On Integrable Systems Related to Semisimple Lie Groups*

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Abstract

Quantum scattering systems related to the noncompact semisimple Lie groups G in the sense that the Hamiltonian of the system can be written as a function of the Casimir operator of G are considered. The S-matrix for such systems are defined in terms of an interwining operator of underling symmetry group G.

1. Introduction

Integrable systems provide the key to our understanding of more realistic interactions. They appear in different areas of physics both in classic and quantum domains. Some of these systems like the Kepler system do not differ essentially from the one observed in nature.

During last decates there was a great development of integrable theories, both clasical and quantum. It was shown group theory allows one to introduce a unified framework equally applicable to the both classical and quantum integrable systems associated with root systems [1], [2]. In the quantum case, the Hamiltonian H of such systems are functions of quadratic Casimir operator of underling symmetry group G. Hence, this connection allows one to find the wavefunctions, spectra and S-matrices, without a direct solution of the Schrödinger equation. However, in this description, as in the conventional approach, the S-matrix is defined through the asymptotic behaviour of the scattering wave functions. Hence, the natural question arises as to whether S-matrices can be calculated purely group-theoretically. The answer is in the affirmative [3]. We show how one can in principle evaluate the S-matrix without ever writing a Schodinger equation or

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wavefunction or ever mentioning the concepts of space and time. Moreover, this method has led to the hope that one may be able to classify and may determine explicitly all the S-matrices for systems with symmetry group.

2. The S-matrix

We now set out the method which will enable us to determine the S-matrix explicitly. As usually, we use a time independent approach, i.e. work only with the Schrdinger energy eigenvalue equation. Suppose Hamiltonian H of quantum scattering system is constructed from Casimir operator C of some noncompact group G

$$H = f(C) \tag{1}$$

For example, Hamiltonian of the two-dimensional Kepler problem $H = p^2/2 + \beta/r$ is related to the Casimir invariant $C = -J_1^2 - J_2^2 + J_3^2$ of noncompact group $SO(2,1)$ by [4]

$$H = -\frac{\beta^2}{2(C + 1/4)}$$

Recall the generators J_i, $i = 1, 2, 3$, satisfy the commutation relations

$$[J_1, J_2] = -iJ_3, \ [J_2, J_3] = iJ_1, \ [J_3, J_1] = iJ_2$$

and they are expressed in terms of quantum integrals M and $\mathbf{A}$ (angular momentum and Runge-Lentz vector, respectively) as $J_i = (2H)^{-1/2} A_i, \ i = 1, 2, \ J_3 = M$ where

$$
\begin{aligned}
M &= x_1 p_2 - x_2 p_1, \\
A_1 &= \frac{1}{2}(-Mp_2 - p_2 M) + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \\
A_2 &= \frac{1}{2}(Mp_1 + p_1 M) + \frac{x_2}{\sqrt{x_1^2 + x_2^2}}.
\end{aligned}
$$

The scattering eigenstates form a basis for the principal series of the unitary irreducible representations (UIR's) of $SO(2,1)$ (for details see [4] and references therein).

Our aim is to determine the S-matrix which connects any arbitrary incoming Ψ^{in} to an outgoing Ψ^{out} state

$$\Psi^{out} = S\Psi^{in}$$

The state vectors Ψ^{in} and Ψ^{out} are assumed to satisfy the free Schrodinger equation (corresponding to the same value E of the energy)

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$$H_o \Phi_\alpha = E \Phi_\alpha \ ,$$

with α being a complete set of variables that commute with H_o.

However, we find it expedient to use a definition of the $S-$matrix in terms of exact states rather than free ones

$$\Psi^- = S\Psi^+ \ , \tag{2}$$

where $\Psi^\pm$ are the eigenstates of the full Hamiltonian H which are labeled by the same quantum numbers as Φ

$$H\Psi_\alpha^\pm = E\Psi_\alpha^\pm$$

The relation of the states Ψ^+ and Ψ^- to Ψ^{in} and Ψ^{out} are following : If $\Psi^+(t)$ and $\Psi^-(t)$ are the wave packets which are centered about the stationary states Ψ^+ and Ψ^- respectively, we have

$$\lim_{t\to-\infty} \Psi^+(t) = \Psi^{in}(t), \ \lim_{t\to+\infty} \Psi^-(t) = \Psi^{out}(t)$$

Here $\Psi^{in}(t)$ and $\Psi^{out}(t)$ are wave packets constructed from the free states. In other words, the states Ψ^+ and Ψ^- are the solutions of the Lippman-Schwinger equations.

