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Topological Unitarity Identities in the Theory of Scalar Charged Particles Interacting Through a Pure C-S Gauge Field

Müge BOZ
Physics Department, Hacettepe University
06532 Ankara-TURKEY
Vladimir Ya. FAINBERG
P. N Lebedev Institute of Physics
Moscow-RUSSIA

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Abstract

Starting from the theory of relativistic scalar charged particles interacting via C-S field, it is verified that the gauge invariant sum of the imaginary parts of all Feynman diagrams with C-S photon lines on the mass shell vanish.

1. Introduction

In this work, we consider the theory of relativistic scalar charged particles interacting via Chern-Simons (C-S) gauge field. It is well-known that in this model there are no solutions for the classical free A_μ field which implies that real free particles of the gauge field do not exist. In fact, this is one of the reasons why the C-S field is called as topological field. This would mean that the complete set of vector physical states in the total Hilbert space of the system does not contain these topological particles, which in turn leads to very important topological unitarity identities requiring that the gauge invariant sum of imaginary parts of all Feynman diagrams with on-shell internal gauge field lines vanishes. Verification of this argument, for the case of scattering of two oppositely charged particles, is the main aim of this work.

This paper is organized as follows: We construct the S-matrix for the relativistic scalar Chern-Simons theory and deduce the unitarity identities in section 2. Then, in section 3 we verify that for the scattering of two oppositely charged particles, the gauge invariant sum of the imaginary parts of all Feynman diagrams with Chern-Simons internal photon lines on the mass shell is equal to zero. We discuss our results in Section 4.

2. S-Matrix for the relativistic scalar Chern-Simons Theory

The classical action of the systems interacting with Chern-Simons fields in α-gauge is:

$$
\begin{aligned}
S &= S_{\mathrm{CS}} + S_g + S_m \\
S_{CS} &= \int d^3x(\frac{1}{4}\varepsilon_{\mu\nu\lambda}F^{\mu\nu}A^{\lambda}) \\
S_g &= \frac{1}{2\alpha}\int d^3x(\partial_\mu A^\mu)^2 \\
S_m &= \int d^3x[(D^\mu\varphi)^*(D_\mu\varphi) - m^2(\varphi^*\varphi) + \lambda(\varphi^*\varphi)^2],
\end{aligned}
\tag{1}
$$

where $D_\mu = \partial_\mu - ieA_\mu(x)$ and λ is the constant of self-interaction. We choose the metric as $\eta_{\mu\nu} = (1, -1, -1)$.

The S-matrix operator of the theory of scalar charged particles coupled to a C-S gauge field has been constructed in [1]; and shown to be the same as the S-matrix of the relativistic scalar QED in which the Feynman diagrams with external photon lines are not considered and the propagators of C-S particles are substituted in place of the ones for photons.

Let us briefly summarize the results of [1] for completeness:

$$
\widehat{S} = Texp[i\widehat{S}_{int}(\widehat{\varphi}, \widehat{\varphi}^*, A)]. \tag{2}
$$

Here; $\widehat{S}_{int}(\widehat{\varphi}, \widehat{\varphi}, A) = \int : [igA_\mu(x)(\widehat{\varphi}\partial_\mu\widehat{\varphi}^* - \partial^\mu\widehat{\varphi}\widehat{\varphi}^*) + g^2 {A_\mu}^2\varphi^*\varphi + \lambda(\varphi^*\varphi)^2] : d^3x$, where $\widehat{S}$ is the S matrix operator and $\widehat{\varphi}$, $\widehat{\varphi}^*$ are the operators for the free fields φ and φ^* in the interaction representation and the following decomposition is chosen:

$$
\begin{aligned}
\widehat{\varphi} &= \frac{1}{2\pi}\int \frac{d^2p}{\sqrt{2E_p}}[a(p)e^{ipx} + b^+e^{-ipx}] \\
\widehat{\varphi}^* &= \frac{1}{2\pi}\int \frac{d^2p}{\sqrt{2E_p}}[a^+(p)e^{-ipx} + b(p)e^{ipx}],
\end{aligned}
\tag{3}
$$

where $\mathrm{p}_0 = E_p = \sqrt{\overrightarrow{p}^2 + m^2}$ a, b and a^+, b^+ are the annihilation and the creation operators with the usual commutation relations. As was mentioned above, the propagator of the C-S photon exists, although the introduction of the operators A_μ(x) for the free field C-S field is possible only in case of gauge $\alpha \neq 0^*$. Therefore, one can bring into use such operators $\hat{A}_\mu$(x) in interaction representation with the property that only the vacuum expectation values from the T-product of even number of operators A_μ(x) are

