

Generalized 2D Yang–Mills theories: large– N limit and phase structure*

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Received 20.04.2000

Abstract

After review the 2D Yang–Mills theories (YM_2) and its large– N behaviour, the Generalized 2D Yang–Mills theories (gYM_2) and their partition functions on a general two–dimensional Riemann surface are discussed. The large– N behaviour of these models is studied in weak regime, and in strong regime, we restrict ourselves to ϕ^4 gYM_2 . We show that this model has a third order phase transition, similar to ordinary YM_2 theory.

1. Introduction

The two–dimensional Yang–Mills theory (YM_2) is a theoretical tool for understanding the main theory of particle physics, QCD_4 , and there have been many efforts to understand this theory in recent years. The partition function of these theories on Σ_g , a two–dimensional Riemann surface of genus g , has been first calculated in the context of lattice gauge theory [1,2]. On the other hand the string interpretation of $2d$ Yang–Mills theory was discussed in [3] and [4] by studying the $1/N$ expansion of the partition function for $SU(N)$ gauge group. It was shown that the coefficients of this expansion are determined by a sum over maps from a two–dimensional surface onto the two–dimensional target space.

Now the interesting point is that the pure YM_2 , as a 2D counterpart of the theory of strong interactions, is not unique, and it is possible to generalize it without losing properties such as invariance under area–preserving diffeomorphisms and lack of propagating

*Talk presented in Regional Conference on Mathematical Physics IX held at Feza Gürsey Institute, Istanbul, August 1999.

degrees of freedom. In ref. [5], these generalized models have been introduced in the framework of the BF theory, and the partition function of these theories has been obtained by regarding the general Yang-Mills action as a perturbation of the topological theory at zero area. In [6], the large N -limit of such theories (called generalized 2D Yang-Mills theories (gYM₂) in [6]), when coupled to fundamental fermions of $SU(N)$ have been studied. And the authors of [7] have generalized the Migdal's suggestion about the local factor of plaquettes, and have shown that this generalization satisfies the necessary requirements. In this way they have found the partition function and the expectation values of Wilson loops of gYM₂'s. For a review see [8].

One of the important features of YM₂, and also gYM₂'s, is their behaviour in the case of large gauge groups, e.g., the large N behaviour of $SU(N)$ (or $U(N)$) gauge theories. On one hand, it is interesting to study the relation between these theories at large N and the string theory, as mentioned above. On the other hand, these kinds of investigations are useful in exploring more general properties of large- N QCD, such as the phase structure of QCD. Therefore, studying the large N behaviour of the free energy of these theories is very important.

In this talk, I want to report the new results that have been obtained in the study of the large- N limit of gYM₂ theories. The plan of the talk is following. In section 2, we will bring a brief review about the calculation of the partition function of YM₂ on Σ_g in the continuum approach. In section 3, the large- N behaviour of YM₂ will be discussed, and it will be shown that this model has a third order phase transition, a fact that has been known earlier in the context of lattice formulation [9]. In section 4, we will introduce the gYM₂ and the partition function of these theories, for an arbitrary model, will be found. In section 5, we will study the large- N behaviour of these models and the free energy of them will be obtained for all gYM₂ models in the weak regime ($A < A_c$ areas). Some specific models will be also investigated in the strong regime.

2. Partition function of YM₂ on Σ_g

The two-dimensional pure Yang-Mills action is defined as following

$$S = \int \text{tr}(F^2) d\mu, \quad (1)$$

where $F^{\mu\nu}$ is the field-strength tensor and $d\mu = \sqrt{g(x)} d^2x$. If one works in the Schwinger-Fock gauge:

$$A_\mu^a(x) = \int_0^1 ds s x^\nu F_{\nu\mu}^a(sx), \quad (2)$$

which has the following property:

