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The Canonical Thermodynamic Formalism:Chaotic States of Spin and Gauge Systems*

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Abstract

The three-site antiferromagnetic Ising, Q-state Potts ($Q < 2$) and $Z(N)$ gauge models in some temperatures and external fields have a chaotic behavior. We investigate the thermodynamic formalism of these models on the hierarchical lattices. Using the canonical thermodynamic formalism of dynamical systems, the nonanalytic behavior in the distribution of Lyapunov exponents is investigated and the phase transition point on the "chaotic free energy" for multisite interaction antiferromagnetic Ising, Q-state Potts and $Z(2)$ gauge models. We also consider Fisher Zeros of the Baxter-Wu model.

1. Introduction

The macroscopic states of matter are determined by the properties and motion of the microscopic constituents. Since the number of the constituence is extemenly large, it is impossible to obtain a complete description of physical systems. In statistical mechanics for the equilibrium state (stable point) we deal with averages taken over an extremely long time and introduce the order paramiters (magnetization, concentration, expectation value of a single plaquette...). In many cases the physical system has no stable point (equlibrium state), but it has fixed points (periodic points). One can introduce two-sublattice structure for a spin-1 Ising model in the thermodynamic limit [1]. There are physical systems with infinite number of fixed points, we call them chaotic systems. For such systems we apply the thermodynamic formalism. There exist two methods for investigating such chaotic models which are the microcanonical thermodynamic formalism

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[2] and canonical thermodynamic formalism [3]. The dictionary of these two formalisms are given in Ref.[4]. In this paper we regard only the canonical thermodynamic formalism.

In Sec. 2 we provide the definition of the three-site antiferromagnetic Ising (TSAI), Q-state Potts and $Z(2)$ gauge models, and the exact recursion relations for partition functions on the Bethe-like lattices are obtained. In Sec. 3, using the canonical thermodynamic formalism, we construct the entropy function and chaotic free energy for TSAI, Potts and gauge models. In Sec. 4 we present the dense Mandelbrot set for Fisher Zeros of TSAI.

2. The Recursion Relations

The advantage of the Bethe-like lattices with finite coordination numbers (Bethe, Husimi, plaquette...) is that for models based on them one can obtain the exact recursion relation for corresponding partition function. Using the dynamical systems or recursive equations one can obtain order parameters for the three-site antiferromagnetic Ising, Q -state Potts and $Z(N)$ gauge models on hierarchical lattices.

1. The Hamiltonian for TSAI model is

$$H = -J_3^{'} \sum_{\triangle} \sigma_i \sigma_j \sigma_k - h^{'} \sum_i \sigma_i,$$

where σ_i takes values ± 1, the first sum goes over all triangular faces of the Husimi tree and the second over all sites. Additionally, we use the notation $J_3 = \beta J_3^{'}$, $h = \beta h^{'}$, $\beta = 1/kT$, where h is the external magnetic field, T is the temperature of the system and $J_3 < 0$ corresponding to an antiferromagnetic coupling (in all our numerical calculations we put $J_3^{'} = -1$).

The partition function will be written as

$$Z = \sum_{\{\sigma\}} exp \left\{ J_3 \sum_{\triangle} \sigma_i \ \sigma_j \sigma_k + h \sum_i \sigma_i \right\}, \tag{1}$$

where the summation goes over all configurations of the system.

When the Husimi tree is cut apart at the central triangle, it separates into three identical branches, each of which contains $\gamma - 1$ branches.

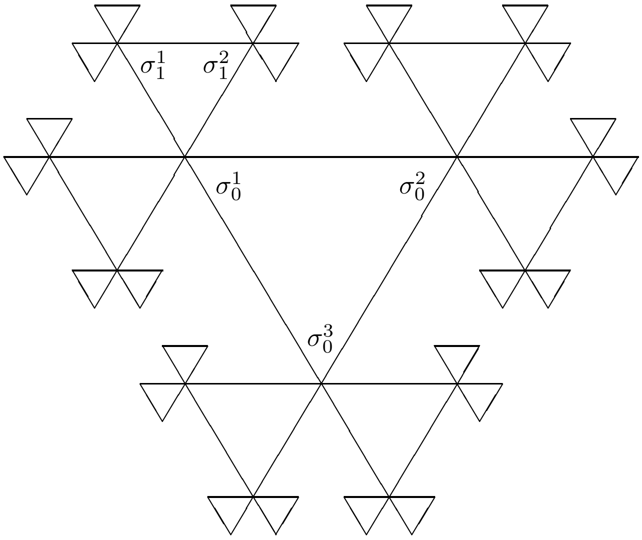

Figure 1. The Husimi lattice with $q = 3$.

