

A detour in de Sitter space while calculating vacuum fluctuations*

Mahmut HORTAÇSU

*Physics Department, Faculty of Sciences and Letters
ITU 80626, Maslak, Istanbul-TURKEY*

Received 20.04.2000

Abstract

We study vacuum fluctuations of different gravitational waves by taking a detour in de Sitter space in the intermediate steps. We find finite contributions when the coupling constant is dimensionless, whereas null result persists in the cases with dimensional coupling constants. We also test our method for exactly solvable cases, and show that no spurious perturbative effects are introduced through our method.

1. Introduction

It is known that the behaviour of vacuum fluctuations of a scalar field in the background metric of de Sitter type spaces is different compared to that in Minkowski type metrics [1]. There are many examples where one finds nonvanishing vacuum fluctuations for the former case, whereas they are null in the latter one.

Here we will be dealing with gravitational waves, built about Minkowski and de Sitter universes. Our main concern will be whether taking a detour in de Sitter space, ending back in Minkowski will result in nonvanishing vacuum expectation values for any component of the stress-energy tensor. When the same calculation is performed staying totally in Minkowski space, some of our results may be ambiguous in the way the ultra-violet divergences are handled . Going to de Sitter space makes some of these divergences milder.

As a first concrete example, we will use the metric given by Nutku and Penrose [2]. This calculation will be a perturbative one. Then we will perform the same calculation for other cases where we know that the exact result gives zero vacuum fluctuations. Here we

*Talk presented in Regional Conference on Mathematical Physics IX held at Feza Gürsey Institute, Istanbul, August 1999.

check whether our finite result is due to spurious effects resulting from the perturbative method or the different limiting procedures used. We will find that our method does not give rise to contradictions to the exact result when the latter is available. The second part will be a test on the validity of our method.

2. Section I

The Nutku-Penrose metric [2] reads

$$ds^2 = 2dudv - u^2|d\zeta + \frac{v}{u}\Theta(v)f(\bar{\zeta})d\bar{\zeta}|^2. \quad (1)$$

Here v is the retarded time, u is similar to the radial distance, ζ is the angle in the stereographic projection. f is the Schwarzian derivative of a holomorphic function $h(\zeta)$. There is only one nonvanishing component of the Weyl scalar,

$$\Psi_4 = \frac{\delta(v)}{u}F(\zeta, \bar{\zeta}) \quad (2)$$

where F is a function of h given above. This metric describes a spherical wave for $v > 0$.

We worked with several examples for h . For simplicity we take first

$$h = e^{\alpha\zeta}, \quad (3)$$

this special case given first by Pullin and Gleiser [3]. Then the metric reduces to

$$ds^2 = 2dudv - \frac{1}{4}[dx^2(2u - v\alpha^2\Theta(v))^2 + dy^2(2u + v\alpha^2\Theta(v))^2] \quad (4)$$

where we changed to real coordinates, $\zeta = \frac{x+iy}{\sqrt{2}}$.

We calculate the vacuum fluctuations for the scalar field around this background perturbatively. This means we have to calculate the vacuum expectation value of the stress-energy tensor of a scalar field propagating in this background metric. This is done by taking the coincidence limit for the two-point function, regularising this term, and performing the necessary derivatives on the two point function $\langle \phi(\chi)\phi(\chi) \rangle_R$ to convert it to $\langle T_{\mu\nu} \rangle$. Here with the subscript R we denote the regularised form of the two-point function and χ is a generic coordinate.

To calculate the two-point function, we need the solution of

$$(\square + \frac{1}{6}R)G_F(\chi, \chi') = \frac{\delta(\chi - \chi')}{\sqrt{-g}}. \quad (5)$$

We expand $\square = \frac{1}{\sqrt{-g}}(\partial_\mu(g^{\mu\nu}\sqrt{-g})\partial_{\nu\mu})$ in powers of α . We define L as the box operator times the square root of minus the metric determinant, $L = \sqrt{-g}\square$ and expand L up to second order in powers of α^2 .

$$L^{II} = 2u^2\partial_u\partial_v + 2u\partial_v - \partial_x^2 - \partial_y^2 - \alpha^2\frac{v}{u}[\partial_x^2 - \partial_y^2] - \alpha^4[\frac{v}{2}\partial_u - \frac{v^2}{4}\partial_v + \frac{3v^2}{2u^2}(\partial_x^2 + \partial_y^2)]. \quad (6)$$

Converting this problem to the standard Sturm-Liouville problem, we can obtain G_F by summing over the eigenfunctions

$$G_F = \sum_{\lambda} \frac{\phi_{*\lambda}(x)\phi_{\lambda}(x')}{\lambda} \tag{7}$$

where

$$L\phi_{\lambda} = \lambda\phi_{\lambda}. \tag{8}$$

The zeroth order expression gives the standard free-field solution. In first order, $\mathcal{O}(\alpha^2)$, we get

$$G_F^{(1)} = [(x-x')^2 - (y-y')^2] \left[A_1 \left(\frac{s^2 v \Theta(v) - s'^2 v' \Theta(v')}{(s-s')D} \right) + \right. \tag{9}$$

$$\left. A_2 \left(\frac{s^2 \Theta(v) + s'^2 \Theta(v')}{(s-s')^2 D} \right) + A_3 \left(\frac{s^3 \Theta(v) - s'^3 \Theta(v')}{(s-s')^3 D} \right) \right]. \tag{10}$$

Here $D = (u-u')(v-v') - \frac{uu'}{2} [(x-x')^2 + (y-y')^2]$, $s = \frac{1}{u}$ and $A_i, i = 1-3$ are constants.