On the other hand, by the assumption (see Eq.1), the state vectors Ψ^+ and Ψ^- are the eigenstates of the Casimir operator C of the symmetry group G

$$C\Psi^\pm = q\Psi^\pm$$

where $q = f^{-1}(E)$. Thus, the scattering eigenstates $\{\Psi_\alpha^+\}$ and $\{\Psi_\alpha^-\}$ form the bases for the Weyl-equivalent representation of the algebra $\mathfrak{g}$ of the symmetry group G, which we denote by U^χ and $U^{\tilde\chi}$, respectively (The representations U^χ and $U^{\tilde\chi}$ have the same Casimir-eigenvalues. Such representations are called Weyl-equivalent). Moreover, it follows from Eq.2 that, the representations U^χ and $U^{\tilde\chi}$ are related by a similarity transformation

$$U^{\tilde\chi}(b) = SU^\chi(b)S^{-1} \ , \text{ for all } \ b \in \mathfrak{g}$$

The S-matrix for the system under consideration is then subject to the constraint equation

$$SU^\chi(b) = U^{\tilde\chi}(b)S \ , \text{ for all } \ b \in \mathfrak{g} \tag{3}$$

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or

$$SU^\chi(g) = U^{\tilde{\chi}}(g)S \text{ , for all } g \in G \tag{4}$$

Here $U^\chi(g)$ and $U^{\tilde{\chi}}(g)$ are the corresponding representations of the group G. The Eq. 3 (or Eq.4) actually is used in deriving the S-matrix.

Now we are in a position to outline our general method for obtaining S-matrix for scattering problems under consideration. In order to determine the S-matrix, we can proceed in two ways. If the principal series of the algebra $\mathfrak{g}$ in the scattering basis is known, we can get the recurrence relations for the S-matrix by applying both sides of Eq.3 to the basis vectors. By solving the recurrence relations, one can find the explicit form of S-matrix as function of parameters specifying the representation of $\mathfrak{g}$. An alternative way employs Eq.4. By using a realization of principal series of G on a Hilbert spaces of some functions it is possible to derive from Eq.4 the functional relations for the kernel of operator S which allow one to determine it. This global approach, which is complimentary to the infinitesimal treatment allows one to obtain the integral expression for the S-matrix.

3. Integrable Systems with SO(2,1) symmetry Group

To gain a better understanding of our approach, we first illustrate it for scattering models with $SO_o(2,1)$ symmetry group. To be able to use Eq.3 in the computation of the S-matrix, we have to know an abstract realization of the principal series of $su(1,1) \cong so(2,1)$ algebra.

We recall that the principal series of $su(1,1)$ are characterized by pair $\chi = (\rho, \epsilon)$, where ϵ is equal to 0 or $1/2$,while $0 \le \rho\langle\infty$. The representations specified by the labels $\chi = (\rho, \epsilon)$ and $\tilde{\chi} = (-\rho, \epsilon)$ are Weyl-equivalent. We can take the eigenvector $|\chi; m\rangle$ of J_3 , with $m = n + \epsilon$, $n = 0, \pm1, \pm2, ...$ as the scattering basis of the carrier space of the representation. The principal series of $su(1,1)$ is given by [5]

$$J_3^\chi |\chi; m\rangle = m |\chi; m\rangle \tag{5}$$

$$J_\pm^\chi |\chi; m\rangle = (1/2 - i\rho \pm m) |\chi; m \pm 1\rangle \tag{6}$$

with the Casimir invariant $C = -1/4 - \rho^2$. Here $J_\pm^\chi = iJ_1^\chi \mp J_2^\chi$

