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The Theoretical Analysis of The Second Order Coherence $g^{(2)}(\tau)$ and Power Stabilization of Two-Mode He-Ne Laser

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Abstract

In this study, second-order coherence and its frequency-dependent characteristics and the photon statistical properties of the two-mode He-Ne laser field are investigated.

Key Words: Two – mode laser, Second – order coherence, $g^{(2)}(\tau)$, He – Ne lasers.

1. Introduction

In this study, second-order coherence $g^{(2)}(\tau)$ of the two-mode laser field at the steady state and associate frequency-dependent formula of $g^{(2)}(\tau)$ for free running two-mode laser are investigated from the quantum theory of the light [1,2]. The frequency-dependent characteristics and its quasi-periodicity of $g^{(2)}(\tau)$ of a two-mode laser field are theoretically analyzed [3].

2. Theory

2.1. Frequency-Dependent Formula of $g^{(2)}(\tau)$

If the two-mode annihilation operators are $\hat{a}_1$ and $\hat{a}_2$, the wave functions with space variables are $U_1(x)$ and $U_2(x)$, the mode volumes are V_1=V_2=V, then the electric field operator of a two-mode laser with frequencies ω_1 and ω_2 can be written as follows [4,5]:

$$\hat{E}(x,t) = \hat{E}^{+}(x,t) + \hat{E}^{-}(x,t), \tag{1}$$

where

$$\hat{E}^{+}(x,t) = \sum_{k=1}^{2} U_k(x)\hat{a}_k \exp(-j\omega_k t) \tag{2}$$

$$\hat{E}^{-}(x,t) = \sum_{k=1}^{2} U_k^{*}(x)\hat{a}_k^{(+)} \exp(j\omega_k t) \tag{3}$$

and

$$U_k(x) = j\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2} \exp(jk_j.x) \quad k = 1, 2. \tag{4}$$

For an ideal multimode laser field, the density matrix can be written as a coherent state with random phase

$$\rho = \prod_{k=1}^{N} \rho_k \times \prod_{k=N+1} |O_k \times O_k|, \tag{5}$$

where

$$\rho_k = |\beta_k \times \beta_k| = \frac{1}{2\pi}\int_0^{2\pi} d\Phi_k |\beta_k \times \beta_k| \tag{6}$$

with

$$\beta_k = |\beta_k| \exp(j\Phi_k). \tag{7}$$

Here, $|\beta_k|^2 = \langle \hat{n}_k \rangle$ is average photon number of the $k-$th mode, Φ_k is the random phase, and $|O_k\rangle$ is the state vector of the $k-$th mode of the vacuum state. The creation operator $\hat{a}_k^{(+)}$ and the annihilation operator $\hat{a}_{k'}$ satisfy [6] the relation

$$\left\langle \hat{a}_k^{(+)}\hat{a}_{k'} \right\rangle = Tr\left\{\rho\hat{a}_k^{(+)}\hat{a}_{k'}\right\} = \langle \hat{n}_k \rangle \delta_{kk'}. \tag{8}$$

For a two-mode laser field, N=2, k=1,2 and from the definition of the second-order quantum correlation function of the light field

$$\begin{aligned} G^{(2)}(\tau) &= \left\langle \hat{I}(t)\hat{I}(t+\tau) \right\rangle \\ &= \left\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t) \right\rangle, \end{aligned} \tag{9}$$

one has

$$G^{(2)}(\tau) = \left[|U_1|^2\langle \hat{n}_1 \rangle + |U_2|^2\langle \hat{n}_2 \rangle\right]^2 + 2|U_1|^2|U_2|^2\langle \hat{n}_1 \rangle\langle \hat{n}_2 \rangle \cos(\Delta\omega_q \tau). \tag{10}$$

Since the first-order quantum correlation function of the two-mode laser field is given by

$$\begin{aligned} G^{(1)}(\tau) &= \left\langle \hat{E}^{(-)}(t)\hat{E}^{(+)}(t+\tau)\right\rangle \\ &= |U_1|^2\langle\hat{n}_1\rangle\exp(-j\omega_1\tau)+|U_2|^2\langle\hat{n}_2\rangle\exp(-j\omega_2\tau), \end{aligned} \tag{11}$$

