

Canonical Quantisation of the $SL(2, \mathbb{R})/U(1)$ Gauged WZNW Black Hole Model*

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Abstract

Gauged WZNW models are integrable conformal field theories with non-local canonical free-field transformations. Canonical quantisation requires quantum deformations.

1. Introduction

We discuss the non-nilpotent $SL(2, \mathbb{R})/U(1)$ gauged Wess-Zumino-Novikov-Witten (WZNW) theory. It has attracted much interest in the past [1-9] because it is a conformal field theory with a curved target space metric, and it is integrable [10].

The talk is based on refs [10, 11]. We consider the general solution of the equations of motion, look at the symplectic structure of the theory and derive canonical transformations of the physical fields onto free fields. This will be done mainly for periodic boundary conditions, which describe the interesting case of a closed string moving in the background of a black hole target-space metric. We finally quantise parafermionic conserved quantities.

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2. The Solution of the $SL(2, \mathbb{R})/U(1)$ Theory and Conservation Laws

The classical action of the $SL(2, \mathbb{R})/U(1)$ gauged WZNW theory [1, 10] ($z = \tau + \sigma$, $\bar{z} = \tau - \sigma$)

$$S_{\text{WZNW,gauged}}[r, t] = \frac{k}{2\pi} \int_M dz d\bar{z} (\partial_z r \partial_{\bar{z}} r + \tanh^2 r \partial_z t \partial_{\bar{z}} t). \quad (1)$$

describes a conformal and integrable theory [10]. Its target-space metric

$$ds^2 = dr^2 + \tanh^2 r dt^2 \quad (2)$$

shows in Kruskal coordinates

$$u = -\sinh r e^{it}, \quad \bar{u} = \sinh r e^{-it} \quad (3)$$

after Wick rotation, $t \rightarrow it$, a two-dimensional black hole singularity

$$ds^2 = -\frac{du d\bar{u}}{1 - u\bar{u}} \quad (4)$$

with singular curvature tensor.

The equations of motion, which follow from the action (1)

$$\begin{aligned} \partial_z \partial_{\bar{z}} r &= \frac{\sinh r}{\cosh^3 r} \partial_z t \partial_{\bar{z}} t, \\ \partial_z \partial_{\bar{z}} t &= -\frac{1}{\sinh r \cosh r} (\partial_z r \partial_{\bar{z}} t + \partial_z t \partial_{\bar{z}} r) \end{aligned} \quad (5)$$

have the general solution

$$\begin{aligned} \sinh^2 r &= X \bar{X}, \\ t &= i(B - \bar{B}) + \frac{i}{2} \ln \frac{X}{\bar{X}} \end{aligned} \quad (6)$$

with the definitions

$$\begin{aligned} X &= A + \frac{\bar{B}'}{A'} (1 + A\bar{A}), \\ \bar{X} &= \bar{A} + \frac{B'}{A'} (1 + A\bar{A}). \end{aligned} \quad (7)$$

$A = A(z)$, $B = B(z)$, $\bar{A} = \bar{A}(\bar{z})$ and $\bar{B} = \bar{B}(\bar{z})$ are complex (anti-)chiral functions and A' their derivatives, which are only restricted in order to render r and t real. The solution (6,7) is invariant under $GL(2, \mathbb{C})$ transformations of these functions. Therefore, these (anti-)chiral functions are determined by the physical fields at most up to four complex

constants.

The theory is also characterised by conservation laws. The equations of motion (5) guarantee, in particular, conservation and chirality of the energy-momentum tensor (we shall omit the similar anti-chiral parts whenever possible, and use $\gamma = \sqrt{2\pi/k}$)

$$T \equiv T_{zz} = \frac{1}{\gamma^2} ((\partial_z r)^2 + \tanh^2 r (\partial_z t)^2), \quad (8)$$

and in addition of parafermionic observables [1]

$$V_{\pm} = \frac{1}{\gamma^2} e^{\pm i\nu} (\partial_z r \pm i \tanh r \partial_z t), \quad (9)$$

if ν satisfies

$$\partial_z \nu = (1 + \tanh^2 r) \partial_z t, \quad \partial_{\bar{z}} \nu = \cosh^{-2} r \partial_{\bar{z}} t. \quad (10)$$

Since the integrability conditions of these equations just yield the second equation of (5), the general solution (6, 7) also integrates eqs (10).