We could not obtain the finite part of this expression. In second order, we start getting infrared divergences, which can be remedied by taking a massive, instead of a massless scalar field.

Our initial aim was to calculate the vacuum fluctuations for a massless field in the background metric of a spherical wave. We seemed to be hampered, however, by both ultraviolet and infrared divergences, at the coincidence limit. At this point we thought of trading one for the other.

As a remark also note that in conformally flat spaces, there are no fluctuations. We thought, perhaps, by perturbing around the Minkowski space, we somehow carry this property to our perturbative solution. If we perturb around a space with a dimensional parameter, we may get finite fluctuations even when we set this parameter to zero at the end of the calculation.

We know that we can find exact solutions of vacuum Einstein equations for spherical waves in de Sitter space [4,5]. In fact all we have to do is to multiply our old metric by a factor, which gives us

$$ds_D^2 = \left(1 + \frac{\Lambda uv}{6} \right)^{-2} ds_M^2. \tag{11}$$

The only change in G_F will be a factor multiplying the Minkowski result.

$$G_F^D = \left(1 + \frac{\Lambda uv}{6} \right) \left(1 + \frac{\Lambda u'v'}{6} \right) G_F^M. \tag{12}$$

We can expand this extra factor in terms of sums and differences of the primed and unprimed quantities.

$$\left(1 + \frac{\Lambda uv}{6} \right) \left(1 + \frac{\Lambda u'v'}{6} \right) = \left(1 + \frac{\Lambda UV}{6} \right)^2 + \frac{\lambda}{12} (u-u')(v-v') + \lambda^2 [\dots] \tag{13}$$

where $U = \frac{u+u'}{2}, V = \frac{v+v'}{2}$. We see that factors of $(u-u')(v-v')$ in the numerator cancel powers of $(u-u'), (v-v')$ in the denominator, thus improve the ultraviolet behaviour of G_F .

We want to calculate $\langle T_{\mu\nu} \rangle_R$. In first order we get terms that vanish as Λ goes to zero. In second order, before we let Λ and m^2 go to zero, there is a nonvanishing term that goes as

$$\langle T_{vv} \rangle = \lim_{v \rightarrow v'} \partial_v \partial_{v'} G_F^D = \frac{\Lambda \alpha^4 \delta(v)}{6m^2 u^3}. \tag{14}$$

We set the other terms with pole-like ultraviolet divergences to zero. We note that we get an infrared divergence as the mass parameter m goes to zero even in de Sitter space. Still we have, at least one term free of ultraviolet divergences.

At this point we recall that we want to perform the calculation in Minkowski space for massless fields in the first place. There is nothing that prevents us from taking Λ proportional to m^2 . They even have the same dimensions. We let them approach zero at the same rate. Any proportionality constant between these two terms will be absorbed in α .

When this limit is taken, we end up with [6]

$$\langle T_{vv} \rangle = \frac{\alpha^4}{6u^3} \delta(v). \tag{15}$$

When we checked this behaviour for different choices of h , i.e. $h = (\zeta)^{1+\delta+i\epsilon}$ [7], and $h = \left(\frac{\zeta-1}{\zeta+1}\right)^{1+i\sqrt{2}\epsilon}$ [8], we get the same behaviour, zero for first order and finite for second order.

3. Section II

Now we have to check whether what we are doing is justified, or whether we perform operations that will make any term which is proven to be finite in the exact case equal to a finite expression, and thus leading to contradictions. One point of concern is the fact that we are using perturbative methods which may lead to spurious conclusions. Another weak point is the several limits we take, which may introduce extra unjustified contributions. To check our method we can apply our method to plane-impulsive waves where the exact result is known. In addition to plane waves, we will apply our method still to another example where we also know the exact result.

For plane waves we know both by general theorems [9] and exact calculations [10] that there are no vacuum fluctuations. The metric [11,12] we use is

$$ds^2 = 2dudv - |d\bar{\zeta} + q\zeta v\Theta(v)d\zeta|^2 \tag{16}$$

where for plane waves we choose $q = \frac{g}{2}\zeta^2$, g acting as a small perturbation constant. The exact Green function reads

$$G_F = -\frac{\Theta(v-v')}{2\pi\sigma^2} + \frac{\Theta(v'-v)}{2\pi\sigma^2}, \tag{17}$$

where

$$\sigma^2 = 2(u - u')(v - v') - (x - x')^2(1 + gv)(1 + gv') - (y - y')^2(1 - gv)(1 - gv'). \quad (18)$$

We can not find the finite part of this expression at the coincidence limit, since it is in the monomial form. When we calculate the same expression perturbatively to second order, in de Sitter space, we get [13]

$$\langle T_{vv} \rangle \propto -\frac{\Lambda^2}{m^2} \Theta(v). \quad (19)$$

Even when we take Λ proportional to m^2 as before, this expression vanishes in the limit when these terms go to zero at the same rate. Due to dimensional reasons, Λ can not be proportional to m .