The partition function can be written as follows:

$$Z = \sum_{\{\sigma_0\}} exp\left\{ J_3 \sum_{\triangle} \sigma_0^{(1)}\sigma_0^{(2)}\sigma_0^{(3)} + h\sum_j \sigma_0^{(j)} \right\} \ [g_n(\sigma_0^{(1)})]^{\gamma-1} [g_n(\sigma_0^{(2)})]^{\gamma-1} \ [g_n(\sigma_0^{(3)})]^{\gamma-1}, \tag{2}$$

where $\sigma_0^{(j)}$ are spins of central triangle, n is the number of generations ($n \to \infty$ corresponds to the thermodynamic limit and the surface effects are neglected), and the equation for one branch can be written as

$$g_n(\sigma_0) = \sum_{\{\sigma_i \neq \sigma_0\}} exp\left\{ J_3 \sum_{\triangle} \sigma_0\sigma_1^{(1)}\sigma_1^{(2)} + h \sum_{j=1,2} \sigma_1^{(j)} + J_3 \sum_{\triangle} \sigma_i\sigma_j\sigma_k + h\sum_i \sigma_i \right\}, \tag{3}$$

where the summation of last two terms are over generation $n = 2$ and higher. Each branch, in turn, can be cut along any site of the 1th-generation which is nearest to the central site. The expression for $g_n(\sigma_0)$ can therefore be rewritten in the form

$$g_n(\sigma_0) = \sum_{\{\sigma_1\}} exp\left\{ J_3 \sum_{\triangle} \sigma_0\sigma_1^{(1)}\sigma_1^{(2)} + h \sum_{j=1,2} \sigma_1^{(j)} \right\} [g_{n-1}(\sigma_1^{(1)})]^{\gamma-1} [g_{n-1}(\sigma_1^{(2)})]^{\gamma-1}. \tag{4}$$

From Eq.(4) we easily obtain

$$g_n(+) = e^{J_3+2h} g_{n-1}^{\gamma-1}(+) g_{n-1}^{\gamma-1}(+) + 2e^{-J_3} g_{n-1}^{\gamma-1}(+) g_{n-1}^{\gamma-1}(-) + e^{J_3-2h} g_{n-1}^{\gamma-1}(-) g_{n-1}^{\gamma-1}(-),$$

$$g_n(-) = e^{-J_3+2h} g_{n-1}^{\gamma-1}(+) g_{n-1}^{\gamma-1}(+) + 2e^{J_3} g_{n-1}^{\gamma-1}(+) g_{n-1}^{\gamma-1}(-) + e^{-J_3-2h} g_{n-1}^{\gamma-1}(-) g_{n-1}^{\gamma-1}(-).$$

We introduce the following variable:

$$x_n = \frac{g_n(+)}{g_n(-)}. \tag{5}$$

For x_n we can then obtain the recursion relation

$$x_n = f(x_{n-1}), \qquad f(x) = \frac{z\mu^2 x^{2(\gamma-1)} + 2\mu x^{\gamma-1} + z}{\mu^2 x^{2(\gamma-1)} + 2z\mu x^{\gamma-1} + 1}, \tag{6}$$

where $z = e^{2J_3}$, $\mu = e^{2h}$ *and* $0 \leq x_n \leq 1$. The function $f(x)$ is unimodal: it is continuous, continuously differentiable, and has one maximum x^* in $[0, 1]$. Note that $f(x^*) = 1$ for any γ, h and T. This function is nonhyperbolic (hyperbolicity for $1D$ maps means that $1 < |f'| < \infty$ in all points) and maps the interval $[0, 1]$ onto $[z, 1]$.

Through x_n, obtained by Eq.(6), one can express the magnetization of any site of the central triangle:

$$m = \langle \sigma_0 \rangle = \frac{e^h g_n^\gamma(+) - e^{-h} g_n^\gamma(-)}{e^h g_n^\gamma(+) + e^{-h} g_n^\gamma(-)} = \frac{e^{2h} x_n^\gamma - 1}{e^{2h} x_n^\gamma + 1}, \tag{7}$$

and other thermodynamic parameters, so we can say that the x_n in the thermodynamic limit ($n \to \infty$) determine the states of the system. For example, at high values of temperatures (see next subsection) the recursion Eq.(6) tends to a fixed point and therefore the system has an appointed magnetization m.

