Turk J Phys
25 (2001) , 399 – 406.

Null string in curved backgrounds

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Received 07.11.2000

Abstract

We consider null string in curved backgrounds. Some exact solutions of the classical equations of motion and of the constraints for the null string in a large class of cosmological backgrounds are found.

Key Words: null string, Lagrange theory, Hamilton theory, cosmological background.

1. Introduction

Null p-branes correspond to the usual p-branes with their tension T taken to be zero. This relationship between null p-branes and tensionful ones may be regarded as a generalization of the massless-massive particles correspondence. Physically, the limit $T \to 0$ corresponds to the energetic scale with $E >> M_{planck}$, because the string fundamental length $(\alpha')^{1/2} \sim 1/M_{planck}$, where $M_{planck} \approx 10^{19} GeV/c^2$. In other words, the null p-brane is the high temperature phase of p-brane theory and corresponds to the time period of the Early Universe and Big Bang [1].

There exist also an interpretation of the null and free p-branes as theories, corresponding to different vacuum states of a p-brane, interacting with a scalar field background [2]. So, one can consider the possibility of tension generation for null p-branes [3]. Another viewpoint on the connection between null and tensionful p-branes is that the null one may be interpreted as a "free" theory opposed to the tensionful "interactingu" theory [4].

According to modern ideas, elementary particle interactions are described by a grand unified theory with a simple gauge group which is a valid symmetry at the highest energies. As the energy is lowered, the model undergoes a series of spontaneous symmetry breakings. In the context of hot Big Bang cosmology this implies a sequence of phase transitions in the early universe, with critical temperatures related to the corresponding symmetry breaking scales [5]. Phase transitions in the early universe can give rise to topologically stable defects (vacuum domain walls, Strings) [5, 6]. String can lead to very interesting cosmological consequences. In particular, they can generate density fluctuations sufficient to explain galaxy formation and can produce a number of distinctive and unique observational effects [5-7].

The study of null p-brane (null string) dynamics in curved spacetimes, reveals new insights with respect to null p-brane propagation in flat spacetime [8]. For example it has been shown [9] that an energy-momentum tensor describing a fluid of null string (tensionless string) can act as a source for metrics representing Friedman-Robertson-Walker universes in both its matter and radiation dominanted epochs.

Here we consider Lagrange and Hamilton theory of null p-brane in curved background and then obtain exact null string configurations in a variety of backgrounds.

2. Lagrange and Hamilton theory of null p-brane

The action for null p–branes in a cosmological background $G_{MN}(x)$ may be written as

$$S = \int d^{p+1}\xi \, \frac{det(\partial_\mu x^M \, G_{MN}(x) \, \partial_\nu x^N)}{E(\tau, \underline{\sigma})}, \tag{1}$$

where $M,\ N = 0, 1, ..., D-1;\ \mu, \nu = 0, 1, ..., p$ are the indices of the hyperworldsheet of the null p-brane, ($\xi = (\tau, \underline{\sigma}),\ \underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_p)$) and $E(\tau, \sigma^n)$ is a $(p+1)$ –dimensional hyperworldsheet density, which plays the role of Lagrange multiplier analogous to $e(\tau)$ in the action of massless particle [10]. The determinant g of the induced null p–brane metric $g_{\mu\nu}$

$$g_{\mu\nu} = \partial_\mu x^M \, G_{MN}(x) \, \partial_\nu x^N = \begin{pmatrix} \dot{x}^A \, G_{AB}(x) \, \dot{x}^B & \dot{x}^A \, G_{AB}(x) \, \partial_n x^B \\ \partial_m x^A \, G_{AB}(x) \, \dot{x}^B & \hat{g}_{mn}(x) \end{pmatrix},$$

$$\hat{g}_{mn}(x) = \partial_m x^A \, G_{AB}(x) \, \partial_n x^B \tag{2}$$

may be presented in a factorized form

$$g = \dot{x}^M \, \tilde{\Pi}_{MN}(x) \, \dot{x}^N \, \hat{g}, \qquad \hat{g} = \det \hat{g}_{mn}, \tag{3}$$

where $\dot{x}^M = \partial x^M / \partial \tau,\ \partial_m x^A = \partial x^A / \partial \sigma^m$ $(m = 1, 2, ..., p)$. The matrix $\tilde{\Pi}_{MN}(x)$ has the properties of the projection operator

$$\tilde{\Pi}_{MN} = G_{MN}(x) - G_{MB}(x) \, \partial_m x^B \, (\hat{g}^{-1})^{mn} \, \partial_n x^L \, G_{LN}(x). \tag{4}$$