$$G^a = x^\mu A_\mu^a = 0, \quad (3)$$

it can be shown that the Jacobian of the transformation $(A_1^a, A_2^a) \rightarrow (\xi^a = F_{01}^a, G^a)$, cancels precisely against the ghost determinant [10]. In this way, the expectation value

of any operator becomes

$$\langle O \rangle = \int D\xi e^{-\int \xi^2} O(\xi). \quad (4)$$

Now consider a disk D with boundary condition $\text{Pexp}(\oint_{\partial D} A) = g \in G$, then the wave function of this disk is

$$\begin{aligned} K(g, A(D)) &= \int D\xi e^{-\int \xi^2} \delta(\text{Pe}^{\oint_{\partial D} A}, g) \\ &= \sum_{\lambda} d_{\lambda} \chi_{\lambda}(g) e^{-C_2(\lambda)A(D)/N}. \end{aligned} \quad (5)$$

In this relation λ labels the irreducible representations of the gauge group G , d_{λ} is the dimension of the representations, $\chi_{\lambda}(g)$ is the character of the group element g in the representation λ , $C_2(\lambda)$ is the second Casimir, and $A(D)$ is the area of disk D . Now since a sphere can be constructed by gluing two disks D and D' to each other, the partition function on the sphere is:

$$\begin{aligned} Z(S^2) &= \int dg K(g, A(D)) K(g^{-1}, A(D')) \\ &= \sum_{\lambda} d_{\lambda}^2 e^{-C_2(\lambda)A(S^2)/N}, \end{aligned} \quad (6)$$

where $A(S^2) = A(D) + A(D')$.

By using (5), and the relation

$$\int dg \chi(xgyg^{-1}) = d(\lambda)^{-1} \chi(x) \chi(y), \quad (7)$$

one can compute the wave function on a cylinder, and then on surfaces with more boundaries, and at last by gluing the appropriate surfaces, the wave function on a surface with g genus and n boundaries (with boundary conditions g_1, \dots, g_n) is found as following [1,2,10]

$$K(\Sigma_{g,n})(g_1, \dots, g_n, A(\Sigma_{g,n})) = \sum_{\lambda} d_{\lambda}^{2-2g-2n} \chi_{\lambda}(g_1) \dots \chi_{\lambda}(g_n) e^{-C_2(\lambda)A(\Sigma_{g,n})/N}. \quad (8)$$

3. Large- N limit of YM_2 on sphere

To calculate the large- N limit of the partition function of YM_2 on sphere (the only surface without any boundary that has the non-trivial large- N limit), we use the following expressions for the dimensions and Casimirs of the representations of $U(N)$ group:

$$d_r = \prod_{1 \leq i < j \leq N} \left(1 + \frac{n_i - n_j}{j - i}\right),$$

$$C_2(r) = \sum_{i=1}^N [n_i^2 + n_i(N - 2i + 1)], \quad (9)$$

where n_i is the length of the i -th row of the Young tableau, and $n_1 \geq \dots \geq n_N$. In the large- N limit, the parameter $0 \leq x = k/N \leq 1$ is a continuous parameter, and if we define $n_k/N = n(x)$, it can be shown that

$$Z(S^2) = \sum_r d_r^2 e^{-C_2(r)A/N} = \int \prod_{0 \leq x \leq 1} dn(x) e^{S[n(x)]}, \quad (10)$$

in which

$$S[n(x)] = \frac{1}{2}N^2 \int_0^1 dx \left\{ -A[n^2(x) + (1 - 2x)n(x)] + 2 \int_0^1 dy \log \left[1 + \frac{n(x) - n(y)}{y - x} \right] \right\}. \quad (11)$$

