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The Radius Cutoff for Relativistic Clusters

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Abstract

In this work we consider an isothermic relativistic kinetic gas under relativistic conditions, as a model for high center density galaxies. We apply the radius cutoff of the Emden type, in order to remove infinities. Numerically we find some theoretical configurations with very high values of masses concentrated in small volumes. Profiles of some thermodynamic quantities such as density, presure, will be independent from the choice of radius cutoff, whereas temperature and gravitational potential depend essentially on the cutoff value.

1. Introduction

Many galaxies (like NGC4258) are thought to contain massive black holes-exceeding ten milion solar measses-at their centres, and the best evidence comes from obssserving stars rotating rapidly within a small region around a central body (see for example [1-4]). These systems of high velocity stars raise once more the well-known idea of treating them as configurations of kinetic gas. In this work we consider a self-gravitating sphere of relativistic kinetic gas, using general relativistic theory, with the aim of answering some important questions of astrophysics, such as:

It is possible to have obwects of high density? Of very high values of energy?

Is it possible to construct a very massive object (ex. $10^{8-10}M_{sun}$) in a very small volume (ex. R=0.1 pc) ? etc.

In fact, it is well known that the classic types of these spheres are not limited. The classic equation describing these configurations, called the Emden equation [5], is

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi}{d\eta}\right) = e^{-\psi}. \tag{1}$$

Here, $r \equiv \left(\frac{kT}{4\pi\rho_0 mG}\right)$ is the distance from the center and $V \equiv \frac{kT}{m}\psi$ is the classic gravitational potential (T being temperature; ρ_0 mass density at center and m, the particle mass). This equation gives a mass density of the form $\rho \sim \eta^{-2}$ for large η. This causes the total mass integrated over large distances to become infinite. To resolve this difficulty, two methodes can be employed: a cutoff for energy values greater than the cutoff energy; or confine the self-gravitating sphere in a perfectly reflecting spherical box, which could have every possible radius.

The first method has recently been used, with the truncated Boltzmann distribution function by the cutoff energy. Here, the stability against the relativistic collapse is investigated as well [6-7]. The second method was used very early by Emden, in the case of classical configurations and is known by the name of Emden sphere. In our work we treat the analoguous relativistic Emden sphere.

2. Heuristic Treatment

Let us supose that our relativistic gas composed by N stars is confined inside a self-gravitating sphere of mass M and radius $R : M \sim R, (c = G = 1)$. Since these stars are relativistic (mostly neutron stars and black holes), we have $m \sim r$ (respectively their mass and radius). Then $\frac{R^3}{N}$ is the free volume for each particle (star) and $d \sim \frac{R}{N^{1/3}}$ is the mean distance between these particles.

Next, we require the condition $d >> r$ to be accomplished in such a way that the gravitational attraction between the constituants be weak and the risk of their collision and fusion be negligible. It means that :

$$d \sim \frac{R}{N^{1/3}} \sim \frac{M}{N^{1/3}} \sim \frac{Nm}{N^{1/3}} \sim N^{2/3} r. \tag{2}$$

So, the condition $\frac{d}{r} >> 1$ is fulfilled if $N^{2/3} >> 1$ and $N >> 1$.

We advance in our questions and ask the approximate number of stars (constituents) for having such rare collisions that the configuration survive 10^{10} years $\sim 10^{17}$ s. Let us calculate the free mean distance, by considering relativistic velocities $v \sim 1$:

$$l \sim \frac{1}{\pi n r^2} \sim \frac{1}{\pi \frac{N}{R^3} r^2} \sim \frac{R^3}{N r^2} \sim \frac{M^3}{N m^2} \sim \frac{N^3 m^3}{N m^2} \sim N^2 m. \tag{3}$$

The average time between two collisions will be

$$\tau \sim \frac{l}{1} \sim N^2 m. \tag{4}$$

Then $\tau \sim 3 \times 10^{17}$ s and $m \sim 10^{33}$ kg $\sim 10^{-5}\mathrm{s}^{-1}$. Consequently, the last relation gives $N^2 \sim 10^{22}$ and $N \sim 10^{11}$. So, normal-mass galaxies $M \sim 10^{11} M_{sun}$ with lower limited radius $R \sim M \sim 10^{-2}$ parces are obtained.

