) splits into chiral and anti-chiral parts $H = H_+ + H_-$, with

$$H_\pm = -\frac{1}{2} \int_a^b dx\ \langle g_\pm^{-1}\ g'_\pm\ g_\pm^{-1}\ g'_\pm \rangle\ . \tag{29}$$

The 1-form (26) leads to the symplectic form of WZNW theory [4]

$$\Omega = d\theta = \int_a^b dx\ \left[\langle g_+^{-1}\ dg_+\ (g_+^{-1}\ dg_+)'\rangle - \langle dg_-\ g_-^{-1}\ (dg_-\ g_-^{-1})'\rangle\right] + \\ + \langle g_+^{-1}\ dg_+\ dg_-\ g_-^{-1}\rangle|_a^b. \qquad (30)$$

One can check that the differential of the 1-form $\tilde{\theta} = \theta_+ + \theta_-$ gives the same symplectic form Ω, if the $\theta_\pm$ are given by

$$\theta_\pm = \int_a^b dx\ [-\langle g_\pm^{-1}\ g'_\pm g_\pm^{-1}\ dg_\pm\rangle + \\ + \frac{\langle T_0\ g_\pm^{-1}\ g'_\pm\rangle\langle T_0\ dg_\pm\ g_\pm^{-1}\rangle - \langle T_0\ g'_\pm\ g_\pm^{-1}\rangle\langle T_0\ g_\pm^{-1}\ dg_\pm\rangle}{1 + \langle T_0\ g_\pm\ T_0\ g_\pm^{-1}\rangle}]. \qquad (31)$$

Thus, 1-form (26) can also be split into chiral and anti-chiral parts (up to an exact form and boundary terms).

The gauging procedure of $SL(2,\mathbf{R})$ WZNW theory, which leads to the coset model (14) corresponds to Hamiltonian reduction with the constraints

$$\langle T_0\ g'_+ g_+^{-1}\rangle = 0 \quad \text{and} \quad \langle T_0\ g_+^{-1} g'_+\rangle = 0.$$

Using the parameterization of $g_\pm$ functions (21) we obtain the reduced Hamiltonian $H_| = H_{|+} + H_{|-}$, with

$$H_{|\pm} = \int_a^b dx\ [f_\pm'^2 + \alpha'_\pm \beta'_\pm], \quad \text{where} \quad \tanh^2 f = \frac{\beta'_\pm}{\alpha'_\pm} \qquad (32)$$

and the reduced 1-form $\tilde{\theta}_| = \theta_{|+} + \theta_{|-}$, with

$$\theta_{|\pm} = \int_a^b dx\ [f'_\pm df_\pm + \beta'_\pm d\alpha_\pm]. \qquad (33)$$

Differentiating of (33) reproduces the symplectic form of the (14) model and the integrand in (32) coincides with the energy-momentum tensor (25).

To get the canonical form of the chiral Hamiltonians (32) and chiral 1-forms (33) we pass to the new fields $\phi_{\pm,1}$ and $\phi_{\pm,2}$:

$$e^{\mp i\alpha_\pm} = e^{i\phi_{\pm,2}} \frac{\sinh\phi_{\pm,1} + ie^{\phi_{\pm,1}} F_\pm}{\sqrt{\sinh^2\phi_{\pm,1} + e^{2\phi_{\pm,1}} F_\pm^2}}\ , \quad \text{where} \quad F'_\pm = e^{-2\phi_{\pm,1}} \phi'_{\pm,2}\ .$$

$$e^{\mp i\beta_\pm} = e^{i\phi_{\pm,2}} \frac{\cosh\phi_{\pm,1} - ie^{\phi_{\pm,1}} F_\pm}{\sqrt{\cosh^2\phi_{\pm,1} + e^{2\phi_{\pm,1}} F_\pm^2}}\ , \qquad (34)$$

These equations provide

$$
\begin{aligned}
f_\pm'^2 + \alpha_\pm' \beta_\pm' &= \phi_{\pm,1}'^2 + \phi_{\pm,2}'^2, \\
f_\pm' df_\pm + \beta_\pm' d\alpha_\pm &= \phi_{\pm,1}' d\phi_{\pm,1} + \phi_{\pm,2}' d\phi_{\pm,2}\ .
\end{aligned} \tag{35}
$$

Substituting (34) into the general solution (24), we obtain (see [3])

$$
\begin{aligned}
w = \frac{1}{2} e^{i\phi_{+,2}} e^{i\phi_{-,2}} [& e^{\phi_{+,1}} e^{\phi_{-,1}} (1 + 4F_+F_-) - e^{-\phi_{+,1}} e^{-\phi_{-,1}} \\
& + 2i(F_+ e^{\phi_{+,1}} e^{-\phi_{-,1}} + F_- e^{-\phi_{+,1}} e^{\phi_{-,1}})].
\end{aligned} \tag{36}
$$

Acknowledgments

One of the authors (G.J.) would like to thank the Organizing Committee of the IX Regional Conference on Mathematical Physics for the financial support and warm hospitality.