In the large- N limit, the dominant contribution in the partition function (10) comes from the classical representation, i.e., the representation that is satisfied in the following saddle-point equation

$$2A\phi(x) = P \int_0^1 \frac{dy}{\phi(x) - \phi(y)}, \quad (12)$$

where P indicates the principal value of the integral and $\phi(x) = n(x) - x + 1/2$. Introducing the density

$$\rho[\phi(x)] = \frac{dx}{d\phi(x)}, \quad (13)$$

the eq.(12) is reduced to:

$$2Az = P \int_{-a}^a \frac{\rho(\lambda)d\lambda}{z - \lambda}, \quad (14)$$

along with the normalization condition

$$\int_{-a}^a \rho(\lambda)d\lambda = 1. \quad (15)$$

The condition $n_1 \geq n_2 \geq \dots \geq n_N$ imposes the following condition on the density $\rho(\lambda)$:

$$\rho(\lambda) \leq 1. \quad (16)$$

In this way one can find the saddle-point solution for $\rho(\lambda)$ as:

$$\rho(\lambda) = \frac{A}{2\pi} \sqrt{\frac{4}{A} - \lambda^2}, \quad \text{and} \quad a = \sqrt{\frac{2}{A}}, \quad (17)$$

which is the famous semi-circle distribution. Using (17), the free energy becomes [11]

$$F = \frac{1}{N^2} \ln Z(S^2) \simeq \frac{1}{N^2} \ln (\exp(S_{\text{class.}})) = \frac{A}{24} - \frac{1}{2} \log A. \quad (18)$$

The maximum value of the density $\rho(\lambda)$ in (17) is $\rho_{\max} = \rho(0) = \frac{A}{2\pi} \sqrt{\frac{4}{A}}$, which by (16) must be $\rho_{\max} \leq 1$. Therefore the distribution (17) is valid only for the areas

$$A \leq A_c = \pi^2. \quad (19)$$

For studying the YM_2 in strong region ($A \geq A_c$), Douglas and Kazakov [12] assumed the following symmetric ansatz for ρ in this region

$$\rho(\lambda) = \begin{cases} 1, & z \in [-b, b] \\ \tilde{\rho}(\lambda), & z \in [-a, -b] \cup [b, a]. \end{cases} \quad (20)$$

It can be shown that the saddle-point equation for $\tilde{\rho}$ is

$$g(h) = \frac{1}{2}Ah + \log \frac{h-b}{h+b} = P \int_{-a}^a \frac{\tilde{\rho}(s)ds}{h-s}. \quad (21)$$

If one introduces the function $f(h)$ in the complex h -plane

$$f(h) = \int_{-a}^a \frac{\tilde{\rho}(s)ds}{h-s}, \quad (22)$$

where is analytic function on the complex plane except for the cuts at $[-a, -b]$ and $[b, a]$, it has the following property

$$f(h + i\epsilon) = g(h) - i\pi\tilde{\rho}, \quad (23)$$

and one can show that

$$f(h) = \frac{1}{2\pi i} \sqrt{(a^2 - h^2)(b^2 - h^2)} \oint_c \frac{g(s)ds}{(h-s)\sqrt{(a^2 - s^2)(b^2 - s^2)}}, \quad (24)$$

where c is a contour encircling two above mentioned cuts.

In this way, it is found that [12]

$$F'_{\text{strong}}(A) - F'_{\text{weak}}(A) = \frac{1}{\pi^2} \left(\frac{A - A_c}{\pi^2} \right)^2 + \dots, \quad (25)$$

where F' means derivative with respect to A . Therefore, it is shown that the YM_2 has a *third order* phase transition.

4. Partition function of gYM_2 on Σ_g

The main properties of YM_2 theory are:

- 1-Invariance under area-preserving diffeomorphism.
- 2-Lack of propagating degrees of freedom.
- 3-Area-law behaviour ($Z \sim e^A$), which in some sense relates to confinement.

4-Third-order phase transition.

5-...

