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From antibracket to equivariant characteristic classes*

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Abstract

We equip the exterior algebra of Riemann manifold by the odd symplectic structure. Then, by its use we construct the equivariant even (pre)symplectic structure, whose Poincare - Cartan invariants of the second structure define the equivariant Euler classes of the surfaces.

1. Introduction

Since the beginning of eighties equivariant cohomology attracted some interests in physics (see e. g. [1]-[3]). It is stimulated by the applications of localization formulas to evaluation of path integrals for a wide class of topological and supersymmetric field theories [4, 5, 6, 7]. In the series of papers A. Niemi with collaborators developed new approach to the evaluation of quantum-mechanical path integrals, based on localization formulae (see [8], [9] and refs. therein). Equivariant cohomology naturally formulated by use of the language of supermathematics, where the role of supermanifold played by the exterior algebra $\Lambda(M)$ of the given Riemann manifold M. On the other hand, since the realistic field theories are described by the degenerate Lagrangians, it have to be useful to define localization formulae on the surfaces N in the given manifold M, and, therefore, the equivariant characteristic classes of N.

In this paper we construct a family of the equivariant integral densities on the surfaces $\Gamma \subset \Lambda(M)$, generalizing the known construction of the equivariant Euler classes [3, 4]. For this purpose we, at first, construct the odd symplectic structure Ω_1 on the exterior algebra $\Lambda(M)$ of the Riemann manifold M . Then we show, that the Lie derivative of Ω_1 along the vector field, corresponding to S^1- equivariant transformation (where S^1- action

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on M defines the isometry of metrics) defines the S^1*-equivariant even (pre)symplectic structure* Ω_0 (*Section 2*).

The Poincare-Cartan invariants [10] of Ω_0 define the equivariant densities, corresponding to the equivariant characteristic classes of the surfaces (*Section 3*).

Notice, that the initial object our considerations: the odd symplectic structure is mainly due to the Batalin-Vilkovisky quantization formalism [11], which is closely related with integration theory over (super)surfaces[12]. Notice also, that the odd symplectic structure, constructed on the exterior algebra of symplectic manifold, relates the equivariant cohomology with bi-Hamiltonian supersymmetric mechanics (with even and odd symplectic structures) [13, 14].

2. Odd and Even Symplectic Structures

In this section we equip the exterior algebra of Riemann manifold M by the odd symplectic structure and then construct the even S^1-invariant (pre)symplectic structure.

Let (M, g) be the Riemann manifold and ξ its Killing vector defining the S^1 action. Let $\Lambda(M)$ be the exterior algebra of M, parametrized by the local coordinates $z^A = (x^i, \theta^i)$, where x^i denote the local coordinates on M and θ^i denote the basic 1-forms dx^i, $p(\theta^i) = 1$.

Consider the vector fields $\hat{X}$ and $\hat{E}$ on $\Lambda(M)$:

$$\hat{X} = \xi^i \frac{\partial}{\partial x^i} + \xi^i_{,k}\theta^k \frac{\partial}{\partial \theta^i}, \quad \hat{E} = \xi^i \frac{\partial}{\partial \theta^i} + \theta^i \frac{\partial}{\partial x^i} : \quad [\hat{E}, \hat{E}]_+ = 2\hat{X}. \tag{1}$$

It is obvious that $\hat{X}$ corresponds to the Lie derivative of differential forms on M along ξ: $\hat{X} \rightarrow L_\xi$, and $\hat{E}$ corresponds to the S^1-equivariant differential: $\hat{E} \rightarrow d_\xi = d + \imath_\xi$. The last expression in (1) corresponds to the homotopy formula $L_\xi = d\imath_\xi + \imath_\xi d$.
Below we consider $\Lambda(M)$ as a supermanifold and denote by $\mathcal{L}$ and d the Lie derivative and exterior differential on $\Lambda(M)$ respectively. The Berezin integration on $\Lambda(M)$ leads to the integration of differential forms on M.