3. General Considerations

The general relativistic equations for static and spherically symetric configurations can be writen as follows $(c = G = 1)$:

$$8\pi p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}$$
$$8\pi\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{5}$$

where the metric is :

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{6}$$

We consider a simple perfect relativistic gas of material particles with proper mass m at temperature T. So, the velocities of particles are considered very high and the gas statistics follow the Jutner-Synge distribution [8]:

$$N = \frac{n}{4\pi m^2 K_2\left(\frac{m}{kT}\right)kT}\exp\left(\frac{M_l\lambda_l}{kT}\right). \tag{7}$$

M_l and λ_l are the fourvectors characterizing energitic state of the system and n is the numerical density of particles with mass m in the rest frame of the gas. K_2 is the Bessel function of the second order. For the Bessel function $K_n(x)$ we have the following formulae:

$$K_n(x) = \int_0^{\infty}\exp(-x\cosh t)\cosh ntdt \tag{8}$$

$$xK_n'(x) + nK_n(x) = -xK_{n-1}(x). \tag{9}$$

Using the above distribution function, the state of the relativistic perfect gas is represented by the equations (see [8]):

$$p = nkT$$
$$\rho + p = mn\left(\frac{2kT}{m} - \frac{K_2'\left(\frac{m}{kT}\right)}{K_2\left(\frac{m}{kT}\right)}\right)$$
$$n = 4\pi C_1 m^2 kTK_2\left(\frac{m}{kT}\right), \tag{10}$$

where C_1 is a constant.

In the case of absence of the gravitational field, T is the temperature measured by the observer in the rest frame. So, it would have same value measured every where. In the case of the gravitational field, the thermodynamic equilibrium will be realised by a varied temperature, dependent on the gravitational field. It has been found that the temperature T measured by the local observer fulfills the relation $T\sqrt{g_{00}} = \text{const}$(see[9]), which in our case could be written:

$$Te^{\nu/2} = const. \tag{11}$$

Noting $a \equiv \frac{m}{kconst}$, the state equation is transformed to:

$$\begin{aligned} n(r) &= 4\pi m^3 C_1 \frac{K_2(ae^{\nu/2})}{ae^{\nu/2}} \\ p(r) &= 4\pi m^4 C_1 \frac{K_2(ae^{\nu/2})}{(ae^{\nu/2})^2} \\ \rho(r) &= 4\pi m^4 C_1 \frac{K_2(ae^{\nu/2})}{ae^{\nu/2}} \left[\frac{3}{ae^{\nu/2}} + \frac{1}{\frac{K_0}{K_1} + \frac{2}{ae^{\nu/2}}} \right]. \end{aligned} \tag{12}$$

4. Reduction of Parameters

Let us make the following transformation:

$$U(r) \equiv ae^{\nu/2}. \tag{13}$$

The system of differential equations for the field λ and ν takes the form:

$$\begin{aligned} \frac{d\lambda}{dr} &= r \left[e^{\lambda} \left(8\pi\rho - \frac{1}{r^2} \right) + \frac{1}{r^2} \right] \\ \frac{dU}{dr} &= \frac{Ur}{2} \left[e^{\lambda} \left(8\pi p + \frac{1}{r^2} \right) - \frac{1}{r^2} \right], \end{aligned} \tag{14}$$

where

$$\begin{aligned} n(r) &= 4\pi m^3 C_1 \frac{K_2(U)}{U} \\ p(r) &= 4\pi m^4 C_1 \frac{K_2(U)}{U^2} \\ \rho(r) &= 4\pi m^4 C_1 \frac{K_2(U)}{U} \left[\frac{3}{U} + \frac{1}{\frac{K_0}{K_1} + \frac{2}{U}} \right] \\ &= 4\pi m^4 C_1 \left[\frac{3K_0}{U^2} + \frac{6K_1}{U^3} + \frac{K_1}{U} \right]. \end{aligned} \tag{15}$$

Applying the scaling $r = \frac{1}{\sqrt{32\pi^2 m^4 C_1}}\chi$, the system to be resolved becomes:

$$\begin{aligned} \frac{d\bar{\lambda}}{d\chi} &= \chi \left[e^{\bar{\lambda}} \left(\bar{\rho} - \frac{1}{\chi^2} \right) + \frac{1}{\chi^2} \right] \\ \frac{d\bar{U}}{d\chi} &= \frac{\bar{U}\chi}{2} \left[e^{\bar{\lambda}} \left(\bar{p} + \frac{1}{\chi^2} \right) - \frac{1}{\chi^2} \right] \end{aligned} \tag{16}$$

with

$$\bar{p} = \frac{K_0}{U^2} + \frac{2K_1}{U^3}$$
$$\bar{\rho} = \left[\frac{3K_0}{U^2} + \frac{6K_1}{U^3} + \frac{K_1}{U}\right]. \quad (17)$$

It is clear that $\bar{\lambda}(\chi) = \lambda(\chi/\sqrt{32\pi^2 m^4 C_1}), \bar{U}(\chi) = U(\chi/\sqrt{32\pi^2 m^4 C_1}), \bar{p}(\chi) = p(\chi/\sqrt{32\pi^2 m^4 C_1})$ $\bar{\rho}(\chi) = \rho(\chi/\sqrt{32\pi^2 m^4 C_1})$. Bessel functions were used to arrive at the final forms.