These properties are not unique for YM_2 action:

$$Z = \int D\xi e^{-\int \text{tr}(\xi^2)} = \int D\xi DB e^{-\int (\text{tr}(B\xi) + \text{tr}(B^2))}, \quad (26)$$

where B is an auxiliary pseudo-scalar field in the adjoint representation of the gauge group, but rather are shared by a wide class of theories, called generalized Yang-Mills theory, defined by the action

$$Z = \int D\xi DB e^{-\int (\text{tr}(B\xi) + \text{tr}(\Lambda(B)))}, \quad (27)$$

where $\Lambda(B)$ is an arbitrary class function, i.e.,

$$\Lambda(UBU^{-1}) = \Lambda(B), \quad \forall U \in G. \quad (28)$$

By a method which is very similar to those discussed in section 2, one can finally find the partition function of these theories as follows [7,13]

$$Z(\Sigma_g) = \sum_r d_r^{2-2g} e^{-A(\Sigma_g)\Lambda(r)}, \quad (29)$$

in which

$$\Lambda(r) = \sum_{k=1}^p \frac{a_p}{N^{k-1}} C_k(r). \quad (30)$$

In (30), a_k 's are arbitrary constants, and C_k is the k th Casimir of group

$$C_k = \sum_{i=1}^N [(n_i + N - i)^k - (N - i)^k]. \quad (31)$$

5. Large- N limit of gYM_2 on sphere

Our main strategy is the same as one discussed in section 3. In the large- N limit, if we define

$$\phi(x) = -n(x) - 1 + x, \quad (32)$$

the partition function becomes

$$Z = \int \prod_{0 \leq x \leq 1} d\phi(x) e^{S[\phi(x)]}. \quad (33)$$

where

$$S(\phi) = N^2 \left\{ -A \int_0^1 dx G[\phi(x)] + \int_0^1 dx \int_0^1 dy \log|\phi(x) - \phi(y)| \right\}, \quad (34)$$

apart from an unimportant constant, and

$$G(\phi) = \sum_{k=1}^p (-1)^k a_k \phi^k. \quad (35)$$

In this case, the saddle-point equation is

$$g[\phi(x)] = \frac{A}{2} G'(\phi) = \text{P} \int_0^1 \frac{dy}{\phi(x) - \phi(y)}, \quad (36)$$

or in terms of the density ρ is

$$g(z) = \text{P} \int_b^a \frac{\rho(\lambda) d\lambda}{z - \lambda}. \quad (37)$$

Again if we define the function $H(z)$ in the complex z -plane

$$H(z) := \int_b^a \frac{\rho(\lambda) d\lambda}{z - \lambda}, \quad (38)$$

with following property

$$H(z \pm i\epsilon) = g(z) \mp i\pi\rho(z) \quad b \leq z \leq a, \quad (39)$$

Then it can be shown that

$$H(z) = \frac{1}{2\pi i} \sqrt{(z-a)(z-b)} \oint_c \frac{g(\lambda) d\lambda}{(z-\lambda)\sqrt{(\lambda-a)(\lambda-b)}}, \quad (40)$$

where the contour c is a contour encircling the cut $[b, a]$, and excluding z . Note that in this case it is not necessary the limits of integral (38) are symmetric, unlike the YM_2 in which $b = -a$ (see eq.(14)), and in principle the parameters a and b are independent.

5.1. $A < A_c$ region

In the weak-coupling ($A < A_c$) region, one can deform the contour c in (40) to a contour around the point z and the contour c_∞ (a contour at the infinity), and using the $z \rightarrow \infty$ limit of (38), to find the explicit form of ρ and the relations which determine the parameters a and b [14]. When G is an even function, in which $b = -a$, the final results are

$$\rho_w(z) = \frac{\sqrt{a^2 - z^2}}{\pi} \sum_{n,q=0}^{\infty} \frac{(2n-1)!!}{2^n n! (2n+q+1)!} a^{2n} z^q g^{(2n+q+1)}(0), \quad (41)$$

and

$$\sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n! (2n-1)!} a^{2n} g^{(2n-1)}(0) = 1. \quad (42)$$

