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Some String Cosmological Models in Cylindrically Symmetric Inhomogeneous Universe

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Abstract

The aim of this paper is to investigate the behavior of a string in the cylindrically symmetric inhomogeneous cosmological model. It has been assumed that expansion θ in the model is proportional to σ_1^1 of the eigenvalue of the shear tensor σ_ν^μ. Also, various physical and geometrical properties of the model have been discussed.

Key Words: String, Cosmological Models.

1. Introduction

In recent years, there has been considerable interest in string cosmology. Cosmic strings are topologically stable objects which might be formed during a phase transition in the early universe [1]. Moreover, the investigation of cosmic strings and physical processes near such strings has received wide attention because they are believed to give rise to density perturbations leading to the formation of galaxies [2].

Letelier has given the general solution to Einstein's field equations for a cloud of strings with spherical, plane and a particular case of cylindrical symmetry [3]. Then, in 1983, he solved Einstein's field equations for a cloud of massive strings and obtained cosmological models in Bianchi I and Kantowski-Sachs space-times [4]. Banerjee et al. have investigated Bianchi I string cosmological models with and without a source-free magnetic field [5]. Exact solutions of string cosmology for Bianchi type II, VI_0, VIII and IX space-times have been studied by Krori et al. [6]. Chakraborty has solved Einstein's field equations for Bianchi type VI_0 and IX space-times with string plus electromagnetic field and discussed string cosmology for various metrics [7,8]. Nevin has solved Einstein's field equations for spherical or static cylindrical symmetry with string dust source [9]. Bianchi type III string cosmological models with and without source-free magnetic field have been discussed by Tikekar and Patel [10]. Chakraborty and Chakraborty have solved field equations for the spherically symmetric homogeneous anisotropic space-time with string source [11]. Ram and Singh have solved Einstein's field equations for Bianchi VI_0 type space-time with string cloud [12]. Anisotropic cosmological model with string source has been obtained by Bhattacharya and Karade [13]. Krori et al. has studied cosmic strings in some Bianchi type cosmologies and noted that cosmic strings do not occur in Bianchi type V space-time [14]. Tikekar et al. have solved Einstein's field equations for cylindrically-symmetric space-time with string source [15]. Recently, Kılınç and Yavuz have been obtained cosmic string models for inhomogeneous cylindrically symmetric space-times [16]. Some string dust models for Bianchi type I space-time have been discussed in detail by Yavuz and Tarhan [17].

In this paper we consider cylindrically symmetric inhomogeneous space-times in the context of string cosmology and obtain some new exact solutions . Our solutions are different from the other authors' solutions.

2. Field Equations

We consider the metric in form

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2, \tag{1}$$

where A, B, and C are functions of x and t. The Einstein field equations for a cloud of strings are [4]

$$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu = -(\rho u^\mu u_\nu - \lambda x^\mu x_\nu). \tag{2}$$

Here, ρ is the rest energy of the cloud of strings with massive particles attached to them. $\rho = \rho_p + \lambda$, ρ_p being the rest energy density of particles attached to the strings and λ the density of tension that characterizes the strings. The unit time-like vector u^μ describes the four-velocity vector of the matter and the space-like vector x^μ represents the string direction in the cloud, i.e. the direction of anisotropy. We have

$$u^\mu u_\mu = -x^\mu x_\mu = -1, \quad u^\mu x_\mu = 0. \tag{3}$$

From (1), (2) and (3), in the comoving frame, we write

$$u^\mu = (0, 0, 0, A^{-1}) \tag{4}$$

and choose x^μ parallel to $\partial/\partial x$ so that

$$x^\mu = (A^{-1}, 0, 0, 0). \tag{5}$$

The Einstein field equations (2) for metric (1) are

$$G^1_1 \equiv \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{A'}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) + \frac{\dot{B}\dot{C}}{BC} - \frac{B'C'}{BC} = \lambda A^2, \tag{6}$$

$$G^2_2 \equiv \left(\frac{\dot{A}}{A}\right)^{\cdot} - \left(\frac{A'}{A}\right)' + \frac{\ddot{C}}{C} - \frac{C''}{C} = 0, \tag{7}$$

$$G^3_3 \equiv \left(\frac{\dot{A}}{A}\right)^{\cdot} - \left(\frac{A'}{A}\right)' + \frac{\ddot{B}}{B} - \frac{B''}{B} = 0, \tag{8}$$

$$G^4_4 \equiv -\frac{B''}{B} - \frac{C''}{C} + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{A'}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) + \frac{\dot{B}\dot{C}}{BC} - \frac{B'C'}{BC} = \rho A^2, \tag{9}$$

$$G^1_4 \equiv \frac{\dot{B}'}{B} + \frac{\dot{C}'}{C} - \frac{\dot{A}}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) - \frac{A'}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0. \tag{10}$$

where prime and dot denote differentiation with respect to x and t, respectively.