Therefore the action (1) can be written in the following form:

$$S = \int d^{p+1}\xi \, \frac{\dot{x}^M \, \tilde{\Pi}_{MN}(x) \, \dot{x}^N \, \hat{g}}{E(\tau, \underline{\sigma})}. \tag{5}$$

The variation of the action (5) with respect to $E(\tau, \underline{\sigma})$ generates the degeneracy condition for the induced metric $g_{\mu\nu}$

$$g \equiv \det g_{\mu\nu} = 0, \tag{6}$$

which separates the class of $(p+1)$ – dimensional isotropic geodesic hypersurfaces characterized by the null volume. In the gauge

$$\dot{x}^M G_{MN}(x)\partial_m x^N = 0, \qquad \left(\frac{\hat{g}}{E}\right)^{\bullet} = 0 \tag{7}$$

we find the motion equations and constraints in the following form

$$\ddot{x}^M + \Gamma^M_{PQ}(G)\,\dot{x}^P\dot{x}^Q = 0, \tag{8}$$

$$\dot{x}^M G_{MN}(x)\dot{x}^N = 0, \qquad \dot{x}^M G_{MN}(x)\partial_m x^N = 0. \tag{9}$$

Null p–branes theory in curved space–time is characterized by a set of constraints connected with the reparametrization symmetry. To find these constraints consider the canonical momentum of null p–brane $\mathcal{P}_M$ conjugated to its world coordinate x_M

$$\mathcal{P}_M = 2E^{-1}\,\hat{g}\,\tilde{\Pi}_{MN}(x)\,\dot{x}^N. \tag{10}$$

Then we find the following primary constraints:

$$\mathcal{K}_m(\underline{\sigma}) \equiv G_{MN}\partial_m x^N \mathcal{P}^M = 0. \tag{11}$$

The Hamiltonian density produced by the action functional (1) is

$$\mathcal{H}_0 = \frac{1}{4}E\,\hat{g}^{-1}G^{MN}(x)\mathcal{P}_M\mathcal{P}_N \tag{12}$$

and the condition of conservation of the primary constraint

$$\mathcal{P}_{(E)} = 0, \tag{13}$$

where $\mathcal{P}_{(E)}$ is the canonical momentum conjugated to E, generates the following condition:

$$\dot{\mathcal{P}}_{(E)} = \int d^p\xi\,\{\mathcal{H}_0, \mathcal{P}_{(E)}\}_{P.B.} = -\frac{1}{4\hat{g}}G^{MN}(x)\mathcal{P}_M\mathcal{P}_N = 0 \tag{14}$$

produces the secondary constraint

$$\mathcal{K}_\perp(\underline{\sigma}) \equiv G^{MN}(x)\mathcal{P}_M\mathcal{P}_N = 0. \tag{15}$$

Additional constraints do not appear so the total hamiltonian of null p-brane is given by

$$H = \int d^p\xi\,[\lambda^m(G_{MN}(x)\partial_m x^N\mathcal{P}^M) + \frac{E}{4\hat{g}}G^{MN}(x)\mathcal{P}_M\mathcal{P}_N + \omega\mathcal{P}_{(E)}]. \tag{16}$$

The constraints (11) and (15) satisfy the following (equal τ) Poisson bracket algebra

$$\{\mathcal{K}_\perp(\underline{\sigma}), \mathcal{K}_l(\underline{\sigma'})\}_{P.B.} = (\mathcal{K}_\perp(\underline{\sigma}) + \mathcal{K}_\perp(\underline{\sigma'}))\partial_l\delta^{(p)}(\underline{\sigma} - \underline{\sigma'}),$$

$$\{\mathcal{K}_l(\underline{\sigma}), \mathcal{K}_n(\underline{\sigma'})\}_{P.B.} = \mathcal{K}_l(\underline{\sigma})\partial'_n\delta^{(p)}(\underline{\sigma} - \underline{\sigma'}) + \mathcal{K}_n(\underline{\sigma})\partial'_l\delta^{(p)}(\underline{\sigma} - \underline{\sigma'}), \tag{17}$$

$$\{\mathcal{K}_\perp(\underline{\sigma}), \mathcal{K}_\perp(\underline{\sigma'})\}_{P.B.} = 0.$$

The hamiltonian (16) and the reparametrization constraints (11), (15) may be used for the quantization of null p–brane in a curved spacetime.

