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Gauge Transformations in General Relativity–A Report*

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Abstract

We take as our basic principle that the Lagrangian and the Hamiltonian formulations of general relativity should be equivalent. It is then possible to choose the gauge transformations so as to be projectable under the Legendre map from configuration-velocity space (the tangent bundle) to phase space (the cotangent bundle). This projectability requirement then implies that gauge transformations in general relativity (we study in particular spacetimes with a Yang-Mills field) must depend in a specific way on the lapse and shift of the metric. This talk outlines the arguments and presents some of the results of the application of these principles.

1. Introduction

Based on the principle that the Lagrangian and Hamiltonian formulations of a theory should be equivalent, we have previously discussed [1] the relationship between diffeomorphisms and gauge transformations in general relativity. We recently [2] extended this discussion to include spacetimes containing a Yang-Mills field. The basic principle means that gauge transformations can be chosen to be projectable under the Legendre

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map from configuration-velocity space (the tangent bundle) to phase space (the cotangent bundle) — this builds on a rather formal treatment by Pons and Shepley [3]. We show that the diffeomorphism group, which acts on the spacetime manifold, is not part of the gauge group. Instead, the gauge group acts on dynamical variables and depends on them, though each diffeomorphism is indeed included.

In this report, we outline the procedures, results, and some of the implications of our approach. We show that descriptors of the diffeomorphism-induced gauge transformations must depend in a specific way on the lapse and shift of the metric, as well as on the Yang-Mills potential itself. Our results are like those earlier obtained by Salisbury and Sundermeyer [4,5], but we obtain them, we feel, using a more physical point of view. We emphasize that our discussion is purely classical, but we hope that our results will prove significant to programs aiming at quantization, since all gauge variables are retained.

2. Singular Lagrangians

A gauge transformation, broadly speaking, is a transformation of a physical theory which leaves the physics unchanged. Suppose we consider a Lagrangian $L(q, \dot{q})$ (without explicit dependence on the time t; note that q is a generic configuration variable, and the dot means time derivative). An infinitesimal transformation of the configuration variables $\delta q^i(q, \dot{q}, t)$ will be a symmetry—leave the physics invariant—if the resulting variation in the Lagrangian is a total time derivative:

$$\delta L = \frac{dF}{dt} \ . \tag{1}$$

This criterion implies

$$[L]_i \delta q^i + \frac{dG}{dt} = 0 \ , \tag{2}$$

where G is defined by

$$G := \frac{\partial L}{\partial \dot{q}^s} \delta q^s - F \ , \tag{3}$$

and where $[L]_i$ is the Euler-Lagrange function derivative of L:

$$[L]_i = \alpha_i - W_{ij} \ddot{q}^i \ , \tag{4}$$

where

$$W_{ij} := \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \qquad \text{and} \qquad \alpha_i := -\frac{\partial^2 L}{\partial \dot{q}^i \partial q^s} \dot{q}^s + \frac{\partial L}{\partial q^i} \ . \tag{5}$$

The Legendre matrix $\mathbf{W} = (W_{ij})$ in general will be singular when a gauge transformation is possible.

The Legendre map is a map from configuration-velocity space (the tangent bundle) to phase space (the cotangent bundle), denoted by

$$\mathcal{F}L : TQ \longrightarrow T^*Q \ . \tag{6}$$

It is defined by mapping the configuration-velocity space momentum functions $\hat{p}_i(q,\dot{q})$ into the phase space momentum variables:

$$p_i = \hat{p}_i(q,\dot{q}) = \frac{\partial L}{\partial \dot{q}^i} \ . \tag{7}$$

When W is singular, its kernel is spanned by vectors which are the pullbacks of functions in phase space (see [3]):

$$\gamma^i_A = \mathcal{F}L^* \left(\frac{\partial \phi_A}{\partial p_i} \right) \quad \text{with} \quad W_{ij}\gamma^i_A = 0 \ . \tag{8}$$

The ϕ_A are (effective) Hamiltonian primary constraints. We then see that

$$\gamma^i_A \frac{\partial G}{\partial \dot{q}^i} = 0 \ . \tag{9}$$

We can state this result in a significant way: G is the pullback of a function G_H in phase space, and the pullbacks of the momentum-derivatives of G_H generate the symmetry transformations:

$$G = \mathcal{F}L^*(G_H) \ , \tag{10}$$

$$\delta q^i = \mathcal{F}L^* \left(\frac{\partial G_H}{\partial p_i} \right) \ . \tag{11}$$

The canonical Hamiltonian H_C (which is unique on the constraint surface) is that function whose pullback is the Lagrangian energy (a function in configuration-velocity space):

$$E_L = \hat{p}_s \dot{q}^s - L = \mathcal{F}L^*(H_C) \ . \tag{12}$$

To the canonical Hamiltonian one can add constraints and eventually terminate the process which leads to the dynamics (which will be equivalent to the Lagrangian theory) [3,6,7].