Let us consider the magnetization of the central base site. If we set $J_3^{'} = -1$ and $\gamma = 3$ in Eq.(7) and vary the temperature, then for large T we see that m, is a simple monotonically increasing function of h for $h > 0$ (see Fig.2a). If we lower T, at some point we find that there is a single bubble in the plot of m versus h (Fig.2b). As we continue to lower T, new bubbles are formed as parts of the old bubbles (Fig.2c), and for still lower T's we reach a region where for intermediate values of h we have chaos, period-three windows, etc. (see Fig.2d). The reason for this is that we have the presence of frustration effects: for small h the interaction J_3 dominates, while for large h, h dominates. One can also see in three dimension how the chaotic states arise at different temperatures and external magnetic fields [5].

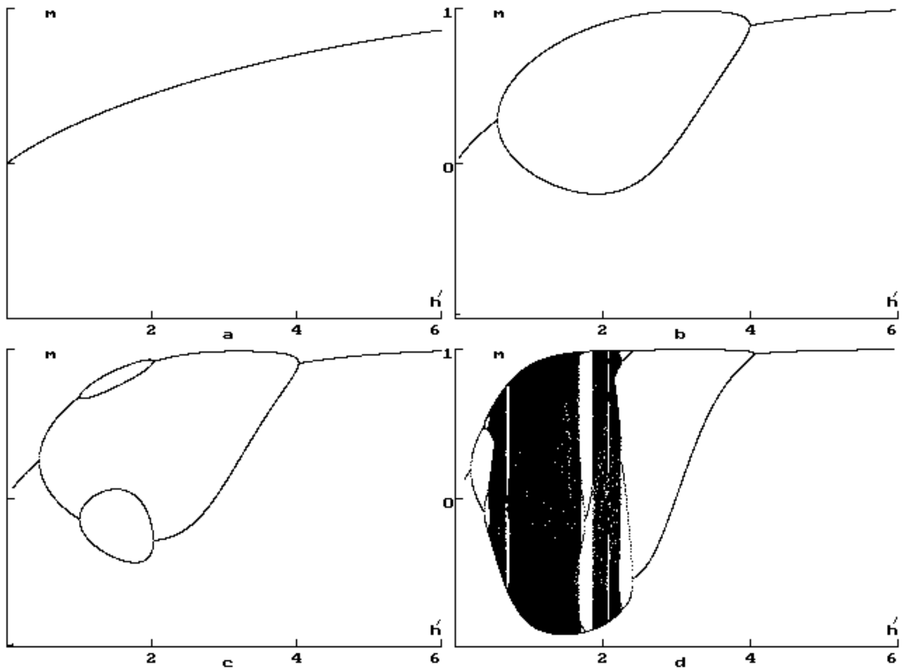

Figure 2. Plots of m versus h' for different temperatures T ($\gamma = 4$). (a) T=3; (b) T=1.3; (c) T=1.15; (d) T=0.7.

2. The Potts model in the magnetic field is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{<i,j>} \delta(\sigma_i, \sigma_j) - H \sum_i \delta(\sigma_i, 1),$$

where σ_i takes the values 1,2,..., Q, the first sum goes over all edges and the second one over all sites on the lattice. Additionally, we use the notation $K = J/kT,\ h = H/kT$.

The partition function and single site magnetization is given by

$$\mathcal{Z} = \sum_{\{\sigma\}} e^{-\mathcal{H}/kT},$$

$$M =< \delta(\sigma_0, 1) >= \mathcal{Z}^{-1} \sum_{\{\sigma\}} \delta(\sigma_0, 1) e^{-\mathcal{H}/kT},$$

where the summation goes over all configurations of the system.

The recursion relation (6) converge to a fixed point at every values of parameters h, K in a ferromagnetic case ($K > 0$) and has only one period doubling in an antiferromagnetic case ($K < 0$) corresponding to a rise of antiferromagnetic order in different sublattices for

$Q \geq 2$. The situation change drastically for $Q < 2$ in contrast to above case. For systems with Q values in the range $Q < 2$ and with antiferromagnetic interactions or for systems with Q values in the range $Q < 1$ and with ferromagnetic interactions one obtains for M versus h bifurcation diagrams with the full range of period doubling cascade, chaos, etc. The Figure 3 shows plots of M versus h for anti-ferromagnetic case($K = -0.5$, $Q = 0.8$, $\gamma = 3$).

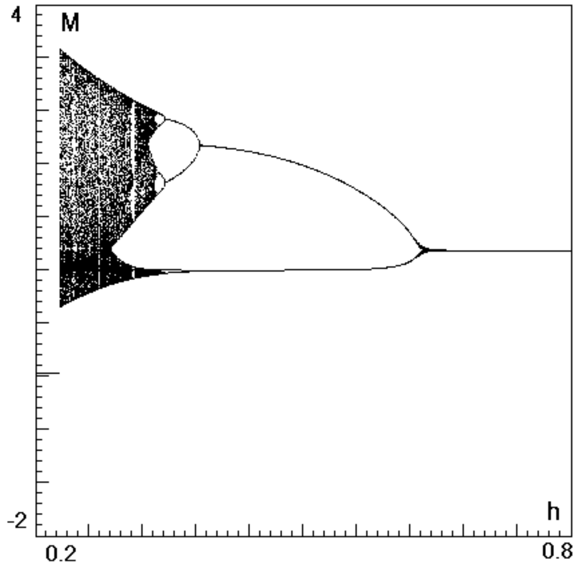

Figure 3. Plot of M (magnetization) versus h (external magnetic field). K=-0.5, Q=0.8, and $\gamma = 3$.