From (16) one obtains the Hamilton equations

$$\dot{x}^M = \frac{1}{2}E\hat{g}^{-1}G^{MN}(x)\mathcal{P}_N, \quad \dot{E} = \omega,$$

$$\dot{\mathcal{P}}_M = -\frac{1}{4}E\hat{g}^{-1}\partial_M G^{LN}(x)\mathcal{P}_L\mathcal{P}_N. \tag{18}$$

It is easy to show that on the classical level equations (18) are equivalent to equations (8), (9).

It seems interesting to study the interactions of null p-branes with antisymmetric tensor fields $T^{N_1\ldots N_l}(x)$ since these interactions conserve lightlike character of world hypersurface of null p-branes.

The interactions with $T^{N_1\ldots N_l}(x)$ fields are introduced by adding a new term having Wess-Zumino-like form into the action (1):

$$\begin{aligned} S \;\; = \;\; & \int d^{p+1}\xi \, [\frac{det(\partial_\mu x^M \; G_{MN}(x) \; \partial_\nu x^N)}{E(\tau, \underline{\sigma})} - \\ & -\widetilde{\lambda}\varepsilon^{\mu_1\ldots\mu_l}\partial_{\mu_1}x^{M_1}\ldots\partial_{\mu_l}x^{M_l}G_{M_1N_1}(x)\ldots G_{M_lN_l}(x)T^{N_1\ldots N_l}(x)]. \end{aligned} \tag{19}$$

From the variational integral (19) one obtains the classical equations of motion and constraints

$$\ddot{x}^L + \Gamma^L_{MN}(G)\dot{x}^M\dot{x}^N + \lambda\varepsilon^{\mu_1\ldots\mu_l}\partial_{\mu_1}x^{M_1}\ldots\partial_{\mu_l}x^{M_l}\times$$

$$\times\left[\partial^L T^{M_1\ldots M_l} - \partial^{M_1}T^{LM_2\ldots M_l} - \ldots - \partial^{M_l}T^{M_1\ldots L}\right] = 0, \tag{20}$$

$$\dot{x}^M G_{MN}(x)\dot{x}^N = 0, \; \dot{x}^M G_{MN}(x)\partial_m x^N = 0, \; \left(\hat{g}E^{-1}(\tau, \underline{\sigma})\right)^\bullet = 0, \tag{21}$$

where $\lambda = \widetilde{\lambda}\left(E\hat{g}^{-1}\right)/2$.

3. Null strings in a maximum symmetric spacetime

It is known that D-dimensional spacetime does not have more than $D(D+1)/2$ independent Killing's vectors. We will consider D-dimensional backgrounds which have exactly $D(D+1)/2$ Killing's vectors and have maximum symmetric backgrounds [11]. The set of maximum symmetric backgrounds have constant curvature and we can always find transformation from one of these backgrounds to another. This remark allows one to study null string (null p-brane) dynamics in some background of constant curvature, where we can solve the equations of motion and constraints for a null string (null p-brane).

We now write down the null string equations of motion and constraints for a maximum symmetric metric given by [11]:

$$g_{MN}(x) = C_{MN} + \frac{k}{1 - kC_{PQ}x^P x^Q} C_{MA}x^A C_{NB}x^B, \tag{22}$$

$$A, B, M, ... = 0, 1, ..., D-1,$$

where $k = -1, 0, 1$ and C_{MN} is a constant $D \times D$-matrix.

The null string equations and constraints are given as:

$$\ddot{x}^M + \Gamma^M_{NL}(G)\dot{x}^N \dot{x}^L = 0, \tag{23}$$

$$G_{MN}(x)\dot{x}^M \dot{x}^N = 0,\ G_{MN}(x)\dot{x}^M x'^{\ N} = 0, \tag{24}$$

where the overdots and primes denote differentiation with respect to τ and σ, respectively. Equations (23) are essentially geodesic equations. The constraint ($G_{MN}(x)\dot{x}^M \dot{x}^N = 0$) implies that we should look for only null geodesics. Thus, knowing the null geodesic in a background spacetime would naturally lead to null string configurations provided all the constraints are satisfied. Using the metric (22) the equations of motion (23) give us

$$x^M(\tau, \sigma) = p^M(\sigma)\tau + x^M(\sigma), \tag{25}$$

where $p^M(\sigma), x^M(\sigma)$ are the initial data. The explicit form of the solutions (25) allows to transform the constraints (24) into those for the Cauchy initial data:

$$p^2(\sigma)[1 - kx^2(\sigma)] + k(p(\sigma)x(\sigma))^2 = 0, \tag{26}$$

$$p^2(\sigma)(p'(\sigma)x(\sigma)) - (p(\sigma)x(\sigma))(p(\sigma)p'(\sigma)) = 0, \tag{27}$$

$$(p(\sigma)p'(\sigma))[1 - kx^2(\sigma)] +$$

$$+k(p(\sigma)x(\sigma))[p'(\sigma)x(\sigma) - p(\sigma)x'(\sigma)] + kp^2(\sigma)(x'(\sigma)x(\sigma)) = 0 \tag{28}$$

$$(p(\sigma)x'(\sigma))[1 - kx^2(\sigma)] + k(p(\sigma)x(\sigma))(x'(\sigma)x(\sigma)) = 0, \tag{29}$$

where

$$p(\sigma)x(\sigma) \equiv C_{MN}\ p^M(\sigma)x^N(\sigma). \tag{30}$$

In a flat spacetime ($k = 0$) we have the well known result. The results (25)-(29) can be generalized for null p-brane.

4. The null string dynamics in a spacetime with nontrivial conformal group

Let $V_{n+1}, n \geq 3$ be a Riemann space with Lorentz signature. We will call the space V_{n+1} as a space with nontrivial conformal group if and only if V_{n+1} is conformally equivalent to a space with metric [12]

$$dS^2 = (dx^0)^2 - (dx^1)^2 - a_{ij}(x^1 - x^0)dx^i dx^j, \quad i, j = 2, 3, ..., n, \tag{31}$$

where a_{ij} is an arbitrary positive definite matrix. In the case $n = 3$ the space with metric (31) can be conformally transformed to the space with metric tensor

$$g^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -f(x-t) & -\varphi(x-t) & 0 \\ 0 & -\varphi(x-t) & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad G = \varphi^2 - f < 0, \tag{32}$$

where $t = x^0, x = x^1, y = x^2, z = x^3$.

The null string equations of motion and constraints in this background turn out to be:

$$x,_{\tau\tau} + \frac{1}{2}\left(\frac{1}{G}\right)'(y,_\tau - \varphi\ z,_\tau)^2 - \frac{\varphi'}{G}(y,_\tau - \varphi z,_\tau)\ z,_\tau = 0, \tag{33}$$

$$\left[\frac{1}{G}(y,_\tau - \varphi\ z,_\tau)\right],_\tau = 0, \quad x,_{\tau\tau} - t,_{\tau\tau} = 0, \tag{34}$$

$$z,_{\tau\tau} + \frac{1}{G}(y,_\tau - \varphi\ z,_\tau)(x-t),_\tau\ \varphi' = 0, \tag{35}$$

$$t,_\tau^2 - x,_\tau^2 + \frac{1}{G}(y,_\tau^2 + f\ z,_\tau^2) - \frac{2\varphi}{G} y,_\tau\ z,_\tau = 0, \tag{36}$$

$$t,_\tau\ t,_\sigma - x,_\tau\ x,_\sigma + \frac{1}{G}(y,_\tau\ y,_\sigma + f\ z,_\tau\ z,_\sigma) - \frac{\varphi}{G}(y,_\tau\ z,_\sigma + z,_\tau\ y,_\sigma) = 0, \tag{37}$$

where $(...),_\tau = \partial(...)/\partial\tau,\ (...),_\sigma = \partial(...)/\partial\sigma,\ (...)' = d(...)/d(x-t)$. The exact solutions of the equations of motion (33)-(35) are:

$$\begin{aligned} t(\tau,\sigma) \quad = \quad & t(\sigma) + [p_0(\sigma) + a(\sigma)p_2(\sigma) - \\ & -\frac{1}{2}a^2(\sigma)(p_1(\sigma) - p_0(\sigma))\ f(\psi(\sigma))]\tau - \frac{1}{2}a^2(\sigma)[F(\psi(\sigma)) - \\ & -F(\psi(0))] - a(\sigma)b(\sigma)[\Phi(\psi(\tau)) - \Phi(\psi(0))], \end{aligned} \tag{38}$$

$$x(\tau,\sigma) = x(\sigma) - t(\sigma) + (p_1(\sigma) - p_0(\sigma))\tau + t(\tau,\sigma), \tag{39}$$

$$y(\tau,\sigma) = y(\sigma) + a(\sigma)[F(\psi(\tau)) - F(\psi(0))] + b(\sigma)[\Phi(\psi(\tau)) - \Phi(\psi(0))], \tag{40}$$