The rotation ω^2 is identically zero. The expansion θ, shear scalar σ^2, acceleration vector $\dot{u}_\mu$ and proper volume V^3 are respectively found to have the following expressions:

$$\theta = u^\mu_{;\mu} = \frac{1}{A}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{11}$$

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{3}\theta^2 - \frac{1}{A^2}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC}\right), \tag{12}$$

$$\dot{u}_\mu = u_{\mu;\nu}u^\nu = \left(\frac{A'}{A}, 0, 0, 0\right), \tag{13}$$

$$V^3 = \sqrt{-g} = A^2BC, \tag{14}$$

where g is the determinant of the metric (1). Using the field equations and the relations (11) and (12) one obtains the Raychaudhuri's equation as

$$\dot{\theta} = \dot{u}^\mu_{;\mu} - \frac{1}{3}\theta^2 - 2\sigma^2 - \frac{1}{2}\rho_p, \tag{15}$$

where

$$R_{\mu\nu}u^{\mu}u^{\nu} = \frac{1}{2}\rho_p. \tag{16}$$

With the help of Eqs. (1)-(5), the Bianchi identity ($T^{\mu\nu}_{;\nu} = 0$) reduced to two equations:

$$\dot{\rho} - \frac{\dot{A}}{A}\lambda + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\rho = 0 \tag{17}$$

and

$$\lambda' - \frac{A'}{A}\rho + \left(\frac{A'}{A} + \frac{B'}{B} + \frac{C'}{C}\right)\lambda = 0. \tag{18}$$

Thus due to all the three (strong, weak and dominant) energy conditions, one finds $\rho \geq 0$ and $\rho_p \geq 0$, together with the fact that the sign of λ is unrestricted, it may take values positive, negative or zero as well.

3. Solutions of The Field Equations

To get a determinate solution, let us assume that expansion (θ) in the model is proportional to the eigenvalue σ_1^1 of the shear tensor σ_ν^μ . This condition leads to

$$A = (BC)^n, \tag{19}$$

where n is a constant. Equations (7) and (8) lead to

$$\frac{\ddot{B}}{B} - \frac{B''}{B} = \frac{\ddot{C}}{C} - \frac{C''}{C}. \tag{20}$$

Using (19) in Eq. (10), yields

$$\frac{\dot{B}'}{B} + \frac{\dot{C}'}{C} - 2n\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\left(\frac{B'}{B} + \frac{C'}{C}\right) = 0. \tag{21}$$

There are three cases:
i) B = f(x) g(t) and C = h(x) k(t);
ii) B = f(x) g(t) and C = f(x) k(t);
iii) B = f(x) g(t) and C = h(x) g(t) which is discussed by Kılınç and Yavuz [16].

$$\textbf{case(i)} : B = f(x)g(t) \;\; and \;\; C = h(x)k(t). \tag{22}$$

From Equations (21) and (22) we have

$$\frac{f'/f}{h'/h} = -\frac{(2n-1)(\dot{k}/k) + 2n(\dot{g}/g)}{(2n-1)(\dot{g}/g) + 2n(\dot{k}/k)} = K(const.). \tag{23}$$

Equation (23) leads to

$$\frac{f'}{f} = K\frac{h'}{h} \tag{24}$$

and

$$\frac{\dot{k}/k}{\dot{g}/g} = \frac{K - 2nK - 2n}{2nK + 2n - 1} = a(const.). \qquad (25)$$

From Eqs. (24) and (25) we have

$$f = \alpha h^K \qquad (26)$$

and

$$k = \delta g^a, \qquad (27)$$

where α and δ are constants of integration. From Eqs. (20) and (22) we get

$$\frac{\ddot{g}}{g} - \frac{\ddot{k}}{k} = \frac{f''}{f} - \frac{h''}{h} = N(const.). \qquad (28)$$

Equation (28) leads to

$$\frac{\ddot{g}}{g} - \frac{\ddot{k}}{k} = N \qquad (29)$$

and

$$\frac{f''}{f} - \frac{h''}{h} = N. \qquad (30)$$

From Eqs. (25) and (29), we find

$$g\ddot{g} + a\dot{g}^2 = -\frac{N}{a-1}g^2 \qquad (31)$$

which leads to

$$g = \beta^{\frac{1}{a+1}} \sinh^{\frac{1}{a+1}} \left[b(t - t_o) \right], \qquad (32)$$

where

$$b = \sqrt{\frac{N(a+1)}{1-a}}.$$

β and t_o are constants of integration. Thus from Eq. (27) we get

$$k = \delta\beta^{\frac{a}{a+1}} \sinh^{\frac{a}{a+1}} \left[b(t - t_o) \right]. \qquad (33)$$