In the above equations, $g^{(n)}$ is the n th derivative of g defined in (36). It can be shown that, [15], for all $G(\phi) = \phi^{2k}$ models, with $k \in \mathbb{Z}$, $\rho(z)$ has a minimum at $z = 0$ and two symmetric maxima at $\pm z_0$, and the derivative of the free energy is

$$F'(A) = \frac{1}{2kA}. \quad (43)$$

As the simplest example of gYM₂, consider $G(\phi) = \phi^4$. Using the above relations, one finds

$$\begin{aligned} \rho(z) &= \frac{2A}{\pi} \left(\frac{a^2}{2} + z^2 \right) \sqrt{a^2 - z^2}, \\ a &= \left(\frac{4}{3A} \right)^{1/4}. \end{aligned} \quad (44)$$

The density ρ has a minimum at $z = 0$ and two maxima at $z_0^{(\pm)} = \pm a/\sqrt{2}$. Now as $\rho(z_0^{(\pm)}) = \sqrt{2}a^3A/\pi$, if $A > A_c = 27\pi^4/256$, then the condition $\rho \leq 1$ is violated. So the solution (44) is valid only in the region $A \leq A_c$.

5.2. $A > A_c$ region for $G(\phi) = \phi^4$ model

In the strong region ($A > A_c$), we first restrict ourselves to $G(\phi) = \phi^4$ model. In this region, we use the following symmetric ansatz for ρ

$$\rho_s(z) = \begin{cases} 1, & z \in [-b, -c] \cup [c, b] =: L' \\ \tilde{\rho}_s(z), & z \in [-a, -b] \cup [-c, c] \cup [b, a] =: L. \end{cases} \quad (45)$$

The saddle-point equation is the same as weak region, i.e.,

$$2Az^3 = \text{P} \int_{-a}^a dw \frac{\rho_s(w)}{z-w}, \quad z \in L. \quad (46)$$

If we define the function $\tilde{H}(z)$ in the complex z -plane with three-cut singularity at $z \in L$

$$\tilde{H}(z) = \int_L dw \frac{\tilde{\rho}_s(w)}{z-w}, \quad (47)$$

then

$$H(z) := \text{P} \int_{-a}^a dw \frac{\rho_s(w)}{z-w} = 2Az^3 - i\pi\rho(z) = \tilde{H}(z) + \log \frac{z+b}{z+c} + \log \frac{z-c}{z-b}. \quad (48)$$

Note also that the expansion of $H(z)$ for large $-z$ is

$$H(z) = \frac{1}{z} + \frac{1}{z^3} \int_{-a}^a \rho_s(\lambda) \lambda^2 d\lambda + \frac{1}{z^5} F'_s(A) + \dots \quad (49)$$

Now it can be shown that the solution of $\tilde{H}(z)$ is

$$\tilde{H}(z) = \frac{1}{2\pi i} \sqrt{(z^2 - a^2)(z^2 - b^2)(z^2 - c^2)} \oint_{c_L} \frac{2A\lambda^3 - \log \frac{\lambda+b}{\lambda+c} - \log \frac{\lambda-c}{\lambda-b}}{(z-\lambda)\sqrt{(\lambda^2 - a^2)(\lambda^2 - b^2)(\lambda^2 - c^2)}} d\lambda, \quad (50)$$

where c_L is a contour encircling the three distinct intervals of L . By deforming the contour c_L to a contour at infinity and three contours encircling the point z and the two intervals $[-b, -c]$ and $[c, b]$, respectively, one can determine $\tilde{H}(z)$ and then $H(z)$ and finally expands $H(z)$ at large z . The coefficient of $1/z$ in this expansion must be taken 1 (see (49)), which results

$$A \left[\frac{3}{4}(a^4 + b^4 + c^4) + \frac{1}{2}(a^2b^2 + b^2c^2 + c^2a^2) \right] - 2 \int_c^b \frac{\lambda^3 d\lambda}{R(\lambda)} = 1, \quad (51)$$

in which

$$R(\lambda) = \sqrt{(a^2 - \lambda^2)(b^2 - \lambda^2)(\lambda^2 - c^2)}. \quad (52)$$

And the coefficient of $1/z^2$ in this expansion must be taken 0, which results

$$A(a^2 + b^2 + c^2) = 2 \int_c^b \frac{\lambda d\lambda}{R(\lambda)}. \quad (53)$$

In this way we have *two* equations (51) and (53), and *three* unknown parameters a , b , and c !