The main part of the work was done during the visit of G.J. at DESY Zeuthen and ICTP and he is grateful to these Institutions. His work was supported also by the grants from: INTAS, RFBR and the Georgian Academy of Sciences.

3. Appendix

The matrices (11) satisfy the relations

$$T_m\ T_n = -\eta_{mn}\ I + \epsilon_{mn}{}^{l}\ T_l, \tag{A.1}$$

where I is the unit matrix, η_{mn} is a metric tensor of $3d$ Minkowski space: $\eta_{mn} = \mathrm{diag}(+,-,-)$, $\epsilon_{012} = 1$. The normalized trace of matrixes $\langle A \rangle := -1/2\ \mathrm{tr}\ (A)$ gives

$$\langle T_m\ T_n \rangle = \eta_{mn}, \qquad \langle T_l\ T_m\ T_n \rangle = \epsilon_{lmn}. \tag{A.2}$$

For $u = u^n T_n$ and $v = v^n T_n$, we have $\langle u\ T_n \rangle = u_n$, $\langle u\ v \rangle = u^n v_n$ and we get the isometry between $sl(2, \mathbf{R})$ algebra and $3d$ Minkowski space.

The ‘left’ and ‘right’ 1-forms

$$L_n = \langle T_n\ dg\ g^{-1} \rangle, \qquad R_n = \langle T_n\ g^{-1}\ dg \rangle \tag{A.3}$$

are related by

$$L_m = \Lambda_m{}^n(g) R_n, \tag{A.4}$$

where $\Lambda_m{}^n(g) = \langle T_m\ g\ T^n\ g^{-1} \rangle$. The 3-form $h = \langle g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg \rangle$ can be expressed in terms of ‘right’‘ (or ‘left’) 1-forms, since from (A.3) we have

$$g^{-1}\ dg = T^n R_n, \qquad dg\ g^{-1} = T^n L_n. \tag{A.5}$$

Using (A.2) we get

$$h = 6\ R_0 \wedge R_1 \wedge R_2. \tag{A.6}$$

The differentials of (A.3) and (A.4) give

$$dL_n = \epsilon_n{}^{lm}\ L_l \wedge L_m, \qquad dR_n = -\epsilon_n{}^{lm}\ R_l \wedge R_m,$$
$$d\Lambda_{mn} = 2\epsilon_n{}^{kl}\ \Lambda_{mk}(g)\ R_l. \tag{A.7}$$

Taking differential of (1) and using (A.6), (A.7) we obtain (3).

The matrices $\Lambda_m{}^n(g)$ belong to the group $SO_\uparrow(2.1)$. The property $\Lambda_0{}^0 \geq 1$ can be seen by direct computation ($\Lambda_0{}^0 = 1 + 2(q_1^2 + q_2^2)$ (see (13)). The property $\langle u\ g\ v\ g^{-1}\rangle > 0$ for time-like u and v follows from the isometry between the $sl(2.\mathbf{R})$ algebra and 3d Minkowski space.

Since $T_0^2 = I$, we have $\exp(\epsilon T_0) = I\cos\epsilon + T_0 \sin\epsilon$. From (A.1) we find

$$\exp(\epsilon T_0)\ T_n\ \exp(\epsilon T_0) = T_n \quad \text{for} \quad n = 1, 2,$$

which provides gauge invariance of (12).

References

[1] P. Forgacs, A. Wipf, J. Balog, L. Feher and L. O'Raifeartaigh, *Phys. Lett.*, **B 227**, (1989) 214. J. Balog, L. Feher, L. O'Raifeartaigh, P. Forgacs and A. Wipf, *Ann. Phys.*, **203**, (1990) 76; *Phys. Lett.*, **B 251** (1990) 361. L. Feher, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf *Phys. Rep.*, **222**, (1992) 1.

[2] A.N. Leznov and M.V. Saveliev, *Lett. Math. Phys.*, **3**, (1979) 489; *Commun. Math. Phys.*, **74**, (1980) 111. A. M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, (Birkhauser Verlag, Basel-Boston-Berlin, 1990).

[3] U. Muller and G. Weigt, *Phys. Lett.*, **B 422**, (1997) 258. hep-th/9805215; *Comm. Math. Phys.*, in press. hep-th/9907057.

[4] M.F. Chu, P. Goddard, I Halliday, D. Olive and A Schwimmer, *Phys. Lett.*, **B 266**, (1991) 71.