Equation (10) can be rewritten as

$$G^{(2)}(\tau)=\left[|U_1|^2\langle\hat{n}_1\rangle+|U_2|^2\langle\hat{n}_2\rangle\right]^2-\left[|U_1|^4\langle\hat{n}_1|^2+|U_2|^4\langle\hat{n}_2\rangle^2\right]+\left|G^{(1)}(\tau)\right|^2. \tag{12}$$

From the definition of the degree of second-order coherence

$$g^{(2)}(\tau)=\frac{G^{(2)}(\tau)}{\left[G^{(1)}(0)G^{(1)}(0)\right]}=\frac{\left\langle\hat{I}(t)\hat{I}(t+\tau)\right\rangle}{\left\langle\hat{I}(t)\right\rangle^2} \tag{13}$$

The general formula of the degree of second-order coherence of a two-mode laser field at the steady state is given by:

$$g^{(2)}(\tau)=1-\frac{\left\langle\hat{I}_1\right\rangle^2+\left\langle\hat{I}_2\right\rangle^2}{\left[\left\langle\hat{I}_1\right\rangle+\left\langle\hat{I}_2\right\rangle\right]}+\left|g^{(1)}(\tau)\right|^2, \tag{14}$$

where

$$\left\langle\hat{I}_1\right\rangle=\hbar\omega_1\langle\hat{n}_1\rangle, \tag{15}$$

$$\left\langle\hat{I}_2\right\rangle=\hbar\omega_2\langle\hat{n}_2\rangle \tag{16}$$

are the output intensities of a two-mode laser at the steady state [7,8,9].

$$\text{If } k=\frac{\left\langle\hat{I}_1\right\rangle}{\left\langle\hat{I}_2\right\rangle} \tag{17}$$

is the relative intensity of the two-mode output, Equation (17) is can be rewritten as

$$g^{(2)}(\tau)=\frac{2k}{(1+k^2)}+\left|g^{(1)}(\tau)\right|^2, \tag{18}$$

where $g^{(1)}(\tau)$ is the degree of first-order coherence of the light. In order to discuss the influence of the frequency width $\delta\nu_H$ of longitudinal mode upon the $g^{(2)}(\tau)$, we employ the semi-classical theory of light to derive the degree of first-order coherence $g^{(1)}(\tau)$. The form of $g^{(1)}(\tau)$ is given by

$$g^{(1)}(\tau) = \frac{\exp(-\pi\delta\nu_H\tau)}{\left\langle \hat{I}_1 \right\rangle + \left\langle \hat{I}_2 \right\rangle} \left| \left\langle \hat{I}_1 \right\rangle^2 + \left\langle \hat{I}_2 \right\rangle^2 + 2\left\langle \hat{I}_1 \right\rangle \left\langle \hat{I}_2 \right\rangle \cos(2\pi\Delta\nu_q\tau) \right|^{1/2} \tag{19}$$

$$g^{(1)}(\tau) = \frac{\exp(-\pi\delta\nu_H\tau)}{1+k} \left|1 + k^2 + 2k\cos(2\pi\Delta\nu_q\tau)\right|^{1/2}, \tag{20}$$

where $\delta\nu_H$ is the frequency width of the longitudinal modes. When $\delta\nu_H \to 0$, or the delay time τ, and optical path difference $\Delta\ell = c\tau$ are very small, Equations (14)-(18) can be simplified to:

$$g^{(2)}(\tau) = 1 + \frac{2\left\langle \hat{I}_1 \right\rangle \left\langle \hat{I}_2 \right\rangle}{\left[\left\langle \hat{I}_1 \right\rangle + \left\langle \hat{I}_2 \right\rangle\right]^2} \cos(2\pi\Delta\nu_q\tau) \tag{21}$$

$$g^{(2)}(\tau) = 1 + \frac{2k}{[1+k]^2} \cos(2\pi\Delta\nu_q\tau) \tag{22}$$

If the output intensity of the two modes are same, $\left\langle \hat{I}_1 \right\rangle = \left\langle \hat{I}_2 \right\rangle$ or k=1, with $\delta\nu_H \to 0$, Equation (14) or Equation (23) can be simplified further to

$$g^{(2)}(\tau) = 1 + \left|g^{(1)}(\tau)\right|^2 = 1 + \frac{1}{2}\cos(2\pi\Delta\nu_q\tau). \tag{23}$$