The Potts model has many specially modulated and chaotic phases when $Q < 2$. The presence of phase transitions is in obvious contradiction to the universality hypothesis. The transition to chaos is provided by Feigenbaum exponents which is well known to be an one-dimensional map.

The recursion relations for the Potts-Bethe map have the form

$$x_n = f(x_{n-1}, K, h), \qquad f(x, K, h) = \frac{e^h + (e^K + Q - 2)x^{\gamma-1}}{e^{K+h} + (Q-1)x^{\gamma-1}}.$$

The Q-state Potts model on the Bethe lattice for $Q < 2$ we impose two restrictions on the parameters in order to apply the canonical thermodynamic formalism to the Potts-Bethe map. The first one is regards only an odd coordination number γ for getting even function of x of the Potts-Bethe map. The second one is the requirement that

$$f(0) = -f(f(0)),$$

from which we obtain the following restrictions on h and K

$$\exp(h) = \frac{1 - \exp(2K) + 2\exp(K) - \exp(K)Q - Q}{2\exp(\gamma K)}.$$

The second one is needed for exhibiting the fully developed chaotic behavior of the Potts-Bethe map in the (h,K) plane [7].

3. It is well known that the vacuum of the non-Abelian gauge theories is non integrable in the classical limit and exhibits dynamical chaos. These chaotic solutions are playing the essential role in the self thermalization of the hot quark-gluon plasma in heavy ion collision. It was shown that the deconfinement phase domains in $SU(2)$ lattice gauge theory have a noninteger dimension near the phase transition.

We are presenting arguments which leads to chaotic states in the $Z(N)$ gauge theory on the lattice. The expectation value of the Palyakov loop operator of the $Z(N)$ theory determines the phase structure of the $SU(N)$ gauge theory at a finite temperature. In particular the restoration of the $SU(2)$ gauge symmetry in standard model and deconfinment transition in QCD at a finite temperature are consequence of phase transitions in $Z(2)$ and $Z(3)$ models respectively. Using the dynamical systems or recursive approaches one can obtain an order parameter for the $Z(N)$ gauge model on a hierarchical lattices. It is necessary to mention that for various reasons it is interesting to generalize lattice gauge actions by including larger interaction loops.

We construct a $Z(2)$ gauge model with three plaquette representation of the action and the cascade of phase transitions according to Feingenbaum scheme leading to chaotic states for some values of parameters of the model. The generalized Bethe lattice is constructed by successive building up of shells. As a zero shell we take the central plaquette and all subsequent shells come out by gluing up two new plaquettes to each link of a previous shell. As a result we get a tree of plaquettes [6] (Figure 4a).

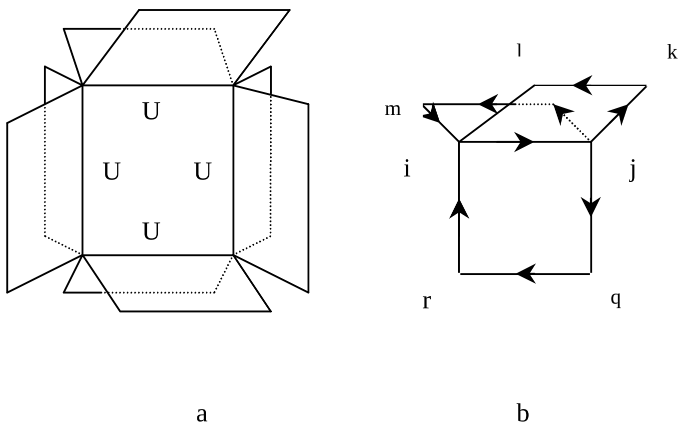

Figure 4. (a) Generalized Bethe lattice of plaquettes. (b) The minimal product of the gauge variable along the three plaquettes. The arrows show the bypass routing of loop.