From Eqs. (24) and (30) we have

$$hh'' + Kh'^2 = \frac{N}{K-1}h^2 \qquad (34)$$

which leads to

$$h = \ell^{\frac{1}{K+1}} \sinh^{\frac{1}{K+1}} \left[r(x - x_o) \right], \qquad (35)$$

where

$$r = \sqrt{\frac{N(K+1)}{K-1}}$$

and ℓ and x_o are constants of integration. Thus from Eq. (26) we have

$$f = \alpha \ell^{\frac{K}{K+1}} \sinh^{\frac{K}{K+1}} \left[r(x - x_o) \right]. \tag{36}$$

Hence

$$B = fg = Q \sinh^{\frac{K}{K+1}} \left[r(x - x_o) \right] \sinh^{\frac{1}{a+1}} \left[b(t - t_o) \right] \tag{37}$$

and

$$C = hk = R \sinh^{\frac{1}{K+1}} \left[r(x - x_o) \right] \sinh^{\frac{a}{a+1}} \left[b(t - t_o) \right], \tag{38}$$

where

$$Q = \alpha \beta^{\frac{1}{a+1}} \ell^{\frac{K}{K+1}}$$

and

$$R = \delta \beta^{\frac{a}{a+1}} \ell^{\frac{1}{K+1}}.$$

Therefore,

$$A = (BC)^n = M \sinh^n \left[r(x - x_o) \right] \sinh^n \left[b(t - t_o) \right], \tag{39}$$

where $M = (QR)^n$.
After suitable transformation of coordinates metric (1) reduces to the form

$$\begin{aligned} ds^2 = M^2 \sinh^{2n}(rX) \sinh^{2n}(bT) \left(dX^2 - dT^2 \right) + Q^2 \sinh^{\frac{2K}{K+1}}(rX) \sinh^{\frac{2}{a+1}}(bT) dY^2 \\ + R^2 \sinh^{\frac{2}{K+1}}(rX) \sinh^{\frac{2a}{a+1}}(bT) dZ^2. \end{aligned} \tag{40}$$

In this case, the physical parameters, λ, ρ, ρ_p and the kinematical parameters θ, σ^2, $\dot{u}_\mu$ and V^3 for this model are, respectively, given by

$$\begin{aligned} \lambda = \frac{1}{M^2 \sinh^{2n}(bT) \sinh^{2n}(rX)} \Big\{ - b^2 \left[\frac{a}{(a+1)^2} + n \right] \coth^{2n}(bT) + b^2 \\ - r^2 \left[\frac{K}{(K+1)^2} + n \right] \coth^2(rX) \Big\}; \end{aligned} \tag{41}$$

$$\begin{aligned} \rho = \frac{1}{M^2 \sinh^{2n}(bT) \sinh^{2n}(rX)} \Big\{ b^2 \left[\frac{a}{(a+1)^2} + n \right] \coth^{2n}(bT) \\ + r^2 \left[\frac{K}{(K+1)^2} + n \right] \coth^2(rX) - r^2 \Big\}; \end{aligned} \tag{42}$$

$$\begin{aligned} \rho_p = \frac{1}{M^2 \sinh^{2n}(bT) \sinh^{2n}(rX)} \Big\{ 2b^2 \left[\frac{a}{(a+1)^2} + n \right] \coth^{2n}(bT) - b^2 \\ + 2r^2 \left[\frac{K}{(K+1)^2} + n \right] \coth^2(rX) - r^2 \Big\}; \end{aligned} \tag{43}$$

$$\theta = \frac{b(n+1)\coth(bT)}{M\sinh^n(bT)\sinh^n(rX)}; \tag{44}$$

$$\sigma^2 = \frac{b^2\coth^2(bT)\left[(a+1)^2(n^2-n+1)-3a\right]}{3(a+1)^2M^2\sinh^{2n}(bT)\sinh^{2n}(rX)}; \tag{45}$$

$$\dot{u}_\mu = (nr\coth(rX),0,0,0); \tag{46}$$

$$V^3 = \sqrt{-g} = (QR)^{2n+1}\sinh^{2n+1}(bT)\sinh^{2n+1}(rX). \tag{47}$$

From (44) and (45), we have

$$\frac{\sigma^2}{\theta^2} = \frac{(a+1)^2(n^2-n+1)-3a}{3(n+1)^2(a+1)^2} = const. \tag{48}$$

$$\textbf{case(ii)} : B = f(x)g(t) \;\; and \;\; C = f(x)k(t). \tag{49}$$

From Eq. (21) we have

$$(4n-1)\frac{f'}{f}\left(\frac{\dot{g}}{g}+\frac{\dot{k}}{k}\right) = 0 \tag{50}$$

The equation (50) leads to three cases:

$$\begin{aligned} &a) n = \frac{1}{4} \\ &b) \frac{f'}{f} = 0 \\ &c) \frac{\dot{g}}{g}+\frac{\dot{k}}{k} = 0. \end{aligned} \tag{51}$$