*See [3] for the more detailed analysis of quantization of Chern-Simons field for arbitrary gauge.

non-vanishing and reduce to the product of the vacuum expectation values of the T-product of two operators A_μ(x). In principle, the situation here is the same as in QED in (2+1) dimensions, where the contributions of scalar and longitudinal photons compensate each other in unitarity conditions.

It is well-known that the $\widehat{S}$-matrix operator is unitary in physical subspace and the scattering amplitude operator $\widehat{T}$ is defined by the expression:

$$\widehat{S} = 1 - \widehat{T} \tag{4}$$

(where the δ function has been suppressed). It is shown in [2] that for arbitrary non-diagonal ($|\beta\rangle \neq |\alpha\rangle$) matrix elements, using the unitarity property of the $\widehat{S}$-matrix one can write that:

$$\langle\beta|2Im\widehat{T}|\alpha\rangle = \langle\beta|\widehat{T}\widehat{T}^{+}|\alpha\rangle \tag{5}$$

$$\langle\beta|2Im\widehat{T}|\alpha\rangle = \sum_{\gamma}\langle\beta|\widehat{T}|\gamma\rangle\langle\gamma|\widehat{T}^{+}|\alpha\rangle, \tag{6}$$

where $\{|\gamma\rangle\}$ is the complete set of states of the charged particles and does not contain the states of topological photon. In this case, we can include in total Hilbert space also the states of topological photons which, however, do not contribute to unitarity conditions. The imaginary part of the photon propagator is non-zero due to the fact that the vacuum expectation values of the T-product of even number of operators is non-vanishing as mentioned above. For this reason, in the framework of the perturbation theory (5) tells us that the diagrams with internal on shell photon lines appear. On the other hand, since $\{|\gamma\rangle\}$ are physical states, in a given order of perturbation theory the Feynman diagrams contributing to the imaginary part of (6) can not have internal on-shell topological photon lines. Then, consistency of these two equations (5) and (6) requires that the gauge invariant sum of the imaginary parts of all Feynman diagrams, with internal on shell free photon lines vanishes. We shall verify this general argument of the theory for the case of scattering of two oppositely charged particles in one loop order.

3. Scattering of two oppositely charged particles in one loop order

In the scattering of two oppositely charged particles, there are 3 creation and 3 annihilation diagrams as shown in (Fig1.)

For the annihilation diagrams, the expression for $A_{\mu\nu}$ (p,q) can be obtain as:

$$A_{\mu\nu}(p,q) = \frac{(2p-k)^{\mu}(q-p+k)^{\nu}}{(p-k)^2-m^2} + \frac{(q-p+k')^{\mu}(2p-k')^{\nu}}{(p-k')^2-m^2} + 2\eta^{\mu\nu} \tag{7}$$

where $p+q=k+k'$ and $k^{\mu}A_{\mu\nu}=A_{\mu\nu}k'^{\nu}=0$

For the creation diagrams, one can get the expression for $A_{\lambda\sigma}(p'q')$ as

$$A_{\lambda\sigma}(p',q')=\frac{(-k+2p')^{\lambda}(q'-p'+k)^{\sigma}}{(k-p')^2-m^2}+\frac{(-k'+2p')^{\sigma}(q'-p'+k')^{\lambda}}{(k-q')^2-m^2}+2\eta^{\lambda\sigma}, \quad (8)$$

where $p'+q'=k+k'$ and $k^{\lambda}A_{\sigma\lambda}=0$

From the combination of these 3 creation and 3 annihilation diagrams, one can obtain 9 diagrams which are shown in (Fig 2.) First we will calculate the imaginary parts of the diagrams (2f), (2g), (2h), (2j):