We are now ready to define the S-matrix. Using the relations $SJ_3^\chi = SJ_3^{\tilde{\chi}}$ (see Eqs.3 and 5 one has $J_3^{\tilde{\chi}}S |\chi; m\rangle = mS |\chi; m\rangle$ and we can conclude that $S |\chi; m\rangle = S_m |\tilde{\chi}; m\rangle$. Let us find the numbers S_m. To this end we apply both sides of equality $SJ_+^\chi = J_+^{\tilde{\chi}}S$ to the basis $|\chi; m\rangle$. We then obtain the recurrence relation

$$(1/2 - i\rho + m)S_{m+1} = (1/2 + i\rho + m)S_m$$

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which implies

$$S_m = c(\rho)\frac{\Gamma(1/2 + i\rho + m)}{\Gamma(1/2 - i\rho + m)} \tag{7}$$

where $c(\rho)$ is a constant of modulus =1. The energy- dependent parameter ρ is determined by the relation between the Hamiltonian H and the Casimir invariant C. For example for the Kepler problem $\rho = \beta/k$.

Note that, the operator S does not mix states belonging to different one-dimensional subspaces H_m which are spanned by $|\chi; m\rangle$. This observation suggests that there might exist a class of one-dimensional potentials related to $SO(2,1)$ for which the S-matrix is given by number S_m. This , in fact exactly what happens in the "potential group" approach to the scattering problems in which the representations of group G describe scattering states with same energy but different potential strengths. For example, the group $SO(2,1)$ appears as "potential group" for the Pöschl-Teller potential, where $H = -(C + 1/4)$ and the eigenvalue m of J_3 is associated with the potential strength. The corresponding S-matrix is a 2x2 diagonal matrix with elements equal to S_m .Moreover, we can extract corresponding one dimensioal potentials from the Casimir operator.

To do this, let us consider,for example, a (reducible!) representation $T(g)$ of $SO(2,1)$ realized in the Hilbert space of square-integrable function $f(\xi)$ on upper sheet of hyperboloid $\xi_o^2 - \xi_1^2 - \xi_2^2 = 1, \xi_o > 0$ with an invariant measure $d\xi = d\xi_1 d\xi_2/\xi_o$. The representation $T(g)$ are defined by $T(g)f(\xi) = f(\xi g)$. To the representation $T(g)$ of $SO(2,1)$ there corresponds the representation of its Lie algebra

$$J_1 = i\xi_o \frac{\partial}{\partial \xi_1}, J_2 = i\xi_o \frac{\partial}{\partial \xi_2}, J_3 = i\left(\xi_2 \frac{\partial}{\partial \xi_1} - \xi_1 \frac{\partial}{\partial \xi_2}\right)$$

Now, we require the representation space to be irreducible. (We note, the representation $T(g)$ is decomposed onto the direct integral of principal series representation $\chi = (\rho, 0)$.) Such a restriction is obtained if all functions are subject to Casimir eigenvalue equation $Cf = (-\rho^2 - 1/4)f$, where

$$C = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \left(\xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2}\right)^2 + \xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2} \quad .$$

The $|\chi; m\rangle$ basis is given by the decomposition according to compact subgroup $SO(2,1) \supset SO(2)$. As a prelude to this decomposition one introduces the spherical coordinates on the hyperbolid; $\xi_o = \cosh \alpha$, $\xi_1 = \sinh \alpha \cos \varphi$, $\xi_2 = \sinh \alpha \sin \varphi$. With introduction of spherical coordinates and substitution of the function $f(\alpha, \varphi)$ by $\omega^{-1/2}\Psi(\alpha)e^{im\varphi}$, where $\omega = \sinh \alpha$ is the weight function in the hyperboloid measure $d\xi = \sinh \alpha d\alpha d\varphi$, the Casimir eigenvalue equation reduces to Schrödinger equation

$$\left(-\frac{d^2}{d\alpha^2} + \frac{m^2 - 1/4}{\sinh^2 \alpha}\right)\Psi(\alpha) = E\Psi(\alpha)$$

where $E = \rho^2$. The Hamiltonian is now given by $H = -(C + 1/4)$, (on one - dimensional subspace H$_m$). Thus, the S-matrix for the Pöschl-Teller potential $V(\alpha) = (m^2 - 1/4)/\sinh^2 \alpha$ is a 2×2 diagonal matrix with elements

$$S_m = \frac{\Gamma(1 - i\rho)\Gamma(1/2 + i\rho + m)}{\Gamma(1 + i\rho)\Gamma(1/2 - i\rho + m)}$$

with $\rho = \sqrt{E}$. Since the Pöschl-Teller potential with $m = 1/2$ corresponds to the free case, we have choosed the phase factor c in 7 as $c(\rho) = \Gamma(1 - i\rho)/\Gamma(1 + i\rho)$.