So we got an infinite dimensional lattice on which three plaquettes are gluing up to each link. In the $Z(2)$ gauge model field variables U_{ij} defined on the links and take values ± 1. The gauge invariant action in the presence of three plaquette interaction may be written in the form

$$S = -\beta_3 \sum_{3p} U_{3p} - \beta_1 \sum_{p} U_p, \tag{8}$$

where

$$U_p = U_{ij}U_{jk}U_{kl}U_{li}$$

is the product of the gauge variables along the plaquette contour and

$$U_{3p} = U_{p_n} U^{'}_{p_{n+1}} U^{''}_{p_{n+1}} = U_{kl}U_{li}U_{ij}U_{jk}U_{km}U_{mn}U_{nj}U_{jo}U_{op}U_{pk}$$

is the minimal product of the gauge variable along the three plaquettes (Figure 4b);β_1, β_3 are the gauge coupling constants.

The partition function of this model is given by

$$Z = \sum_{\{U\}} e^{-S}, \tag{9}$$

where the sum is over all possible configurations of the gauge field variables $\{U\}$.The expectation value of the central single plaquette P will have the following form

$$P \equiv < U_{p_0} >= Z^{-1} \sum_{\{U\}} U_{p_0} e^{-S}. \tag{10}$$

The advantage of the generalized Bethe lattice is that for the models formulated on it an exact recursion relation can be derived. The partition function separates into four identical branches, when we are cutting apart the zero shell (central plaquette) of the generalized Bethe lattice. Then the partition function for nth generation system may be rewritten

$$Z_n = \sum_{\{U_{p_0}\}} e^{\beta_1 U_{p_0}} g_n^4(U_{p_0}), \tag{11}$$

where the sum is over all possible configurations of the field variables defined on the links of a zero plaquette $\{U_{p_0}\}$ and

$$g_n(U_{p_i}) = \sum_{\{U'_{p_{i+1}}, U''_{p_{i+1}}\}} e^{\beta_3 U_{p_i} U'_{p_{i+1}} U''_{p_{i+1}} + \beta_1 U'_{p_{i+1}} + \beta_1 U''_{p_{i+1}}} g^3_{n-1}(U'_{p_{i+1}}) g^3_{n-1}(U'^{\prime}_{p_{i+1}}). \tag{12}$$

One can easily obtain:

$$g_n(+) = 16e^{\beta_3+2\beta_1} g^3_{n-1}(+)g^3_{n-1}(+) + 32e^{-\beta_3} g^3_{n-1}(+)g^3_{n-1}(-) + 16e^{\beta_3-2\beta_1} g^3_{n-1}(-)g^3_{n-1}(-),$$

$$g_n(-) = 16e^{-\beta_3+2\beta_1} g^3_{n-1}(+)g^3_{n-1}(+) + 32e^{\beta_3} g^3_{n-1}(+)g^3_{n-1}(-) + 16e^{-\beta_3-2\beta_1} g^3_{n-1}(-)g^3_{n-1}(-).$$

Note that U_{p_i} takes the values ± 1 and coefficients before exponents arise from a gauge invariance.

Introducing the notation

$$x_n = \frac{g_n(+)}{g_n(-)}. \tag{13}$$

we obtain a recursion formula for x_n

$$x_n = f(x_{n-1}), \qquad f(x) = \frac{z\mu^2 x^6 + 2\mu x^3 + z}{\mu^2 x^6 + 2z\mu x^3 + 1}, \tag{14}$$

where $z = e^{2\beta_3}$, $\mu = e^{2\beta_1}$ and x_0 is boundary condition term.

Through x_n one can express the average value of the central plaquette for nth generation system:

$$P = \frac{8\{e^{\beta_1} g_n^4(+) - e^{-\beta_1} g_n^4(-)\}}{8\{e^{\beta_1} g_n^4(+) + e^{-\beta_1} g_n^4(-)\}} = \frac{e^{\beta_1} x_n^4 - 1}{e^{\beta_1} x_n^4 + 1}, \tag{15}$$

which is the gauge invariant order parameter in a $Z(2)$ theory for a stabile fixed point[6].

The plot of the P for different values of β_1,β_3 is presented in Figure 5.