Let us equip $\Lambda(M)$ by the odd symplectic structure, which takes in terms of the coordinates (x^i, θ^i) the following form

$$\Omega_1 = dx^i \wedge d(g_{ij}\theta^j) = g_{ij} dx^i \wedge D\theta^j, \quad D\theta^i = d\theta^i + \Gamma^i_{kl}\theta^k dx^l, \tag{2}$$

where Γ^i_{kl} are the Cristoffel symbols for the metric g_{ij}.
The corresponding odd Poisson bracket (antibracket) reads :

$$\{f, g\}_1 = g^{ij}(\nabla_i f \frac{\partial g}{\partial \theta^j} - \frac{\partial f}{\partial \theta^i}\nabla_j g), \quad \nabla_i = \frac{\partial}{\partial x^i} - \Gamma^j_{ik}\theta^k \frac{\partial_l}{\partial \theta^j}, \tag{3}$$

and satisfies the conditions

$$\{f, g\}_1 = -(-1)^{(p(f)+1)(p(g)+1)}\{g, f\}_1 \quad (\text{"antisymmetricity"}),$$
$$(-1)^{(p(f)+1)(p(h)+1)}\{f, \{g, h\}_1\}_1 + \text{cycl.perm.(f, g, h)} = 0 \quad (\text{Jacobi id.}).$$

Since the odd symplectic structure (2) is $\hat{X}$-invariant, the vector field $\hat{X}$ can be presented in the Hamiltonian form

$$\hat{X} = \{\Psi, .\}_1, \quad \text{where} \quad \Psi = \xi_i \theta^i. \tag{4}$$

On the other hand, the odd symplectic structure Ω_1 is not $\hat{E}$-invariant:

$$\mathcal{L}_E \Omega_1 = \tilde{\Omega}_0 \neq 0.$$

Consequently,

$$\Omega_0 = \frac{1}{2}(\xi_{i;j} + g_{in} R^n_{jkl} \theta^k \theta^l) dx^i \wedge dx^j + g_{ij} D\theta^i \wedge D\theta^j \tag{5}$$

(R^n_{jkl} is the curvature tensor on M) defines E-invariant (S^1- equivariant) even presymplectic structure:

$$p(\Omega_0) = 0, \; d\Omega_0 = \mathcal{L}_E d\Omega_1 = 0, \quad \mathcal{L}_E \Omega_0 = \mathcal{L}_E \mathcal{L}_E \Omega_1 = 2\mathcal{L}_X \Omega_1 = 0.$$

Therefore, (1) become Hamiltonian vector fields with respect to Ω_0. The corresponding Hamiltonians read

$$\mathcal{H} \equiv \mathcal{L}_E \Psi = \xi^i g_{ij} \xi^j - \xi_{i;j} \theta^i \theta^j, \quad Q = \xi^i \xi_{i;j} \theta^j, \tag{6}$$

while the (pre)symplectic one-form $\mathcal{A} : \, d\mathcal{A} = \Omega_0$ reads

$$\mathcal{A} = \Omega_1(\hat{E}, ...) = \xi_i dx^i + \theta^i g_{ij} D\theta^j. \tag{7}$$

Thus, starting from the odd symplectic structure we constructed the S^1-equivariant even (pre)symplectic structure on the space of differential forms on the Riemann manifold.

3. Equivariant Characteristic classes

In this Section we construct the equivariant characteristic classes for the surfaces in $\Lambda(M)$.

Let $\Gamma \subset \Lambda(M)$ be a closed surface and Ω_0 be nondegenerate on Γ. Let Γ is parametrized by the equations $z^A = z^A(w)$, where w^μ are local coordinates of Γ.
Thus the following density, is correctly defined on Γ

$$\mathcal{D}_\Gamma(w) = \sqrt{\text{Ber } \Omega_0|_\Gamma} \equiv \sqrt{\text{Ber} \frac{\partial_r z^A}{\partial w^\mu} \Omega_{(0)AB} \frac{\partial_l z^B}{\partial w^\nu}}. \tag{8}$$

This density is invariant under canonical transformations of the presymplectic structure (5) [10], and consequently, is S^1-equivariant.
Hence, *the functional*

$$Z^\lambda(\Gamma, F) = \int_\Gamma \text{e}^{F - \lambda \hat{E} \Psi} \mathcal{D}_\Gamma[dw], \tag{9}$$

is S^1-equivariant for any compact Γ, if $F(z)$ and $\Psi(z)$ are the even $\hat{E}$-invariant and odd $\hat{X}$-invariant functions respectively

$$\hat{E}F = 0, p(F) = 0, \qquad \hat{X}\Psi = 0, p(\Psi) = 1. \tag{10}$$

Thus, repeating standard BRST arguments, one can show that the functional (9) is λ -independent.