There are, however, a class of one dimensional systems related to $SO(2,1)$ group which are not in the same above classes in the sense that their S-matrices are differ from 7. In order to complete the program to find the S-matrices, we have to calculate the operator S for all subgroup bases. We find it expedient to use, for this purpose, equation 4.

As is well known, the group $SO(2,1)$ has three subgroups $SO(2)$, $SO(1,1)$ and $E(1)$ generated by J_3, J_1 and $N = J_2 + J_3$, respectively. Hence, we are interested in examining the operator S in $SO(1,1)$ and $E(1)$ bases in which the operators J_1 and N are diagonal, respectively. The basis vectors will be denoted in the usual fashion by the kets $J_1 |\chi; \mu\tau\rangle = \mu |\chi; \mu\tau\rangle$, with $-\infty < \mu < \infty$, $\tau = \pm 1$ and $N |\chi; \lambda\rangle = \lambda |\chi; \lambda\rangle$ with $-\infty < \lambda < \infty$. (Note that, each UIR of $SO(1,1)$ is doubly degenerate in principal series of UIR of $SO(2,1)$ and τ is the multiplicity label.)

We realize the unitary principal series representations $U^\chi(g)$, $\chi = (\rho, 0)$ of $SO(2,1)$ in the space of function $F(\zeta)$ on the upper sheet of the two-dimensional cone C^+, $\zeta_o^2 - \zeta_1^2 - \zeta_2^2 = 0$, $\zeta_o > 0$, homogeneous of degree $j = -\frac{1}{2} + i\rho$

$$F(a\zeta) = a^j F(\zeta) \ , \ a > 0 \ .$$

The representations $U^\chi(g)$ are given by

$$U^\chi(g)F(\zeta) = F(g\zeta)$$

Since functions $F(\zeta)$ are uniquely determined by their values on some contour Γ intersecting every generatrix at one point, then U^χ can be realized in the space of functions on these contours

$$U^\chi(g)f(\eta) = \left(\frac{\omega_g}{\omega}\right)^j f(\eta_g) \ ,$$

where $\zeta = \omega\eta$, $\omega > 0, \eta \in \Gamma$, and $\eta_g \in \Gamma$, $\omega_g > 0$ are determined from equality $\zeta_g = \omega_g \eta_g$.

By virtue of the theorem on kernel, the operator S can be defined as

$$(Sf)(\eta) = \int_\Gamma K(\eta, \eta')f(\eta')d\eta'$$

where $d\eta$ is invariant measure on Γ. Thus, the equation 4 will serve to fix the dependence of the kernel $K(\eta, \eta')$ on η and η'. The equality 4 implies that

$$(SU^{\chi}(g))F(\eta) = (U^{\tilde{\chi}}(g)SF)(\eta).$$

So, the kernel $K(\eta, \eta')$ is constrained to satisfy the functional equation

$$K(\eta_g, \eta'_g) = \left(\frac{\omega_g}{\omega}\right)^{1+j} \left(\frac{\omega'_g}{\omega'}\right)^{1+j} K(\eta, \eta').$$

The kernel K is, up to a constant $\kappa(j)$, uniquely determined and is given by

$$K(\eta, \eta') = \kappa(j) \left[\eta, \eta'\right]^{-1-j} ,$$

where $\left[\eta, \eta'\right] = \eta_o\eta'_o - \eta_1\eta'_1 - \eta_2\eta'_2$. The module of constant κ is fixed by the normalization relation, which gives

$$| \kappa |^2 = \frac{1}{2\pi}\rho\tanh\pi\rho \quad .$$

Let $\Gamma = \Gamma^{\tau}$, $\tau \in (+, -)$ be the section of C^+ by the plane $\zeta_2 = \tau$, then

$$(Sf)_{\tau}(\alpha) = \frac{2^j}{\sqrt{\pi}}\frac{\Gamma(1+j)}{\Gamma(-1/2-j)}c \sum_{\tau'=\pm 1} \int_{-\infty}^{\infty} d\alpha' \left[\cosh(\alpha - \alpha') - \tau\tau'\right]^{-1-j} f_{\tau'}(\alpha') \quad .$$