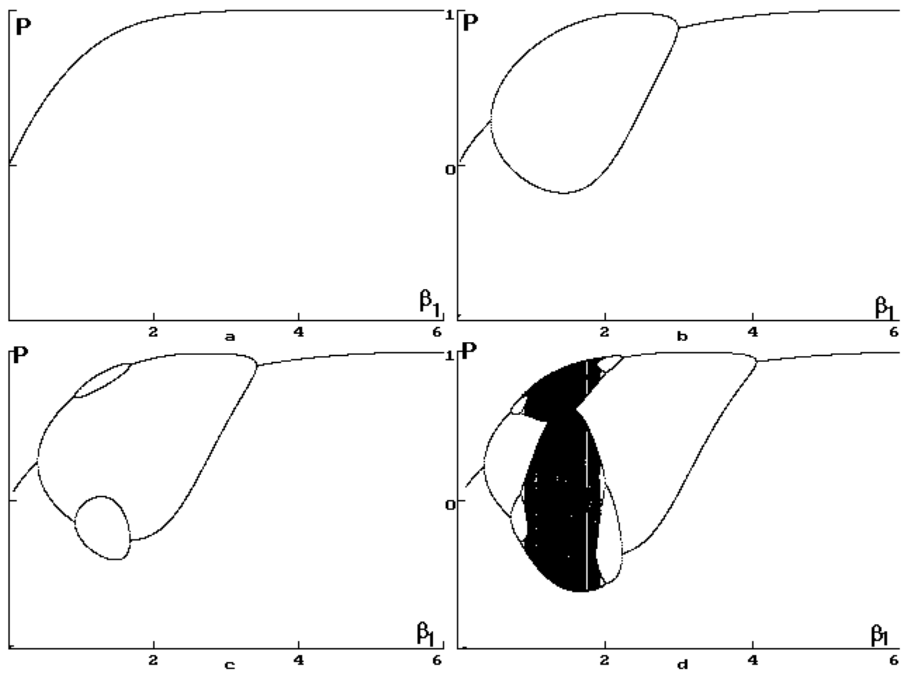

Figure 5. Plot of P versus β_1 for different temperatures T ($\beta_3 = -1/kT$). (a) T=3; (b) T=0.9; (d) T=0.45.

3. Entropy and Free Energy Functions for TSAI, Potts and Gauge Models

For a crisis map we want to describe the scaling properties of the attracting set which in this case is the entire interval I: $[c_0^2, 1]$. For this we need a natural partition and that is provided by the cylinders. For an index n, I is partitioned into 2^n intervals or cylinders, these being the segments with identical symbolic-dynamics sequences of length n taken with respect to the maximum point $x^* = c$. The inverse function $h = f^{-1}$, has two branches and the n-cylinders are all the nth-order preimages of I. The length of the cylinders is denoted by $l_{\epsilon_1,\epsilon_2,\ldots\epsilon_n}$ where $\epsilon \in \{-1, 1\}$.

The partition function $Z(\beta)$ is defined as

$$Z_n(\beta) = \sum_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} l^{\beta}_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} = \sum_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} e^{-\beta ln l_{\epsilon_1,\epsilon_2,\ldots\epsilon_n}}, \tag{16}$$

where $\beta \in (-\infty, \infty)$ is a free parameter, the inverse "temperature". In the limit $n \to \infty$ the sum behaves as

$$Z_n(\beta) = e^{-n\beta F(\beta)}, \tag{17}$$

which defines the free energy, $F(\beta)$. The entropy $S(\lambda)$ is the Legendre transform

$$S(\lambda) = -\beta F(\beta) + \lambda\beta, \tag{18}$$

where the relation between λ and β is found from

$$\lambda = \frac{d}{d\beta}(\beta F(\beta)), \quad \beta(\lambda) = S'(\lambda), \tag{19}$$

and these have the following meaning: In the limit $n \to \infty$, $e^{nS(\lambda)}$ is the number of cylinders with length $l = e^{-n\lambda}$ or, equivalently, with Lyapunov exponent λ. The dimension of the set of points in I having Lyapunov exponent λ is $S(\lambda)/\lambda$.

By using the Eqs.(16), (17) we can numerically calculate the free energy which is shown in Figure 6. One can see from Figure 6 that the free energy has a nonanalytic behavior and shows the existence of the phase transition of first order in this region of β.

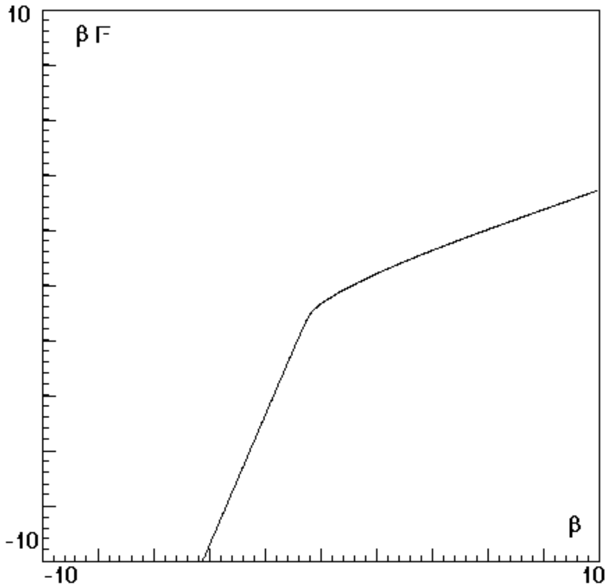

Figure 6. The free energy $F(\beta)$, which have a nonanalytical behavior around $\beta = 0.69$ and shows the existence of the phase transition in the region of β.