The case (a) reduces the number of equation to four but, with five unknowns which requires additional assumption for a viable solution. In the case (b), the model turns to be a particular case of the Bianchi type I model. Therefore, we consider the case

$$\frac{\dot{g}}{g}+\frac{\dot{k}}{k} = 0. \tag{52}$$

From Eqs. (20) and (52) we have

$$\frac{\ddot{g}}{g} = \frac{\ddot{k}}{k} \tag{53}$$

From (52) and (53) we get

$$g = e^{LT}, \qquad k = e^{-LT} \tag{54}$$

where $T = t - t_o, t_o$ and L are constants. From Eqs. (7) or (8) and (54) we have

$$ff'' - \frac{2n}{2n-1}f'^2 - \frac{L^2}{2n+1}f^2 = 0. \tag{55}$$

Solving Eq. (55) we obtain

$$f = \ell^{2n+1} \sinh^{2n+1}[M(x - x_o)], \tag{56}$$

where

$$M = \frac{L}{2n+1}$$

and ℓ and x_o are constants of integration. Hence

$$B = fg = Qe^{LT} \sinh^{2n+1}[M(x - x_o)] \tag{57}$$

and

$$C = fk = e^{-2LT} B = Qe^{-LT} \sinh^{2n+1}[M(x - x_o)] \tag{58}$$

where $Q = \ell^{2n+1}$. Therefore,

$$A = (BC)^n = N \sinh^{2n(2n+1)}[M(x - x_o)], \tag{59}$$

where $N = Q^{2n}$.

After suitable transformation of coordinates the metric (1)reduces to the form

$$ds^2 = N^2 \sinh^{2nm}(MX)(dX^2 - dT^2) + Q^2 \sinh^m(e^{2LT} dY^2 + e^{-2LT} dZ^2) \tag{60}$$

where $m = 2(2n + 1)$.

In this case, the physical parameters λ, ρ, ρ_p and the kinematical parameters $\sigma^2, \dot{u}_\mu$ and V^3 for this model are respectively given by

$$\lambda = -\frac{L^2}{N^2 \sinh^{2nm}(MX)} \left[(4n+1)\coth^2(MX) - 1\right]; \tag{61}$$

$$\rho = \frac{L^2}{N^2(2n+1)\sinh^{2nm}(MX)} \left[(4n+1)(2n-1)\coth^2(MX) - (2n+3)\right]; \tag{62}$$

$$\rho_p = \frac{4L^2}{N^2(2n+1)\sinh^{2nm}(MX)} \left[n(4n+1)(2n-1)\coth^2(MX) - (n+1)\right]; \tag{63}$$

$$\sigma^2 = \frac{L^2}{A^2} = \frac{L^2}{N^2 \sinh^{2nm}(MX)}; \tag{64}$$

$$\dot{u}_\mu = (2n(2n+1)M \coth(MX), 0, 0, 0); \tag{65}$$

$$V^3 = \sqrt{-g} = Q^m \sinh^{m(2n+1)}(MX). \tag{66}$$

4. Discussion

In case (i), with the help of the physical and kinematical parameters, we can determine some physical and geometrical features of the model. The physical and kinematical parameters become infinity as $T \to 0$, i.e., the model starts with a big bang at T =0, and goes on expanding indefinitely. In general, the model represents an expanding, shearing, and nonrotating universe. For large values of T, the space-time is conformally flat. Since $\lim_{T\to\infty} \frac{\sigma}{\theta} \neq 0$, the model does not approach isotropy for large values of T. The solutions identically satisfy the Bianchi identities given by (17) and (18). The physical parameters ρ and ρ_p are non negative for $r \geq b$. The model represents expanding universe for $n > -1$. For $\frac{N(a+1)}{(1-a)} > 0$ and $\frac{N(K+1)}{(K-1)} > 0$, it has to be that either $N > 0, -1 < a < 1$ and $K < -1, K > 1$ *or* $N < 0, a < -1, a > 1$ and $-1 < K < 1$.

In case (ii), the expansion θ is zero. With the help of the physical and kinematical parameters, we can determine some physical and geometrical features of the model. All kinematical quantities are independent of t. The physical and kinematical parameters become infinity as $X \to 0$, i.e., the model starts with a big bang at $X = 0$, and goes on expanding indefinitely. In general, the model represents nonexpanding, nonrotating and shearing universe. The acceleration vector $\dot{u}_\mu$ is zero for $n = 0$ and $n = -1/2$. In the model, the physical parameters diverge for $n = -1/2$.

Choosing suitable values for n, we have $\rho \geq 0$ and $\rho_p \geq 0$. The solutions identically satisfy the Bianchi identities given by (17) and (18).

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