Another example is given by the metric of Halilsoy [14]. For the metric

$$ds^2 = 2dudv - \cosh^2(gu)dx^2 - \cos^2(gu)dy^2, \quad (20)$$

we can calculate the Green Function exactly to give

$$G_F = \frac{g}{8\pi^2} \Theta(u - u') \frac{1}{(\cosh gu' \cos gu' \cosh gu \cos gu)^{\frac{1}{2}}} \quad (21)$$

$$\times \frac{1}{(AB)^{\frac{1}{2}}} \frac{1}{\left[(v - v') - \left(\frac{g(x-x')^2}{2A} - \frac{g(y-y')^2}{2B} \right) \right]} + sym \quad (22)$$

where $A = \tanh gu - \tanh gu'$ and $B = \tan gu - \tan gu'$. When regularised, we find that $\langle T_{\mu\nu} \rangle_R = 0$. Perturbative calculations in both Minkowski and de Sitter spaces give results which can not be made finite by taking the cosmological constant Λ proportional to m^2 .

4. Conclusion

We studied the technique of taking a detour in de Sitter space in several metrics. For the first metric introduced, we got a finite result, whereas for metrics with dimensional coupling constants, both for the plane and for spherical waves, we got null results. In the plane wave case, this was a blessing, since otherwise our perturbative method-via a detour in de Sitter space, would be in conflict with the exact result.

As a final remark we want to report our results on still another metric, with dimensional coupling constant, where the exact result is not known. This metric was given for shock waves by Nutku [15].

$$ds^2 = 2Pdudv + 2uP_\zeta d\zeta dv + 2uP_{\bar{\zeta}} d\bar{\zeta} dv - 2u^2 d\zeta d\bar{\zeta}. \quad (23)$$

Here $P = \frac{1}{|h_\zeta|}$, where h is an arbitrary holomorphic function of the argument $\zeta + gv\Theta(v)$. Our method failed to give a finite fluctuation for this case [16] too. Here g is a coupling constant with dimensions of mass.

All the metrics that failed to give a finite result had dimensional coupling constants. At this point one may wonder whether the presence of dimensional coupling constants is the reason that prohibits finite fluctuations.

Acknowledgement: I am grateful to Prof.Dr. Yavuz Nutku for giving me his metrics before publication and constant moral support. I thank my collaborators N. Özdemir, R. Kaya, Y. Enginer, K. Ülker, B. Yapışkan, B.C. Lütfioğlu. This work is partially supported by TUBITAK, the Scientific and Technical Research Council of Turkey and TUBA, the Academy of Sciences of Turkey.

References

- [1] N.D.Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge 1982;
- [2] Y. Nutku and R. Penrose, *Twistor Newsletter* **34** (1992) 9;
- [3] R. Gleiser and J. Pullin, *Class. Quantum Grav.* **6** (1989) L141;
- [4] Private communication by Y. Nutku.
- [5] P.A.Hogan, *Phys. Lett.A* **171** (1992) 21;
- [6] M.Hortaçsu, *Class. Quant. Grav.* **13** (1996) 2683;
- [7] M.Hortaçsu, *Class. Quant. Grav.* **7** (1990) L165, M.Hortaçsu, *J. Math. Phys.* **34** (1993) 690, M.Hortaçsu and R. Kaya, *J. Math. Phys.* **35** (1994) 3043;
- [8] Y.Enginer, M.Hortaçsu and N.Özdemir, *Int. J. Modern Phys. A*,**13** (1998) 1201;
- [9] S.Deser, *J.Phys. A*, **8** (1975) 1972;
- [10] G.W.Gibbons, *Commun. Math. Phys.* **45** (1975) 191;
- [11] Private communication by Y. Nutku.
- [12] J.Podolsky and K.Vesely, *Phys.Lett. A* **241** (1998) 145;
- [13] M.Hortaçsu and R.Kaya, *Gen. Rel. Grav.* **30** (1998) 1645;
- [14] M.Halilsoy, *Class. Quant. Grav.* **14** (1997) 2231;
- [15] Y.Nutku, *Phys. Rev. D* **44** (1991) 3164;
- [16] M.Hortaçsu, *Class. Quant. Grav.* **9** (1992) 1619; N.Özdemir and M.Hortaçsu, *Class. Quant. Grav.* **12** (1995) 1221; M.Hortaçsu and K.Ülker, *Class. Quant. Grav.* **15** (1998) 1405.