How can the transition be determined accurately?

Let us consider the characteristic Lyapunov exponent as an order parameter which will differ in the two phases. Figure 7a shows this order parameter at negative temperatures and Figure 7b shows at positive temperatures calculated for different sizes of the system. The curves on Figure 7b converge towards a line and result in a first order transition at positive $\beta_c \approx 0.69$, whereas in a class of maps close to $x \to 4x(1-x)$ a phase transitions of first order occur at negative β_c. We would like to mention that it is hard to determine the critical "temperature" of phase transition by numerical methods with high precision. Consequently, the obtained value of critical "temperature" is approximate.

To consider the above results in terms of the entropy function $S(\lambda)$, let us first discuss the general appearance of the entropy function. First of all, it should be positive on some interval $[\lambda_{min}, \lambda_{max}]$. The value $\lambda = ln2$ must belong to that interval, which follows from the fact that the sum of the lengths of all cylinders on a given level is 1. Secondly it is often found that the values of λ_{min} and λ_{max} are given by the logarithms of the slopes at the origin . In our case $\lambda_{min} = 0$ $(f'(x_0 = c_0^2) = 1)$ and $\lambda_{max} = ln2.68$ $(|f'(x_1)| = 2.68)$. Consequently, the slope at the fixed point away from x_0 is larger then the slope at x_0. It also indicates the quite different behavior of the TSAI system then in the cases of a class of maps close to $x \to 4x(1-x)$ where another phase transition exists.

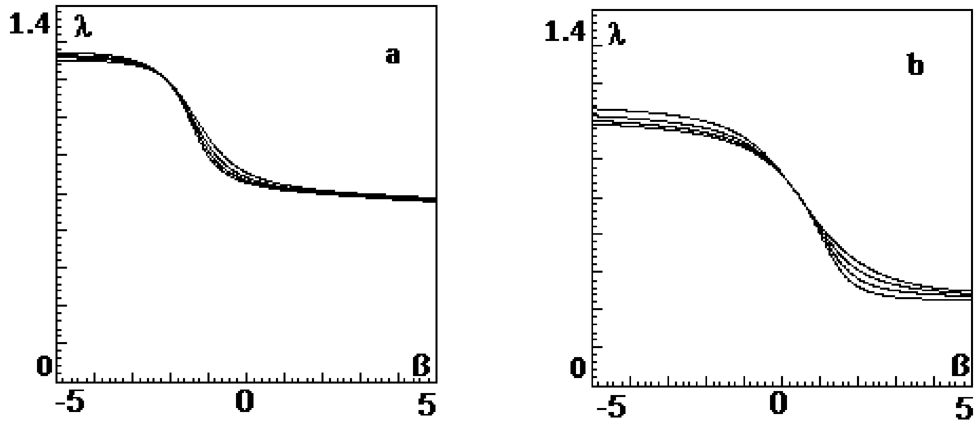

Figure 7. The order parameter $\lambda(\beta)$ calculated for different sizes of the system, corresponding to $n = 9, 11, 13$ and different μ (6). (a) μ=5; (b) μ=105.

The precise form of the entropy function is, as mentioned in the introduction, not easy to obtain with great accuracy. The existence of a first order phase transition implies that there should be a straight line segment in $S(\lambda)$. Of course, with finite-size data, it is impossible determine the straight line segment in $S(\lambda)$. In Figure 8 we present the curve of the entropy function, which corresponds to the size $n = 12$.

The same procedure can be applied for the Entropy in chaotic regimes for Q-state Potts model ($Q < 2$) on the Bethe lattice and for Z(2) gauge model on a generalized Bethe lattice of plaquettes.

4. Dense Mandelbrot Set of Fishers Zeroes

We investigate Fisher's zeroes of partition function of TSAI model on Husimi lattice in the presence of magnetic field and show that at certain values of the magnetic field these singularities lie at a dense set.

The phase transition point of three-site interacting Ising model in the complex plane may be obtained from

$$\mu x_n^{\gamma} + 1 = 0 \tag{20}$$

which expresses the equality of partition function to zero. Thus, in the thermodynamic limit this condition can be checked on the attractor of the complex map. In nonanlytical points this condition gives an unconventional behavior of the magnetization (m). We point out that for real values of parameters the partition function is not equal to zero and the magnetization is not equal to infinity, but if zeroes of partition function touch the real axis at some points one can conclude that these are the phase transition points (nonanalytical points of the free energy).