The vanishing trace of the energy-momentum tensor

$$T_{z\bar{z}} + T_{\bar{z}z} = 0 \quad (11)$$

shows the conformal invariance of the theory. Surprisingly, as in the ungauged theory, the energy-momentum tensor has a Sugawara form

$$T = \gamma^2 V_+ V_-, \quad (12)$$

although the conformal spin-one quantities V_{\pm} are not standard Kac-Moody currents. Using the general solution (6, 7), from the conserved quantities (8, 9) we can easily derive a differential equation of the Gelfand-Dikii type

$$y'' - (\partial_z V_- / V_-) y' - \gamma^2 T y = 0. \quad (13)$$

This equation becomes important for the calculation of the symplectic structure of the theory, because its solutions

$$y_1 = e^B, \quad y_2 = A e^B, \quad (14)$$

and

$$\bar{y}_1 = e^{\bar{B}}, \quad \bar{y}_2 = \bar{A} e^{\bar{B}} \quad (15)$$

for the corresponding anti-chiral Gelfand-Dikii equation, usefully parametrize the general solution of the equations of motion (5). A very symmetrical expression results for the transformed fields u, \bar{u} (3)

$$u = \frac{\bar{y}_1 y'_1 + \bar{y}_2 y'_2}{y_1 y'_2 - y'_1 y_2}, \quad \bar{u} = \frac{y_1 \bar{y}'_1 + y_2 \bar{y}'_2}{\bar{y}_1 \bar{y}'_2 - \bar{y}'_1 \bar{y}_2}. \quad (16)$$

Here for simplicity we shall restrict ourselves in the following to the regular solutions by assuming that for finite z, \bar{z}

$$y_1 y'_2 - y'_1 y_2 \neq 0, \quad \bar{y}_1 \bar{y}'_2 - \bar{y}'_1 \bar{y}_2 \neq 0. \quad (17)$$

The already mentioned $\text{GL}(2, \mathbb{C})$ invariance now becomes

$$\begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \rightarrow \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} \bar{y}_2 \\ \bar{y}_1 \end{pmatrix} \rightarrow \begin{pmatrix} d & -c \\ b & a \end{pmatrix} \begin{pmatrix} \bar{y}_2 \\ \bar{y}_1 \end{pmatrix}, \quad (18)$$

$$\begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C}).$$

3. The Symplectic Structure of the $\text{SL}(2, \mathbb{R})/\text{U}(1)$ Theory

In ref. [10] we have shown that the different (anti-)chiral functions of the theory can be calculated, in principle, in terms of the physical fields by solving the Gelfand-Dikii equations after we have chosen for the physical fields the initial values at ‘time’ τ_0

$$u(\sigma, \tau_0) = u_0(\sigma), \quad \bar{u}(\sigma, \tau_0) = \bar{u}_0(\sigma), \quad \dot{u}(\sigma, \tau_0) = u_1(\sigma), \quad \dot{\bar{u}}(\sigma, \tau_0) = \bar{u}_1(\sigma), \quad (19)$$

and fixed the $\text{GL}(2, \mathbb{C})$ invariance. Although this procedure does not give explicit functions in terms of the physical fields, we can, nevertheless, get the Poisson brackets. In [10] we have calculated the variations $\delta y_k(z), \delta \bar{y}_k(\bar{z})$ explicitly as functions of the variations $\delta u(\tau, \sigma), \delta \bar{u}(\tau, \sigma), \delta \pi_u(\tau, \sigma)$ and $\delta \pi_{\bar{u}}(\tau, \sigma)$ by solving the varied Gelfand-Dikii equations (not writing here the anti-chiral part)

$$\delta y_k'' - (\partial_z V_- / V_-) \delta y_k' - \gamma^2 T \delta y_k = \delta(\partial_z V_- / V_-) y_k' + \gamma^2 \delta T y_k \quad (20)$$

and using the varied initial states.

We give here an example for periodic boundary conditions

$$u(\sigma + 2\pi, \tau) = u(\sigma, \tau), \quad \bar{u}(\sigma + 2\pi, \tau) = \bar{u}(\sigma, \tau). \quad (21)$$

Assuming the canonical Poisson brackets of the physical fields

$$\begin{aligned} \{u(\sigma, \tau), \dot{\bar{u}}(\sigma', \tau)\} &= \{\bar{u}(\sigma, \tau), \dot{u}(\sigma', \tau)\} = 2\gamma^2(1 + u\bar{u})\delta_{2\pi}(\sigma - \sigma'), \\ \{\dot{u}(\sigma), \dot{\bar{u}}(\sigma')\} &= 2\gamma^2(\dot{u}\bar{u} - u\dot{\bar{u}})\delta_{2\pi}(\sigma - \sigma'), \end{aligned} \quad (22)$$

where $\delta_{2\pi}$ is the periodic δ -function defined by

$$\delta_{2\pi}(\sigma - \sigma') \equiv \sum_{n=-\infty}^{\infty} \delta(\sigma - \sigma' + 2\pi n). \quad (23)$$