where $f_{\tau}(\alpha) \equiv F(\cosh\alpha, \sinh\alpha, \tau)$. This then gives the integral representation of the matrix elements of S in $SO(1,1)$ basis. Taking into account that

$$|\chi; \mu+\rangle = \begin{pmatrix} e^{i\mu\alpha} \\ 0 \end{pmatrix} \quad , \quad |\chi; \mu-\rangle = \begin{pmatrix} 0 \\ e^{i\mu\alpha} \end{pmatrix} \quad ,$$

we have $\left\langle \tilde{\chi}; \mu'\tau' \right| S|\chi; \mu\tau\rangle = \delta(\mu - \mu')S_{\tau\tau'}$, where

$$S_{++} = S_{--} = \frac{c}{\pi}\cosh\pi\mu\Gamma(1/2 + i\rho + i\mu)\Gamma(1/2 + i\rho - i\mu) \quad ,$$

$$S_{+-} = S_{-+} = -i\frac{c}{\pi}\sinh\pi\rho\Gamma(1/2 + i\rho + i\mu)\Gamma(1/2 + i\rho - i\mu) ,$$

If Γ is the section of C^+ by the plane $\zeta_o + \zeta_1 = 1$, then

$$(Sf)(x) = \frac{2^j}{\sqrt{\pi}}\frac{\Gamma(1+j)}{\Gamma(-1/2-j)} \int_{-\infty}^{\infty} \left|x - x'\right|^{-2-2j} f(x')dx' \quad .$$

where $f(x) \equiv F\left((1+x^2)/2, (1-x^2)/2, x\right)$. As a result $\left\langle \widetilde{\chi}; \lambda' \middle| S \middle| \chi; \lambda \right\rangle = \delta(\lambda - \lambda')S_\lambda$ with $S_\lambda = c\lambda^{2i\rho}$.

Thus, we have come to a very important conclusion; there exist three classes of one-dimensional scattering systems related to $SO(2,1)$ group with S-matrices given by

(i) class 1, (related to reduction $SO(2,1) \supset SO(2)$)

$$S_m = \begin{pmatrix} R_m & 0 \\ 0 & R_m \end{pmatrix}, \qquad R_m = c(\rho)\frac{\Gamma(1/2+i\rho+m)}{\Gamma(1/2-i\rho+m)} \tag{8}$$

(ii) class 2, (related to reduction $SO(2,1) \supset SO(1,1)$)

$$S_\mu = \begin{pmatrix} R_\mu & T_\mu \\ T_\mu & R_\mu \end{pmatrix} \tag{9}$$

where

$$R_\mu = c(\rho)\cosh \pi\mu\Gamma(1/2+i\rho+i\mu)\Gamma(1/2+i\rho-i\mu) \qquad ,$$

$$T_\mu = -ic(\rho)\frac{1}{\pi}\sinh \pi\rho\Gamma(1/2+i\rho+i\mu)\Gamma(1/2+i\rho-i\mu) \ .$$

(iii) class 3, (related to reduction $SO(2,1) \supset E(1)$)

$$S_\lambda = \begin{pmatrix} R_\lambda & 0 \\ 0 & R_\lambda \end{pmatrix}, \quad R = c(\rho)\lambda^{2i\rho}. \tag{10}$$

It should be noted that, the potential functions $V(x)$ of the second class admit a double degeneracy of the wave function for every positive value of E. The double degeneracy corresponds to the fact that one may construct wave packets which are partly transmitted and partly reflected by the potential $V(x)$. According to 9 the reflection and transmission coefficients are

$$|R_\mu|^2 = \frac{\cosh^2 \pi\mu}{\cosh^2 \pi\mu + \sinh^2 \pi\rho}$$

$$|T_\mu|^2 = \frac{\sinh^2 \pi\rho}{\cosh^2 \pi\mu + \sinh^2 \pi\rho},$$

respectively. It also worth to note that, according to 8 and 10 , the reflection coefficient $|R_m|^2 = |R_\lambda|^2 = 1$ for all potentials of class 1 or 3; hence the reflection is total. This is a result of very general properties, shaved by all one dimensional Hamiltonians which have continuous nondegenerate spectrum