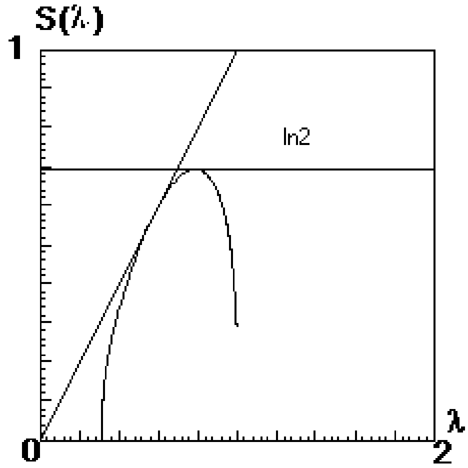

Figure 8. The entropy function corresponding to $n = 12$.

We verify Eq. (20) on the attractor of the map. As an initial condition for the map we take $x_0 = 1$ (free boundary condition) and for investigation of the attractor's behavior we take $n = 10^5\tilde{}10^6$ iterations. Thus we require (20) for at least one x_n of the attractor. The resulting diagrams are shown in Figure 9. Our results are stable with respect to the variation of n. In this picture we draw only the upper part of the complex plane, because the partition function of three-site interacting Ising model has $T \longrightarrow e^{i\pi}T$ symmetry. One can see that partition function zeros lie on a fractal set. The dense region clearly indicates the phase transitions condensation. This region disappears at sufficiently high and low external magnetic fields h. The frustration of the three-site interaction on a triangle is the main reason of such condensation. Note that the phase structure of the root site magnetization are slightly different because Husimi tree is not translation invariant.

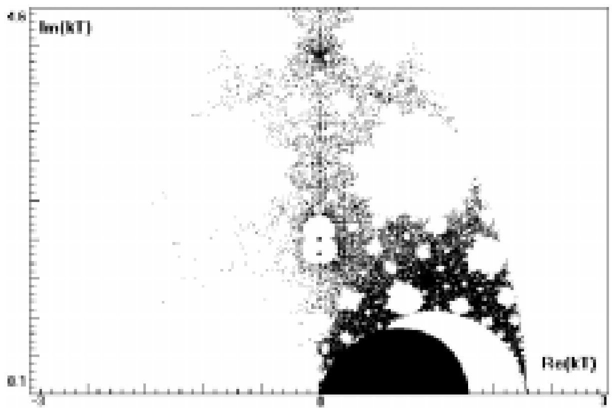

Figure 9. Complex temperature phase diagram (h=3, J=-1, γ=4).

5. Conclusion

The advantage of statistical mechanics over many other fields of science is that the number of degrees of freedom is usually enormously large. This means that the law of large numbers singles out certain members of the ensemble as typical for given external conditions, and their properties are then described by the thermodynamical functions which contain most of the relevant information about macroscopic systems. There has been considerable interest in the thermodynamical description of fractals. A fractal is an object having structure on all length scales described by power laws with noninteger exponents.

The most important objects in studying fractals are the Julia and the dense Mandelbrot sets in the complex plane. A challenge is to find out the Fisher's and Yang-Lee zeroes for multisite antiferromagnetic Ising model and non-integer valued Potts model which can be obtained using recurrence relations for Bethe-like hierarchical lattices.

In the past year, for instance, conformal invariance methods have been extended to arbitrary mixed copolymers which are an example of conformal multifractality [8]. The great advantage of such work is that non-perturbative predictions for the spectrum of multifractal exponents are obtained from consideration of a quenched limit in the formulae determining the weights of scaling operators when coupled to 2D quantum gravity.

In short, the real challenge is to connect conformal or superconformal multifractality with real statistical systems, e.g. with the non-integer valued Potts model connected with branched polymers or the quenched random connectivity models.

Another interesting problem involving multifractal behaviour concerns the starting conditions for which synchronisation can be achieved in dynamical systems. Using the scaling matrix method the critical behaviour of period-doubling bifurcation in two coupled 1D maps had been studied and a new critical behaviour was found at each critical line segment in the linear-coupled case. This coupled systems are described by more than two Feigenbaum exponents.

Inspired by the scaling properties of multifractals Tsallis has proposed a generalized nonextensive form of entropy [9]

$$S_q = k \frac{1 - \sum_i p_i^q}{q - 1} \tag{21}$$

which recovers the usual Boltzmann-Gibbs entropy in the limit $q \to 1$. A number of investigations based on Tsallis thermostatistics are made by the Ege University group [10]. It is of some interest to derive the recurrence relations based on Tsallis grand partition function and investigate the corresponding thermodynamics of multisite interaction Ising, Potts and gauge models.

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