The periodic Poisson brackets of the parameter functions are than non-linearly realised

$$\begin{aligned}
 \{\ln y_1(z), \ln y_1(z')\} &= \{\ln \bar{y}_1(\bar{z}), \ln \bar{y}_1(\bar{z}')\} = 0, \\
 \{\ln y_1(z), \ln y_2(z')\} &= \frac{\gamma^2}{2} \left(\epsilon_{2\pi}(z - z') - \frac{z - z'}{2\pi} \right) \\
 &\quad - \frac{\gamma^2}{2} E(z, z') + \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz'' E(z'', z'), \\
 \{\ln y_1(z), \ln \bar{y}_1(\bar{z})\} &= -\frac{\gamma^2}{4\pi}(z - z')
 \end{aligned} \tag{24}$$

where

$$\epsilon_{2\pi}(z) = 2n + 1 \quad \text{for} \quad 2\pi n < z < 2\pi(n + 1), \quad n \in \mathbb{Z} \tag{25}$$

and

$$E(z, z') = \frac{\exp\left(\frac{\alpha_1 - \alpha_2}{2} \epsilon_{2\pi}(z - z')\right) \frac{y_2(z)y_1(z')}{y_1(z)y_2(z')}}{\sinh \frac{\alpha_1 - \alpha_2}{2}}. \tag{26}$$

However, the field-theoretic case with asymptotic boundary conditions has a much simpler form

$$\begin{aligned}
 \{\ln y_1(z), \ln y_1(z')\} &= \{\ln \bar{y}_1(\bar{z}), \ln \bar{y}_1(\bar{z}')\} = 0, \\
 \{y_1(z), y_2(z')\} &= \frac{\gamma^2}{2} \epsilon(z - z') - \frac{\gamma^2}{2} \frac{y_2(z) y_1(z')}{y_1(z) y_2(z')}.
 \end{aligned} \tag{27}$$

$\epsilon(z)$ is here the sign function

$$\epsilon(z) \equiv \begin{cases} -1 & \text{for } z < 0, \\ 0 & \text{for } z = 0, \\ 1 & \text{for } z > 0. \end{cases} \tag{28}$$

We see that the zero modes in the periodic model complicate the algebra considerably. Although we have unique mappings between the physical fields u, \bar{u} and the parameter functions y_k, \bar{y}_k , our calculations are not finished until we have found a free-field realization of this symplectic structure.

4. The Canonical Free-Field Transformation

There are several methods to find relations between $y_k(z), \bar{y}_k(\bar{z})$ and the chiral, respectively anti-chiral components $\phi_k(z), \bar{\phi}_k(\bar{z})$ of canonical free fields ($k = 1, 2$)

$$\psi_k(\sigma, \tau) = \phi_k(z) + \bar{\phi}_k(\bar{z}). \tag{29}$$

The easiest and most straightforward approach identifies the energy-momentum tensors

$$T(z) = (\partial_z \phi_1)^2 + (\partial_z \phi_2)^2 = \frac{1}{\gamma^2} \frac{y_1' y_2' - y_1' y_2''}{y_1 y_2' - y_1' y_2}. \quad (30)$$

Here we assume that

1. the free fields ψ_1, ψ_2 are local expressions of the parameter functions y_k , and
2. the energy-momentum tensor has, indeed, the free-field form in terms of the free fields.

We again take into consideration the periodic boundary conditions and obtain the result

$$\begin{aligned} \phi_1 + i\phi_2 &= \frac{1}{\gamma} \ln \frac{y_1'}{y_1 y_2' - y_1' y_2}, & \phi_1 - i\phi_2 &= \frac{1}{\gamma} \ln y_1, \\ \bar{\phi}_1 - i\bar{\phi}_2 &= \frac{1}{\gamma} \ln \frac{\bar{y}_1'}{\bar{y}_1 \bar{y}_2' - \bar{y}_1' \bar{y}_2}, & \bar{\phi}_1 + i\bar{\phi}_2 &= \frac{1}{\gamma} \ln \bar{y}_1. \end{aligned} \quad (31)$$