It is clear that the range of the degree of second-order coherence $g^{(2)}(\tau)$ of a two-mode laser field is

$$\frac{1}{2} \leq g^{(2)}(\tau) \leq \frac{3}{2}. \tag{24}$$

If the influence of the drift effect $\Delta\nu$ of longitudinal-mode frequency on two-mode output intensities in a free running laser is considered, the general Equations (14)-(18) for the frequency dependence of $g^{(2)}(\tau)$ in a two-mode laser field can be modified as

$$g^{(2)}(\Delta\nu, \tau) = 1 - \frac{\hat{I}_1^2(\Delta\nu) + \hat{I}_2^2(\Delta\nu)}{\left[\hat{I}_1(\Delta\nu) + \hat{I}_2(\Delta\nu)\right]^2} + \left|g^{(1)}(\Delta\nu, \tau)\right|^2 \tag{25}$$

$$g^{(2)}(\Delta\nu, \tau) = \frac{2k(\Delta\nu)}{[1 + k(\Delta\nu)]^2} + \left|g^{(1)}(\Delta\nu, \tau)\right|^2, \tag{26}$$

where

$$\Delta\nu = \nu_1 - \left(\nu_D - \frac{\Delta\nu_q}{2}\right) = \nu_2 - \left(\nu_D + \frac{\Delta\nu_q}{2}\right). \tag{27}$$

The corresponding $g^{(1)}(\Delta\nu, \tau)$ is given by

$$g^{(1)}(\Delta\nu,\tau)=\frac{\exp(-\pi\delta\nu_H\tau)}{\hat{I}_1(\Delta\nu)+\hat{I}_2(\Delta\nu)}\left|\hat{I}_1(\Delta\nu)^2+\hat{I}_2(\Delta\nu)^2+2\hat{I}_1(\Delta\nu)\hat{I}_2(\Delta\nu)\cos(2\pi\Delta\nu_q\tau)\right|^{1/2} \tag{28}$$

$$g^{(1)}(\Delta\nu,\tau)=\frac{\exp(-\pi\delta\nu_H\tau)}{1+k(\Delta\nu)}\left|1+k(\Delta\nu)^2+2k(\Delta\nu)\cos(2\pi\Delta\nu_q\tau)\right|^{1/2}, \tag{29}$$

where

$$k(\Delta\nu)=\frac{\hat{I}_1(\Delta\nu)}{\hat{I}_2(\Delta\nu)} \tag{30}$$

is the instantaneous intensity ratio of the two-mode output at the drift amount $\Delta\nu$ of longitudinal-mode frequency. When $\delta\nu_H$ can be neglected or τ, $\Delta\ell$ are very small, Equations (25)-(26) can be simplified as

$$g^{(2)}(\Delta\nu,\tau)=1+\frac{2\hat{I}_1(\Delta\nu)\hat{I}_2(\Delta\nu)}{\left[\hat{I}_1(\Delta\nu)+\hat{I}_2(\Delta\nu)\right]^2}\cos(2\pi\Delta\nu_q\tau); \tag{31}$$

$$g^{(2)}(\Delta\nu,\tau)=1+\frac{2k(\Delta\nu)}{[1+k(\Delta\nu)]^2}\cos(2\pi\Delta\nu_q\tau). \tag{32}$$

For a two-mode He-Ne laser working at steady state, if the two-mode intensities are equal, k=1, and the degree of second-order coherence $g^{(2)}(\tau)$ is given by

$$g^{(2)}(\tau)=\frac{1}{2}+\exp(-2\pi\delta\nu_H\tau)\cos^2(\pi\Delta\nu_q\tau). \tag{33}$$

[10-13].