5. The boundary conditions

At center

To resolve numerically the equations (16) and (17) with the Runge-Kuta method, we must determine the boundary conditions at center of the spherical configuration : $U_0, \bar{p}_0, \bar{\rho}_0, \bar{\lambda}_0$. Due to the fact that in origin the metrics is locally flat, we have $\bar{\lambda}_0 = \lambda_0 = 0$. Equations (17) show that if we know U_0, then $\bar{p}_0$ and $\bar{\rho}_0$ are determined. Consequently the only necessary condition at origin for resolving equations (16) and (17) is $U_0 = \frac{m}{KT_0}$.

At the surface

We consider the surface at R, such that the condition of the Emden sphere $\nu(R) = -\lambda(R)$ be fulfilled. So, $ae^{\nu(R)/2} = ae^{-\lambda(R)/2}, U(R) = ae^{-\lambda(R)/2}$; $a = a(R) = U(R)e^{\lambda(R)/2}$. We conclude that R could determine only the constant a. In the configuration (16), (17) the constant a has a no influence and R can not play any role in the profile $U(\chi)$ and $\lambda(\chi)$ (see Fig. 1., Fig. 2).

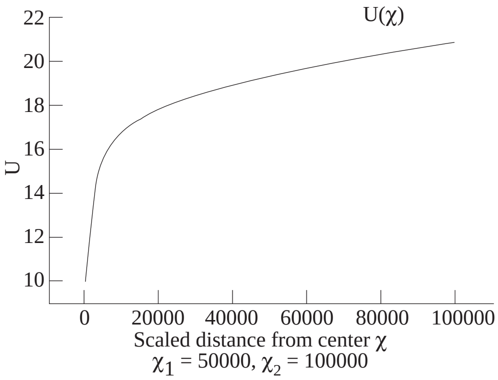

Figure 1. The profile $U(\chi)$ for condition at center $U(0) = 10$. The numerical calculation stops at the arbitrary values χ_1 and χ_2.

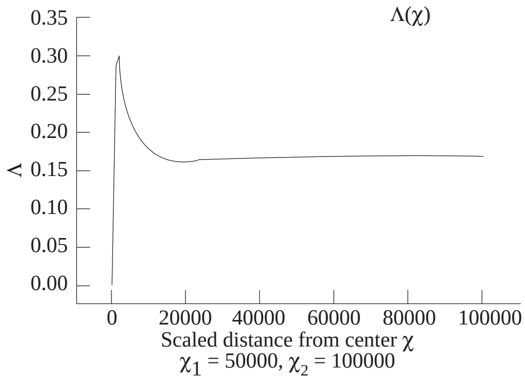

Figure 2. The profile $\lambda(\chi)$ for condition at center $U(0) = 10$. The numerical calculation stops at the arbitrary values χ_1 and χ_2.

Then, for resolving (16), (17), let us determine U_0 and χ_1 (the surface). U_0 is the only one parameter conditioning the profiles $\bar{U}(\chi), \bar{\lambda}(\chi)$ and afterwards $\bar{p}(\chi), \bar{\rho}(\chi)$ with the help of Eqs. (17). These profiles could not depend on the choice of χ_1; the last is needed only to stop the integration. If we first choose χ_1 and afterwards carry out another integration with $\chi_2 > \chi_1$, in the second case we obtain the same profile for $0 < \chi < \chi_1$ as in the first case. We do not find the same conclusion for the value of a and ν_0, which will depend directly on χ_1:

$$a = a(R) = U(R)e^{\lambda(R)/2} = \bar{U}(\chi_1)e^{\lambda(\bar{\chi}_1)/2}. \tag{18}$$

Knowing that

$$U_0 = ae^{\nu_0/2} = \bar{U}(\chi_1)e^{\lambda(\bar{\chi}_1)/2}e^{\nu_0/2}, \tag{19}$$

then

$$\nu_0 = 2\log\frac{U_0}{\bar{U}(\chi_1} - \bar{\lambda}(\chi_1) = \nu(T_0, R\sqrt{32\pi^2 m^4 C_1}) = \nu_0(T_0, R, p_0). \tag{20}$$

The last set of equations is due to the fact that $4\pi^2 m^2 C_1 = \frac{p_0}{\bar{p}_0}$. We made also use of the relation : $\chi_1 = \sqrt{32\pi^2 m^4 C_1}R$. Knowing that the temperature is included in the a value, it is also dependent on R.