As expected, the non-local Poisson bracket relations (24) yield for the fields $\phi_k, \bar{\phi}_k$ the local free-field Poisson brackets

$$\begin{aligned} \{\phi_k(\tau + \sigma), \phi_l(\tau + \sigma')\} &= -\frac{\delta_{kl}}{4} \left(\epsilon_{2\pi}(\sigma - \sigma') - \frac{\sigma - \sigma'}{2\pi} \right), \\ \{\bar{\phi}_k(\tau - \sigma), \bar{\phi}_l(\tau - \sigma')\} &= \frac{\delta_{kl}}{4} \left(\epsilon_{2\pi}(\sigma - \sigma') - \frac{\sigma - \sigma'}{2\pi} \right), \\ \{\phi_k(\tau + \sigma), \bar{\phi}_l(\tau - \sigma')\} &= -\frac{\delta_{kl}}{8\pi} (\sigma + \sigma'). \end{aligned} \quad (32)$$

Solving now (31) for y_k, \bar{y}_k yields the non-local free-field representation

$$\begin{aligned} y_1(z) &= \exp(\gamma\chi(z)) \\ y_2(z) &= -\frac{\exp(\gamma\chi(z))}{2 \sinh(\gamma p_1)} \int_0^{2\pi} dz' \gamma\chi'(z') \exp(-\gamma p_1 \epsilon_{2\pi}(z - z') - 2\gamma\phi_1(z')) \\ \bar{y}_1(\bar{z}) &= \exp(\gamma\bar{\chi}(\bar{z})) \\ \bar{y}_2(\bar{z}) &= -\frac{\exp(\gamma\bar{\chi}(\bar{z}))}{2 \sinh(\gamma p_1)} \int_0^{2\pi} d\bar{z}' \gamma\bar{\chi}'(\bar{z}') \exp(-\gamma p_1 \epsilon_{2\pi}(\bar{z} - \bar{z}') - 2\gamma\bar{\phi}_1(\bar{z}')) \end{aligned} \quad (33)$$

where the zero mode momentum is given by

$$p_1 = \int_0^{2\pi} \dot{\psi}_1(\tau, \sigma) d\sigma. \quad (34)$$

We have checked that the free-field Poisson brackets yield, consistently, the Poisson brackets of the $y_k(z)$, $\bar{y}_k(\bar{z})$, and we could show that these results also follow from the Gelfand-Dikii equations, in case, their coefficients are expressed in terms of the free fields and the initial state problem is solved anew.

This proves that the non-local free-field transformations of the physical fields r , t , or u , \bar{u} are canonical transformations, and we can show they are one to one.

The results could be summarised, finally, in terms of local Bäcklund transformations which would be identical, both, for the periodic as well as for the asymptotic case. Instead, we give here the complete canonical transformation of the fields $u(\sigma, \tau)$, $\bar{u}(\sigma, \tau)$ onto the free fields

$$\begin{aligned} u &= e^{\gamma(\phi+\bar{\chi})} \left(1 + \Phi\bar{\Phi}\right) - \frac{1}{4}e^{-\gamma(\bar{\phi}+\chi)} + \frac{i}{2} \left(e^{\gamma(\phi-\bar{\phi})}\Phi + e^{-\gamma(\chi-\bar{\chi})}\bar{\Phi}\right), \\ \bar{u} &= e^{\gamma(\bar{\phi}+\chi)} \left(1 + \Phi\bar{\Phi}\right) - \frac{1}{4}e^{-\gamma(\phi+\bar{\chi})} - \frac{i}{2} \left(e^{\gamma(\chi-\bar{\chi})}\Phi + e^{-\gamma(\phi-\bar{\phi})}\bar{\Phi}\right). \end{aligned} \quad (35)$$

This transformation is non-locally defined by

$$\begin{aligned} \Phi(z) &= -\frac{1}{2\sinh(\gamma p_1/2)} \int_0^{2\pi} dz' \gamma \phi'_2(z') \exp\left(-\frac{\gamma p_1}{2} \epsilon_{2\pi}(z-z') - 2\gamma \phi_1(z')\right), \\ \bar{\Phi}(\bar{z}) &= -\frac{1}{2\sinh(\gamma p_1/2)} \int_0^{2\pi} d\bar{z}' \gamma \bar{\phi}'_2(\bar{z}') \exp\left(-\frac{\gamma p_1}{2} \epsilon_{2\pi}(\bar{z}-\bar{z}') - 2\gamma \bar{\phi}_1(\bar{z}')\right). \end{aligned} \quad (36)$$

This result was rederived by the Hamilton reduction method and it is given in formula (2.36) of G. Jorjadze's contribution to these proceedings.