The result of this treatment of the relationship between Lagrangian and Hamiltonian treatments is that the generators of gauge transformations must be chosen to be projectable under the Legendre transformation in order to be well-defined in phase space. We give concrete realizations of this requirement in the next section.

3. Einstein-Yang-Mills Theory

The Yang-Mills field variables are denoted A^i_μ, where i is an internal index and μ is a spacetime index. (In electromagnetism, the i doesn't appear, and A_μ is the electromagnetic four-potential.) The Yang-Mills Lagrangian density is given by

$$\mathcal{L}_{YM} = -\frac{1}{4} F^i_{\mu\nu} F^k_{\sigma\tau} g^{\mu\sigma} g^{\nu\tau} C_{ik} \sqrt{|^{(4)}g|} \ ; \tag{13}$$

where the Yang-Mills field is

$$F^i_{\mu\nu} = A^i_{\nu,\mu} - A^i_{\mu,\nu} - C^i_{jk} A^j_\mu A^k_\nu \; ; \tag{14}$$

where $|^{(4)}g|$ is the absolute value of the determinant of the four-metric, $g^{\mu\sigma}$ being its inverse; and where C^i_{jk} are the structure constants of the Yang-Mills group, C_{ij} being a group-invariant metric.

The momentum functions are (the dot is $\partial/\partial t$)

$$\hat{P}^\alpha_i = \frac{\partial \mathcal{L}_{YM}}{\partial \dot{A}^i_\alpha} = F^j_{\sigma\tau} g^{\alpha\sigma} g^{0\tau} C_{ij} \sqrt{|^{(4)}g|} \; . \tag{15}$$

It is clear, because of the antisymmetry of the field, that the (effective!) primary constraints are

$$0 = \hat{P}^0_i = \frac{\partial \mathcal{L}_{YM}}{\partial \dot{A}^i_0} \; . \tag{16}$$

Consequently, projectable gauge transformations must be independent of $\dot{A}^i_0$.

In Yang-Mills theory, a gauge transformation has descriptors Λ^i, generating the transformation

$$\delta_R[\Lambda] A^i_\mu = -\Lambda^i_{,\mu} - C^i_{jk} \Lambda^j A^k_\mu =: -(\mathcal{D}_\mu \Lambda)^i \; , \tag{17}$$

thus defining $\mathcal{D}_\mu$, the Yang-Mills covariant derivative. (The subscript R stands for rotation, electromagnetic theory being the common example.) The effect on the field (this involves an application of the Jacobi identity) is

$$\delta_R F^i_{\mu\nu} = -C^i_{jk} \Lambda^j F^k_{\mu\nu} \; . \tag{18}$$

$\mathcal{L}_{YM}$ is invariant if the Yang-Mills metric obeys $C^s_{ij} C_{sk} = C^s_{ik} C_{sj}$, as it will in two important cases: In a semi-simple group, the metric is usually taken to be $C_{ij} = C^s_{it} C^t_{js}$, and if the group is Abelian, then $C_{ij} = \delta_{ij}$.

Note that δ_R is independent of $\dot{A}^i_0$ and so is, indeed, projectable.