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4. The S-matrices of systems assosiated with semisimple Lie Groups

We note that the operator S satisfying the Eqs.3 or 4 is called the intertwining operator between representations U^χ and $U^{\widetilde{\chi}}$. Therefore, the S-matrix for scattering system described by a Hamiltonian expressed in terms of the Casimir operator of some group G is nothing but the intertwining operator between the Weyl-equivalent principal series representations of G. The explicit expressions of the intertwining operators for semi-simple Lie groups in terms of kernels are introduced in Ref. [6] (see also Refs. [7],[8],[9]) and have been extensively studied in Refs. [10],[11],[12] in a different context. These may be useful in the study of integrable many-body problems related to semi-simple Lie groups [2].

For these concepts to become rigorous it is necessary to introduce some notation. Let G be a connected semi-simple Lie group of matrices, and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ be the Cartan decomposition on $\mathfrak{g}$ with respect to a Cartan involution θ, and let $\mathfrak{a}$ be a maximal commutative subalgebra in $\mathfrak{p}$. Let $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$ is the sum of eigenspaces of the restricted roots that are positive relative to some ordering on the dual of $\mathfrak{a}$, and $\mathfrak{v}$ is $\theta\mathfrak{n}$. Then the Iwasawa decomposition $G = ANK$ is valid, and each element g of G can be written uniquely as

$$g = \exp h(g).n.k(g) \;, \quad \exp h(g) \in A \;, \; n \in N \;, \; k(g) \in K \;,$$

where A, N, K are the subgroups of G with Lie algebras $\mathfrak{a}, \mathfrak{n}, \mathfrak{f}$, respectively; K is compact, N is nilpotent and A is a vector group. We write M and M' for the centralizer and normalizer of A in K. Then $P = MAN$ is also a subgroup of G which is called the minimal parabolic subgroup of G. A finite-dimentional (continuous) irreducible unitary representations of P has the form

$$man \rightarrow \lambda(a)\sigma(m)$$

where λ is a unitary character of A and σ is an irreducible unitary representation of M. The principal series of unitary representations of G is parametrized by (λ, σ) and is obtained by inducing these representations of P to G. Let E_σ be the (finite-dimensional) Hilbert space on which σ is realized. Let further $L^2_\sigma(K, E_\sigma)$ be the Hilbert space of square integrable functions f on the maximal compact subgroup K which obey the condition

$$f(m\kappa) = \sigma(m)f(\kappa)$$

for each m in M. Then the representation $U^{(\lambda,\sigma)}(g)$, $g \in G$, can be realized on $L^2_\sigma(K, E_\sigma)$, where it acts as follows

$$U^{(\lambda,\sigma)}(g)f(\kappa) = e^{\rho H(\kappa g)}\lambda(\exp H(\kappa g))f(k(\kappa g))$$

where ρ half the sum of the positive restricted roots with multiplicities.

Another important decomposition of G is due to Gelfand-Naimark-Bruhat, i.e., almost every element of G can be written in a unique way as a product

$$g = m(g)a(g)nv(g), \qquad m(g) \in M, \ \ a(g) \in A, \ \ n \in N, \ \ v(g) \in V,$$

where V is the subgroup of G with Lie algebra $\mathfrak{v}$. Therefore the representation $U^{(\lambda,\sigma)}(g)$, $g \in G$, can be also realized on Hilbert space $L^2(V, E_\sigma)$, where G acts as follows

$$U'^{(\lambda,\sigma)}(g)f(x) = \mu^{1/2}(xg)\lambda(xg)\sigma(xg)f(v(xg))$$

where $x \in V$ and $\mu(a) = e^{2\rho \ln a}$ is a positive character on A.