5. Quantum Parafermions

The parafermions are especially simple in terms of the free fields (29), which have the usual mode expansion

$$\begin{aligned} \phi_i(z) &= \frac{1}{2}q_i + \frac{1}{4\pi}p_i z + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{a_n^{(i)}}{n} e^{-inz}, \\ \bar{\phi}_i(\bar{z}) &= \frac{1}{2}q_i + \frac{1}{4\pi}p_i \bar{z} + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{\bar{a}_n^{(i)}}{n} e^{-in\bar{z}}. \end{aligned} \quad (37)$$

In order to guarantee the periodicity properties of the parafermionic currents and the closure of their algebra we need a special mode distribution, e.g., the classical chiral parafermions are defined as

$$V_{\pm}(z) = \frac{1}{\gamma} (\partial_z \phi_1 \pm i \partial_z \phi_2) \exp(\pm 2i\gamma \varphi_2(z)), \quad (38)$$

where φ_i is ϕ_i with the linear term in z removed and the whole q zero mode of the free field (29) is included. Then we get periodic parafermionic currents which have a closed canonical algebra

$$\begin{aligned} \{V_{\pm}(z), V_{\pm}(z')\} &= \gamma^2 V_{\pm}(z) V_{\pm}(z') \epsilon_{2\pi}(z - z'), \\ \{V_{\pm}(z), V_{\mp}(z')\} &= -\gamma^2 V_{\pm}(z) V_{\mp}(z') \epsilon_{2\pi}(z - z') + \frac{1}{\gamma^2} \delta'_{2\pi}(z - z') \\ &\quad + \frac{ip_2}{2\pi\gamma} \delta_{2\pi}(\sigma - \sigma'), \\ \{p_2, V_{\pm}(z')\} &= \mp 2i\gamma V_{\pm}(z'), \end{aligned} \quad (39)$$

which provides the Virasoro algebra, and conformal weight one for the V_{\pm}

$$\begin{aligned} \{T(z), V_{\pm}(z')\} &= -(\partial_{z'} V_{\pm}(z') \delta_{2\pi}(z - z') - V_{\pm}(z') \delta'_{2\pi}(z - z')) \\ &\quad \mp \frac{i\gamma p_2}{2\pi} V_{\pm}(z') \delta_{2\pi}(z - z'). \end{aligned} \quad (40)$$

For quantisation let us define normal ordered periodic operators as follows

$$V_{\pm}(z) =: (\alpha \partial_z \phi_1 \pm i\beta \partial_z \phi_2) \exp(\pm 2i\gamma \varphi_2(z)) : \quad (41)$$

Classically, we have $\alpha = \beta = \gamma^{-1}$. At the quantum level the parafermionic algebra closes [11], provided the deformation parameters satisfy the condition

$$\alpha^2 - \beta^2 - \frac{\beta^2 \gamma^2 \hbar}{\pi} = 0. \quad (42)$$

Using the definition

$$h^{\pm}(z) = \frac{1}{2} \left(\epsilon_{2\pi}(z) - \frac{z}{\pi} \right) \mp \frac{i}{2\pi} \ln \left(4 \sin^2 \frac{z}{2} \right), \quad (43)$$

the deformed quantum parafermionic algebra takes the regularised form

$$\begin{aligned} \frac{V_{\pm}(z) V_{\pm}(z')}{e^{i\hbar\gamma^2 h^+(z-z')}} - \frac{V_{\pm}(z') V_{\pm}(z)}{e^{-i\hbar\gamma^2 h^-(z-z')}} &= 0, \\ \frac{V_{\pm}(z) V_{\mp}(z')}{e^{-i\hbar\gamma^2 h^+(z-z')}} - \frac{V_{\mp}(z') V_{\pm}(z)}{e^{i\hbar\gamma^2 h^-(z-z')}} &= -\frac{\hbar\gamma\beta^2 p_2}{2\pi} \delta_{2\pi}(z - z') + i\hbar\beta^2 \delta'_{2\pi}(z - z'). \end{aligned} \quad (44)$$

The parafermionic operators are primary operators only if the energy-momentum-tensor takes an improvement term. This means that quantum mechanically a dilaton arises. More details will be given in [11].

6. Conclusion

Having completely integrated the classical $SL(2, \mathbb{R})/U(1)$ theory, its full quantisation remains still a challenge. Although the classical solution of the $SL(2, \mathbb{R})/U(1)$ gauged WZNW model bears strong resemblance to Liouville or Toda theories its quantum structure might be different. The possible interrelation of the quantum mechanically arising dilaton with the space-time black hole of the theory seems to be especially interesting.

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