The analytic expression for the imaginary parts of the diagrams (f) and (g) is (suppressing the overall constant):

$$\begin{aligned} ImA_f+ImA_g &= 2\int d^3kA_{\mu\nu}(p,q)\varepsilon_{\mu\lambda\rho}k^{\rho}\delta^{+}(k^2)A_{\lambda\sigma}(p'q')\varepsilon_{\nu\sigma\delta}k'^{\delta}\delta^{+}(k'^2) \\ &= 2^5\int d^3k\frac{\varepsilon_{\rho\mu\lambda}k^{\rho}p^{\mu}p'^{\lambda}\varepsilon_{\delta\nu\sigma}k'^{\delta}q^{\nu}q'^{\sigma}\delta^{+}(k^2)\delta^{+}(k'^2)}{[(p-k)^2-m^2][(p'-k)^2-m^2]}. \end{aligned} \quad (9)$$

Here $\delta^{+}(k^2)=\delta(k)\theta(k_0)$. Changing the variable $p-k\to k$, we can rewrite (9) as:

$$ImA_f+ImA_g=-2^5\int d^3k\frac{\varepsilon_{\rho\mu\lambda}k^{\rho}p^{\mu}p'^{\lambda}\varepsilon_{\delta\nu\sigma}k^{\delta}q^{\nu}q'^{\sigma}\delta^{+}(k+p)^2\delta^{+}(q-k)^2}{[(k^2-m^2)][(p'-p-k)^2-m^2]}. \quad (10)$$

Following the usual manuplations, and performing the dk_0 integration, we can write:

$$ImA_f+ImA_g=\frac{2^5p_0^2}{2P_0}\int d^2k\frac{\delta(p_0^2-(\vec{p}+\vec{k})^2)(\vec{k}\times\vec{p}-\vec{p'})^2}{[(k^2-m^2)][(p+k-p')^2-m^2]}, \quad (11)$$

where $P_0=p_0+q_0=2p_0$. Carrying out the integral, and after some lengthy calculations, we obtain:

$$ImA_f+ImA_g=4\pi(1-\cos\theta)(-m+E), \quad (12)$$

where θ is the scattering angle in the center of mass frame

The imaginary parts of the diagrams of (h) and (j) is:

$$\begin{aligned} ImA_h+ImA_j &= 2\int\frac{d^3k\varepsilon_{\mu\lambda\rho}k^{\rho}(q-p+k')^{\mu}(2p-k')^{\nu}\varepsilon_{\nu\sigma\delta}k'^{\delta}(-k+2p')^{\lambda}(q'-p'+k)^{\sigma}\delta^{+}(k^2)\delta^{+}(k'^2)}{(p-k')^2-m^2\}\{(k-p')^2-m^2} \\ &= 2^5\int d^3k\frac{\varepsilon_{\mu\lambda\rho}k^{\rho}q^{\mu}p'^{\lambda}\varepsilon_{\nu\sigma\delta}k'^{\delta}p^{\nu}q'^{\sigma}\delta^{+}(k^2)\delta^{+}(k'^2)}{\{(p-k')^2-m^2\}\{(k-p')^2-m^2\}}. \end{aligned} \quad (13)$$

Changing the variable $p - k' \to k$, we get:

$$ImA_h + ImA_j = -2^5 \int d^3k \frac{\varepsilon_{\mu\lambda\rho} k^\rho q^\mu p'^\lambda \varepsilon_{\nu\sigma\delta} k^\delta p^\nu q'^\sigma \delta^+((k+q)^2)\delta^+((p-k)^2)}{[k^2 - m^2][(p' - q - k)^2 - m^2]}. \quad (14)$$