Let the surface Γ is defined by the equations $f^a(z) = 0$, $a = 1, ...\mathrm{codim}\Gamma$. The functional (9) can be represented in the following dual form (compare with [15])

$$Z^\lambda(\Gamma, F) = \int_{\Lambda(M)} \mathrm{e}^{F(z) - \lambda(\hat{E}\Psi)} \delta(f^a) \sqrt{\mathrm{Ber}\{f^a, f^b\}_0} \mathcal{D}_0 dz, \tag{11}$$

where

$$\{f(z), g(z)\}_0 = \nabla_i f(z)(\xi_{i;j} + R_{ijkl}\theta^k\theta^l)^{-1}\nabla_j g(z) + \frac{1}{2}\frac{\partial_r f(z)}{\partial\theta^i} g^{ij} \frac{\partial_l g(z)}{\partial\theta^j}, \tag{12}$$

is the Poisson bracket, corresponding to (5) and $\mathcal{D}_0(z) \equiv \mathcal{D}_{\Lambda(M)}(z) = \sqrt{Ber\Omega_{(0)AB}}$. Notice, that the functional (9) is invariant under reparametrizations of $\Lambda(M)$ and independent on the choice of the functions f^a.

The functional $Z^\lambda(\Gamma, 0)$ is invariant under smooth deformations of Γ [15], i. e., it defines topological invariant of Γ . For example, $Z^\lambda(\Lambda(M), 0)$ coincide with the Euler number of M. In the limit $\lambda \to 0$ it gives the Poincare-Hopf formula, while in the limit $\lambda \to \infty$ (where Ψ is defined by (6)) Gauss-Bonnet one for the Euler number of M. Hence, (8) defines the S^1-equivariant characteristic class of Γ.

Example : Let the supersurface $\Gamma \subset \Lambda(M)$ is associated with the vector bundle $V(N)$ ($V(N) \subset T(M), N \subset M$) and parametrized by the equations

$$x^i = x^i(y^a), \quad \theta^i = P^i_\alpha(y)\eta^a, \tag{13}$$

where $w^\mu = (y^a, \eta^\alpha)$ are local coordinates of Γ, $p(y) = 0, p(\eta) = 1$.

Thus

$$\Omega_0|_\Gamma = \frac{1}{2}(\xi_{[a,b]} + g_{\alpha\delta}R^\delta_{\beta ab}\eta^\alpha\eta^\beta)dy^a \wedge dy^b + g_{\alpha\beta}D\eta^\alpha \wedge D\eta^\beta. \tag{14}$$

where

$$\xi_{[a,b]} = \xi_{i;j}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}; \quad g_{\alpha\beta} = P^i_\alpha g_{ij} P^j_\beta; \quad D\eta^\alpha = d\eta^\alpha + A^\alpha_{a\beta}\eta^\beta dy^a,$$

and $A^\alpha_{a\beta}$ is the induced connection on $V(N)$, compatible with $g_{\alpha\beta}$,

$$A^\alpha_{a\beta} = g^{\alpha\delta}P^i_\delta g_{ij}\left(P^j_{\beta,a} + \Gamma^j_{lk}P^k_\beta\frac{\partial x^l}{\partial y^a}\right)$$

while $R^\delta_{\beta ab}$ defines its curvature tensor.

Hence, S^1-equivariant Euler class of N reads

$$\mathcal{D}_\Gamma(w) = \sqrt{\mathrm{Ber}\Omega_0|_\Gamma} = \left(\frac{\det(\xi_{[a,b]} + g_{\alpha\delta}R^\delta_{\beta ab}\eta^\alpha\eta^\beta)}{\det g_{\alpha\beta}}\right)^{\frac{1}{2}}. \tag{15}$$

For $\Gamma = \Lambda(N)$ (i. e. when $P^i_\alpha = \frac{\partial x^i}{\partial y^a}$), the expression (15) coincides with the known equivariant Euler classes on N [3].

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