6. Passing to r

When we pass to r, we need to determine two center conditions: U_0, p_0 (or ρ_0) and the surface R. Afterwards, with the value U_0 we determine profiles $\bar{U}(\chi), \bar{\lambda}(\chi), \bar{T}(\chi), \bar{p}(\chi), \bar{\rho}(\chi)$. We stop the integration at $\chi_1 = R\sqrt{8\pi\frac{p_0}{\bar{p}_0}}$. Finally we have $U(r) = \bar{U}(r\sqrt{8\pi\frac{p_0}{\bar{p}_0}})$, $\lambda(r) = \bar{\lambda}(r\sqrt{8\pi\frac{p_0}{\bar{p}_0}})$ etc., as dilatations of $U(\chi), \lambda(\chi)$ etc.

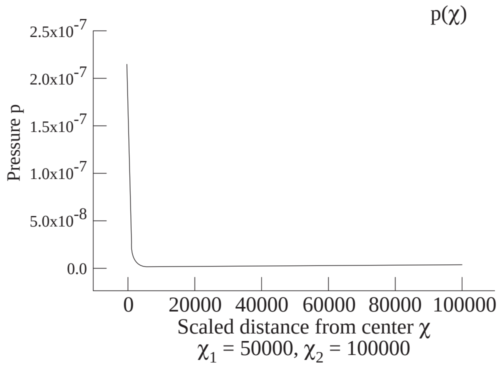

Figure 3. The profile $p(\chi)$ for condition at center $U(0) = 10$. The numerical calculation stops at the arbitrary values χ_1 and χ_2.

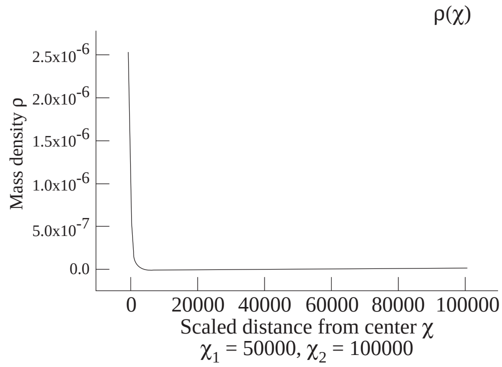

Figure 4. The profile $\rho(\chi)$ for condition at center $U(0) = 10$. The numerical calculation stops at the arbitrary values χ_1 and χ_2.

The numerical integration is done using the Runge-Kuta method with the boundary conditions: $U_0 = 10, R$ of the order of 0.1 parces, $\rho_0 = 10^3 g/cm^3$ to find $M = 10^{11} M_{sun}$.

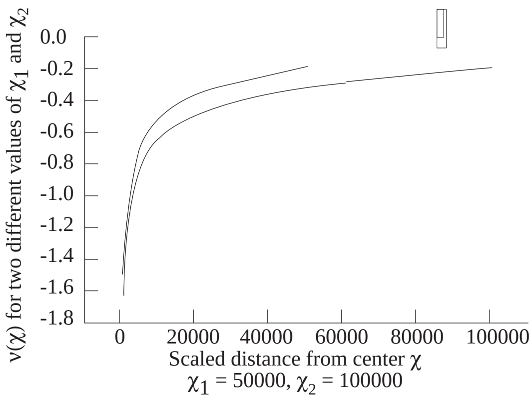

Figure 5. The profile $\nu(\chi)$ for two different boundary values $\chi_1 \neq \chi_2$. We remark the dependence of the profile from the surface $R = \frac{1}{\sqrt{32\pi^2 m^4 C_1}}\chi_1$.

7. Conclusions and Acknowledgements

We considered galaxies as configurations of relativistic kinetic gas, selfgravitating in their own strong gravitational fields. We made use of the radius cutoff for removing the infinities. We found that the profile of some interior quantities characterizing the gas can be reduced through scaling, in one sole model. We find masses of $\sim 10^{8-10} M_{sun}$ confined in a sphere of 0.1 parces.

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