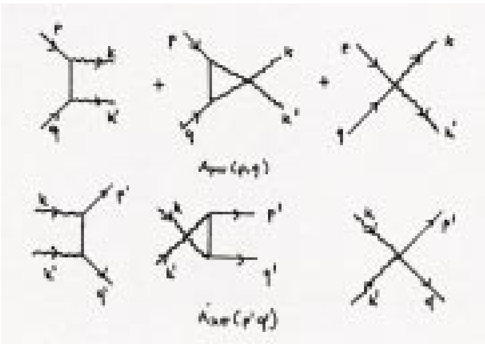

Figure 1. Annihilation and Creation Diagrams

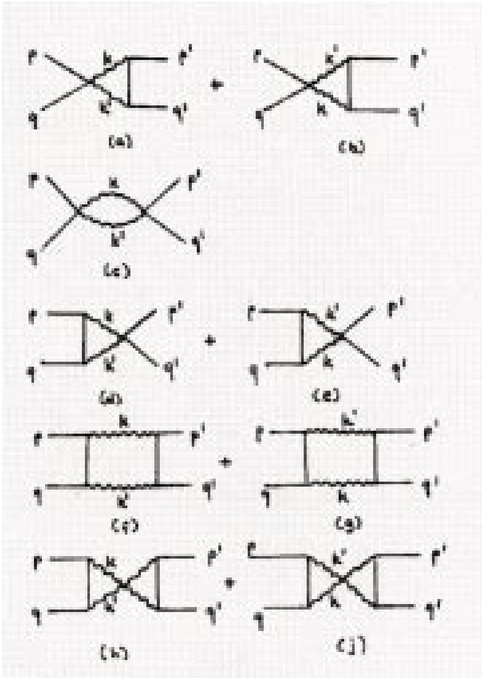

Figure 2. One-Loop diagrams

We note that (14) is the same as (10) with the substition $p \leftrightarrow q$. Following the similar manuplations, we obtain the result for this:

$$ImA_h + ImA_j = 4\pi(1 + \cos\theta)(-m + E). \quad (15)$$

Now, we treat the diagrams (2d) and (2e)

$$
\begin{aligned}
ImA_d + ImA_e &= 2\int d^3k \frac{2\eta^{\lambda\sigma}\varepsilon_{\mu\lambda\rho}k^\rho(2p-k)^\mu\varepsilon_{\nu\sigma\delta}k'^\delta(2q-k')^\nu\delta^+(k^2)\delta^+(k'^2)}{(p-k)^2-m^2} \\
&= 2^4\int d^3k \frac{\{(kk')(pq)-(kq)(pk')\}\delta^+(k^2)\delta^+(k'^2)}{(p-k)^2-m^2}.
\end{aligned}
\tag{16}
$$

The result of this integration is:

$$
ImA_d + ImA_e = 2(2m-3E)\pi. \tag{17}
$$

Next, for the diagram (2c) we obtain:

$$
\begin{aligned}
ImA_c &= \int d^3k 2\eta^{\mu\nu}\varepsilon_{\mu\lambda\rho}k^\rho 2\eta^{\lambda\sigma}\varepsilon_{\nu\sigma\delta}k'^\delta\delta^+(k^2)\delta^+(k'^2) \\
&= 2^3\int d^3k(kk')\delta^+(k^2)\delta^+(k'^2) = 4\pi E
\end{aligned}
\tag{18}
$$

Finally, we calculate the imaginary parts of the diagrams (2a) and (2b)

$$
\begin{aligned}
ImA_a + ImA_b &= 2^2\int d^3k \frac{\varepsilon_{\nu\lambda\rho}k^\rho\varepsilon_{\nu\sigma\delta}(-k+2p')^\lambda(2q'-k')^\sigma k'^\delta\delta^+(k^2)\delta^+(k'^2)}{(k-p')^2-m^2} \\
&= 2^4\int d^3k \frac{\{(kk')(p'q')-(kq')(p'k')\}\delta^+(k^2)\delta^+(k'^2)}{(p'-k)^2-m^2}
\end{aligned}
\tag{19}
$$

Here we note that this is the same expression with (16), only with the replacement $p \to p'$, $q \to q'$. We now have:

$$
ImA_a + ImA_b = 2(2m-3E)\pi \tag{20}
$$

Thus, using equations (12), (15), (17), (18) we obtain the sum of the imaginary parts of the diagrams, which vanish:

$$
ImA_{total} = 8\pi(-m+E) + 4\pi E + 4\pi(2m-3E) = 0. \tag{21}
$$

4. Conclusions

We proved that for relativistic scalar charged particles interacting via C-S gauge field, due to topological properties of the pure C-S photons, the gauge invariant sum of imaginary parts of all the Feynman diagrams with on-shell internal gauge field lines vanish. Each of these diagrams is not gauge invariant by itself; only their sum is. Moreover, if we consider the same diagrams in three different channels; namely s, t, u ($s = (p+q)^2$, $t = (p-p')^2$ and $u = (p-q')^2$), we see that all of them are the different boundary values of a single analytical function of the two invariant variables (for instance, s and t).

In the near future we hope to utilize dispersion approach and topological unitarity identities for the calculation of one-loop scattering amplitudes in C-S theory.

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