4. Diffeomorphism-Induced Transformations

We begin by expressing the spacetime metric in the form

$$(g_{\mu\nu}) = \begin{pmatrix} -N^2 + N^c N^d g_{cd} & g_{ac} N^c \\ g_{bd} N^d & g_{ab} \end{pmatrix} \; , \tag{19}$$

where N is the lapse function and N^c the shift variables. The Lagrangian density for General Relativity (we are ignoring boundary terms) is [8]

$$\mathcal{L}_{GR} = N\sqrt{g} \left({}^3R + K_{ab} K^{ab} - (K^a_a)^2 \right) \; , \tag{20}$$

with the following definitions: Three-dimensional indices $(a, b, \ldots)$ are raised or lowered with the three metric g_{ab}, and ^3R is the three-curvature formed from g_{ab}. The extrinsic curvature is defined by

$$K_{ab} := \frac{1}{2N}(\dot{g}_{ab} - N_{a|b} - N_{b|a}) \ , \tag{21}$$

where the super dot means $\partial/\partial t$, and $|$ means the covariant derivative based on g_{ab}. Note also that the relation between the determinants of the three-metric and the spacetime metric is:

$$^{(4)}g = -N^2 g \ . \tag{22}$$

Note that $\mathcal{L}_{GR}$ is independent of $\dot{N}, \dot{N}^a$, and this leads to the result that a projectable variation must be independent of $\dot{N}, \dot{N}^a$.

One of the principles of General Relativity is that the theory should be independent of the coordinates in which it is expressed. Thus, we consider a general (infinitesimal) diffeomorphism:

$$\delta_D[\epsilon]x^\mu = -\epsilon^\mu \ . \tag{23}$$

The effect of this variation on the components of a tensor such as the metric is the formula for the Lie derivative:

$$\delta_D g_{\mu\nu} = g_{\mu\nu,\sigma}\epsilon^\sigma + g_{\sigma\nu}\epsilon^\sigma_{,\mu} + g_{\mu\sigma}\epsilon^\sigma_{,\nu} \ . \tag{24}$$

In particular (with $e^{ac}g_{cb} = \delta^a_b$),

$$\delta_D N = \dot{N}\epsilon^0 + N_{,a}\epsilon^a + N\dot{\epsilon}^0 - NN^a\epsilon^0_{,a} \ , \tag{25}$$

$$\begin{aligned}\delta_D N^a = {} & \dot{N}^a\epsilon^0 + N^a_{,c}\epsilon^c + N^a\dot{\epsilon}^0 \\ & -(N^2e^{ac} + N^aN^c)\epsilon^0_{,c} + \dot{\epsilon}^a - N^c\epsilon^a_{,c} \ .\end{aligned} \tag{26}$$

To eliminate the dependence on $\dot{N}, \dot{N}^a$, we require that the functions ϵ^μ which generate the variation be dependent on N, N^a:

$$\epsilon^0 = \frac{\xi^0}{N} \qquad , \qquad \epsilon^a = \xi^a - \frac{N^a}{N}\xi^0 \ , \tag{27}$$

where the descriptors ξ^μ are independent of the lapse and shift. Another way of expressing this result is

$$\epsilon^\mu = \delta^\mu_a\xi^a + n^\mu\xi^0 \ , \tag{28}$$

where the unit normal to the $t =$const hypersurfaces is

$$n^\mu = \left(\frac{1}{N}, -\frac{N^a}{N}\right) \ . \tag{29}$$

The Lie derivative of the Yang-Mills potential is

$$\delta_D A^i_0 = \dot{A}^i_0\epsilon^0 + A^i_0\dot{\epsilon}^0 + A^i_a\dot{\epsilon}^a + A^i_{0,a}\epsilon^a \ . \tag{30}$$

This variation is not, as it stands, projectable, even taking into account equation (27). To make the variation projectable, that is independent of A^i_0 as well as independent of N, N^a, one must add a Yang-Mills gauge transformation:

$$\begin{aligned}(\delta_D+\delta_R)A^i_0 &= \dot{A}^i_0\epsilon^0 + A^i_0\dot{\epsilon}^0 + A^i_a\dot{\epsilon}^a + A^i_{0,a}\epsilon^a \\ &\quad -\dot{M}^i - C^i_{st}M^s A^t_0 \ . \end{aligned} \tag{31}$$

One convenient choice for the descriptor of the Yang-Mills part is

$$M^i = A^i_\sigma n^\sigma \xi^0 \ . \tag{32}$$

In summary, the general projectable variation, namely the variation which is a member of the gauge group, is a combination of a variation based on diffeomorphisms and on Yang-Mills gauge transformations. The descriptors are functions which do not depend on the variables N, N^a, A^i_0. We have denoted the descriptors of the diffeomorphism-induced variations by ξ^μ and the descriptors of the general Yang-Mills transformation by Λ^i. The general variation thus is generated by

$$\epsilon^\mu = \delta^\mu_a\xi^a + n^\mu\xi^0 \qquad , \qquad A^i_\sigma n^\sigma\xi^0 + \Lambda^i \ . \tag{33}$$