Let $W = M'/M$ be Weyl group of pair (G, A); that is, W is a finite group. The action of the elements of W upon linear forms λ and upon finite dimensional representations σ of M is defined by

$$w\lambda(a) \ = \lambda(w^{-1}aw), \qquad a \in A,$$

$$w\sigma(m) = \sigma(w^{-1}mw), \quad m \in M,$$

where $w \in M'$ is a representative of the coset of M'/M. Moreover, $w\lambda$ and $w\sigma$ do not depend on the choice of the representative w. The representations $U^{(\lambda,\sigma)}(g)$ and $U^{(w\lambda,w\sigma)}(g)$ are equivalent and the operator $A(w, \lambda, \sigma)$ defined by

$$A(w, \lambda, \sigma)f(\kappa) = \int_K \mu^{1/2}(\kappa'w)\lambda^{-1}(\kappa'w)\sigma^{-1}(\kappa w)f(\kappa'\kappa)d\kappa'$$

(in 'compact picture') or

$$A(w, \lambda, \sigma)f(x) = \int_{V \cap wNw^{-1}} f(vw^{-1}x)dv$$

(in 'non-compact' picture) implements this equivalence

$$A(w, \lambda, \sigma)U^{(\lambda,\sigma)}(g) = U^{(w\lambda,w\sigma)}A(w, \lambda, \sigma) \quad \text{for all} \ \ g \ \in \ G.$$

Thus, there exist a one-to-one correspondence between the intertwining operators and the Weyl group elements.

Since the S matrix for problems with dynamic symmetry group G are related to intertwining operator between the Weyl-equivalent principal series representations of G, we have

$$S = A(w_o, \lambda, \sigma) \ ,$$

where w_o is a Weyl group element satisfying the relation $w_o N w_o^{-1} = V$. This result is of central importance for this paper.

As an example, let us compute the S-matrices for the problems related to principal series representations of the group of 2 by 2 complex matrices of determinant one $SL(2, C)$. We can choose

$$M \ni m = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad A \ni a = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad N \ni n = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix},$$

$$V \ni v = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \quad w_o = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with θ and t real and with z and ξ complex. Let $\sigma_n(m) = \exp(in\theta), \quad \lambda_\rho(a) = \exp(i\rho t)$. The representations specified by the labels $\chi(\rho, n)$ and $\widetilde{\chi}(-\rho, -n)$ are Weyl-equivalent. Then an easy calculation gives

$$(Sf)(z) = \int_V K(z, z') f(z') dz' \tag{11}$$

$$K(z, z') = (2\pi)^{-1} i^{2n} (i\rho + |n|) |z' - z|^{2n-2-2i\rho} (z' - z)^{-2n}$$

in 'non-compact picture' and

$$(Sf)(u) = \int_{SU(2)} K(u, u') f(u') du' \tag{12}$$

$$K(u, u') = i^{2n} (i\rho + |n|) \left| (u'u^{-1})_{21} \right|^{2n-2-2i\rho} \left[(u'u^{-1})_{21} \right]^{-2n}$$

in 'compact picture'.

Formulas 11 and 12 allow one to obtain the integral formulas for the matrix elements of the intertwining operator in various subgroup bases. For example, in $SU(2)$ basis we have

$$\langle \widetilde{\chi}; j'm' | S | \chi; jm \rangle = \sqrt{(2j+1)(2j'+1)} \int_{SU(2)} K(u, u') D^j_{nm}(u') \overline{D^{j'}_{-nm}(u)} du du'$$

where du is the normalized invariant measure on $SU(2)$. Consequently, we obtain that $\langle \widetilde{\chi}; j'm' | S | \chi; jm \rangle = \delta_{mm'} \delta_{jj'} S_j$, where

$$S_j = i^{2|n|} \frac{\Gamma(1 + i\rho + j)\Gamma(-i\rho + |n|)}{\Gamma(1 - i\rho + j)\Gamma(i\rho + |n|)}$$

These results may have applications in a number of scattering problems with $SO(3, 1)$ symmetry. One application is in the study of the scattering processes with spin degrees

of freedom. Such a spin-dependent scattering problem was discussed in [13],where the S-matrix for the simplest choice (i.e. for the spin-1/2 case) has been evaluated by obtaining the explicit wave functions and by studying their asymptotic behaviour. The results can also be used to investigate the scattering processes in the Kaluza-Klein monopole field (see [14] and references therein). In this case the quantity n defines electric charge.

Thus it has become clear that, besides its mathematical beauty, the theory of the intertwining operators may provide a method to construct S- matrices for models associated with semisimpe Lie algebras or other stuctures such as graded Lie albebras, infinite dimensional algebras, q-algebras, etc.

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