The point is that: we must consider such variations of ρ which have two properties:

- 1- Have fixed values in the endpoints of different regions.
- 2- These fixed values must be *equal*.

In Douglas and Kazakov investigation, YM_2 , as we have only two regions and ρ is symmetric, the second requirement satisfies automatically when the first one imposes. But here we have three distinct regions, and the second requirement must be imposed by hand. To impose it, consider again the action (34) for ϕ^4 and in terms of ρ

$$S(\rho_s) = N^2 \left[-A \int_{-a}^a dz \rho_s(z) z^4 + \int_{-a}^a dz \int_{-a}^a dw \rho_s(z) \rho_s(w) \log|z-w| \right], \quad (54)$$

with condition $\int_{-a}^a \rho_s(z) dz = 1$. To consider this condition, we introduce the Lagrange multiplier μ and the functional \tilde{S} as following

$$\tilde{S} = S + N^2 \mu \left[\int_{-a}^a dz \rho_s(z) - 1 \right]. \quad (55)$$

Then $\delta\tilde{S} = 0$ leads to

$$N^2 \int_{-a}^a dz \left[-Az^4 + 2 \int_{-a}^a dw \rho_s(w) \log|z-w| + \mu \right] \delta\rho_s + N^2 \left[\int_{-a}^a dz \rho_s(z) - 1 \right] \delta\mu = 0, \quad (56)$$

and the above equation gives

$$\frac{A}{2}z^4 - \int_{-a}^a dw \rho_s(w) \log|z-w| = \frac{\mu}{2}, \quad z \in L. \quad (57)$$

Differentiating eq.(57) with respect to z , gives our previous saddle-point equation (46). But here we have another requirement, that is the eq.(57) must be equal for $z = b$ and $z = c$. Subtracting eq.(57) at these two points results

$$\int_c^b dz \left[2Az^3 - P \int_{-a}^a dw \frac{\rho_s(w)}{z-w} \right] = 0. \quad (58)$$

Using eqs.(48) and (50), eq.(58) can be rewritten as

$$A \int_c^b R(z) dz + \int_c^b dz P \int_c^b \frac{R(z)\lambda d\lambda}{(z^2 - \lambda^2)R(\lambda)} = 0, \quad (59)$$

and this is our *third* equation.

By expanding a, b , and c near critical point A_c

$$\begin{aligned} c &= s(1-y), \\ b &= s(1+y), \\ a &= s\sqrt{2+e}, \end{aligned} \quad (60)$$

where at $A = A_c$, $e = 0$, $y = 0$, and $s = \frac{a_c}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\frac{4}{3A_c} \right)^{1/4}$, and after lengthy calculations, we can find $F'_s(A)$ and then finally [14]

$$F'_s(A) - F'_w(A) = \frac{1}{27A_c} \left(\frac{A - A_c}{A_c} \right)^2 + \dots \quad (61)$$

In this way we see that the ϕ^4 gYM₂ theory has a *third order phase transition*, similar to ordinary YM₂ theory.

It can be shown that this kinds of phase transition also exists in ϕ^6 and $\phi^2 + \alpha\phi^4$ gYM₂ theories [15].

Acknowledgement

I would like to thank the organizers of the IX. Regional Conference on Mathematical Physics, and also the Institute for Studies in Theoretical Physics and Mathematics for their financial supports.

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