5. Summary, Conclusions, Musings

The canonical Hamiltonian has the form

$$H_C = \int d^3x\, N^A\mathcal{H}_A \ , \tag{34}$$

where

$$N^A = (N, N^a, -A^i_0) \ . \tag{35}$$

The primary constraints are

$$P_A = (p, p_a, -P_i) = 0 \ , \tag{36}$$

and the secondary constraints are

$$\dot{P}_A = \{P_A, H_C\} = -\mathcal{H}_A \ . \tag{37}$$

There are no more constraints. The explicit forms of the $\mathcal{H}_A$ are

$$\begin{aligned}\mathcal{H}_0 &= \frac{1}{2\sqrt{g}}C^{ij}g_{ab}P^a_iP^b_j + \frac{\sqrt{g}}{4}C_{ij}e^{ac}e^{bd}F^i_{ab}F^j_{cd} \\ &\quad +\frac{1}{\sqrt{g}}(p_{ab}p^{ab} - (p^c_c)^2) - \sqrt{g}\,{}^3R \ , \end{aligned}\tag{38}$$

$$\mathcal{H}_a = P^b_iF^i_{ab} - 2p^b_{a|b} \ , \tag{39}$$

$$\mathcal{H}_i = \mathcal{D}_aP^a_i \ . \tag{40}$$

Finally, we give a table of the variables:

Configuration:	g_{ab}	A^i_a	N	N^a	A^i_0
Momentum:	p^{ab}	P^a_i	p	p_a	P_i

The generators of projectable variations have the form (see [2] for details)

$$G(t) = G^{(0)}_A \xi^A + G^{(1)}_A \dot{\xi}^A. \tag{41}$$

Although there is a bit of freedom in the choice of the functions in this equation, the simplest form is

$$G[\xi] = P_A \dot{\xi}^A + (\mathcal{H}_A + P_C N^B \mathcal{C}^C_{AB})\xi^B \ , \tag{42}$$

where

$$\{\mathcal{H}_A, \mathcal{H}_B\} = \mathcal{C}^C_{AB} \mathcal{H}_C \ . \tag{43}$$

The structure functions $\mathcal{C}^C_{AB}$ are calculable from equations (38,39,40) (see [2]).

For example, a global time translation has descriptors determined by $\epsilon^\mu = \delta^\mu_0$, and it is found that the generator is the canonical Hamiltonian, as one would expect. Note that the gauge variables N, N^a, A^i_0 are in a sense chosen, and time translation works on equivalence classes of solutions with the same N, N^a, A^i_0.

The conclusions we can draw from these results are the following:

1) The Lagrangian and the Hamiltonian formulations are, yes, equivalent. That means that the physical contents of the two formulations are the same. This is not a trivial statement, for the counting of degrees of freedom in the two formulations may differ in somewhat subtle ways (see [1,3]).

2) The basis for making a choice of the generators of the gauge transformations is the principle that they must be chosen to be projectable under the Ledendre map in order to ensure comment (1) above.

3) Yang-Mills gauge transformations present no problems. However, General Relativity and the Einstein-Yang-Mills combination are meant to be invariant under diffeomorphisms. Diffeomorphisms do indeed determine gauge transformations, but the gauge transformations, to be projectable, must depend in explicit ways on the lapse N and shift N^a; plus they must involve an associated Yang-Mills transformation depending on the gauge variables N, N^a, A^i_0.

4) We emphasize this last point: The gauge group acts on the dynamical variables. In contrast, the diffeomorphism group acts on the underlying manifold; diffeomorphisms are not the same as the transformations in the gauge group, though by suitable choice of the gauge variables, any diffeomorphism does appear.

5) We've obtained similar results in related formulations of general relativity, in particular in the triad formulation [9] and in the Ashtekar complex formulation (where the reality conditions have to be treated specially) [10].

Thus the gauge group in General Relativity is properly a group acting on dynamical variables, and is thus significantly broadened from the diffeomorphism group, which acts on the spacetime manifold. We hope our insights into the structure of this group will help in strategies to quantize the theory.

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