$$z(\tau,\sigma) = z(\sigma) + a(\sigma)[\Phi(\psi(\tau)) - \Phi(\psi(0))] + b(\sigma)(p_1(\sigma) - p_0(\sigma))\tau, \tag{41}$$

where

$$\begin{aligned} a(\sigma) &= \frac{\varphi(\psi(0))\ p_3(\sigma) - p_2(\sigma)}{(p_1(\sigma) - p_0(\sigma))\ G(\psi(0))}, \\ b(\sigma) &= \frac{\varphi(\psi(0))\ p_2(\sigma) - f(\psi(0))\ p_3(\sigma)}{(p_1(\sigma) - p_0(\sigma))\ G(\psi(0))}, \end{aligned} \tag{42}$$

$$\begin{aligned} \psi(\tau) &= x(\sigma) - t(\sigma) + (p_1(\sigma) - p_0(\sigma))\tau, \\ F(\xi) &= \int f(\xi)d\xi, \quad \Phi(\xi) = \int \varphi(\xi)d\xi. \end{aligned} \tag{43}$$

Finally, substituting (38)-(41) into (36), (37), one obtains

$$p_0^2(\sigma) - p_1^2(\sigma) - 2a(\sigma)\ p_2(\sigma)(p_1(\sigma) - p_0(\sigma)) +$$

$$+ a^2(\sigma)(p_1(\sigma) - p_0(\sigma))^2 f(\psi(0)) + b^2(\sigma)(p_1(\sigma) - p_0(\sigma))^2 = 0, \tag{44}$$

$$p_0(\sigma)\ t,_\tau(\sigma) - p_1(\sigma)\ x,_\sigma(\sigma) - (p_1(\sigma) - p_0(\sigma))[a(\sigma)\ y,_\sigma(\sigma) + b(\sigma)\ z,_\sigma(\sigma)] = 0, \tag{45}$$

where $t(\sigma), x(\sigma), y(\sigma), z(\sigma), p_0(\sigma), p_1(\sigma), p_2(\sigma), p_3(\sigma)$ are the initial data.

As an example of exact solutions, we consider the circular null string configuration

$$\begin{aligned} t(\sigma) &= x(\sigma) = 0, \quad y(\sigma) = R\cos\frac{\sigma}{R}, \quad z(\sigma) = R\sin\frac{\sigma}{R}, \\ p_0(\sigma) &= -p_1(\sigma) = 1, \quad p_2(\sigma) = p_3(\sigma) = 0. \end{aligned} \tag{46}$$

Inserting Equations (46) into Equations (38)-(45) we have a very simple null string solution:

$$t = \tau, \quad x = -\tau, \quad y = R\cos\frac{\sigma}{R}, \quad z = R\sin\frac{\sigma}{R}. \tag{47}$$

This solution describes null string dynamics in plane gravitational waves with arbitrary profile functions $f(x-t), \varphi(x-t)$. It would be interesting to consider other null string configurations and apply it to study null string dynamics in shock waves.

Acknowledgments

The author would like to thank L.Arifov and A.Zheltukhin for useful discussions. This work is supported by Grand Ukraine N F4/1751.

References

[1] A.A. Zheltukhin, *Phys. Lett.*, **B233** (1989) 112.

[2] I.A. Bandos, A.A. Zheltukhin, *Fortschr. Phys.* **41** (1993) 619.

[3] I.A. Bandos, D. Sorokin, M. Tonin, D. Volkov, *Phys. Lett.*, **B 319** (1993) 445.

[4] J. Gamboa, *Class. Quant. Grav.*, **7** (1990) 1647.

[5] A. Vilenkin, *Phys. Repts.*, **121** (1985) 263.

[6] T.W.B. Kibble, *Phys. Repts.*, **67** (1980) 183.

[7] Ya.B.Zel'dovich, *Month. Notic. Roy. Astron. Soc.,* **192** (1980) 663.

[8] S. Kar, *Phys. Rev.,* **D 53** (1996) 6842.

[9] S.N. Roshchupkin, A.A. Zheltukhin, *Class.Quant. Grav.*, **12** (1995) 2519.

[10] A.A. Zheltukhin, *JETP Lett.*, **46** (1987) 208.

[11] S.Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity* (John Wiley and Sons, New York, 1972) p. 412.

[12] A.Z. Petrov, *New Methods in General Relativity* (Moscow: Nauka, 1966) p.285.