However, for a free running two-mode He-Ne laser, the ratio of the two-mode laser intensities at $\Delta\nu$ is given by

$$k(\Delta\nu)=\frac{\hat{I}_1(\Delta\nu)}{\hat{I}_2(\Delta\nu)}=\exp\left[\frac{8\ln 2}{\Delta\nu_D^2}\Delta\nu_q\Delta\nu\right], \tag{34}$$

where $\Delta\nu_D$ and $\Delta\nu_q$ are the laser line width and space of longitudinal modes, $\Delta\nu$ is the amount of the frequency drift of the two-mode frequency ν_1,ν_2 relative to the frequency-symmetric point $\left(\nu_D\pm\frac{\Delta\nu_q}{2}\right)$. Substituting Equation (34) into Equations (25)-(26) and (28)-(29), one has the general formula for frequency-dependent $g^{(2)}(\tau)$:

$$g^{(2)}(\Delta\nu,\tau)=\frac{1+\exp(-2\pi\delta\nu_H\tau)\left[\cos(2\pi\Delta\nu_q\tau)+\cosh\left(8\ln 2\Delta\nu_q\frac{\Delta\nu}{\Delta\nu_D^2}\right)\right]}{1+\cosh\left(8\ln 2\Delta\nu_q\frac{\Delta\nu}{\Delta\nu_D^2}\right)} \tag{35}$$

If $\Delta\nu$ does not vary with the time t, Equation (35) becomes the general formula of $g^{(2)}(\tau)$ of the two-mode He-Ne laser field at the steady state. When $\delta\nu_H \to 0$, or τ,$\Delta\ell$ are very small, Equation (35) reduces to

$$g^{(2)}(\Delta\nu,\tau) = 1 + \frac{\cos(2\pi\Delta\nu_q\tau)}{1+\cos\left(8\ln 2\Delta\nu_q\frac{\Delta\nu}{\Delta\nu_D^2}\right)}, \tag{36}$$

where

$$\Delta\nu_q = \frac{c}{2nL} = \frac{c}{2L} \tag{37}$$

and L is the length of the laser cavity and n is the index of refraction of the active medium, (for gases $n \cong 1$).

2.2. Frequency-Dependent Characteristics of $g^{(2)}(\tau)$

Assuming,

$$\Delta\ell = 2mL \quad m = 0, \pm 1, \pm 2, ... \tag{38}$$

(that is, the optical path difference $\Delta\ell$ is even multiples of the laser cavity length L), from Equations (25)-(26) and (28)-(29), one obtains the frequency-dependent $g^{(2)}(\tau)$ of the two-mode laser field as

$$g^{(2)}(\Delta\nu, 2mL) = \exp\left(-4m\pi\delta\nu_H\frac{L}{c}\right) + \frac{2k(\Delta\nu)}{[1+k(\Delta\nu)]^2} \tag{39}$$

and

$$g^{(2)}(\Delta\nu, 2mL) = 1 + \frac{2k(\Delta\nu)}{[1+k(\Delta\nu)]^2}. \tag{40}$$

Similarly, if

$$\Delta\ell = (2m+1)L \quad m = 0, \pm 1, \pm 2, ... \tag{41}$$

(the optical path difference is odd multiples of the laser cavity length L), the frequency-dependent $g^{(2)}(\tau)$ of the two-mode laser field is given by

$$g^{(2)}\left(\Delta\nu,(2m+1)L\right) = \frac{2k(\Delta\nu) + [1-k(\Delta\nu)]^2 \exp\left[-2(2m+1)\pi\delta\nu_H\frac{L}{c}\right]}{[1+k(\Delta\nu)]^2} \tag{42}$$

and

$$g^{(2)}\left(\Delta\nu,(2m+1)L\right) = 1 + \frac{2k(\Delta\nu)}{[1+k(\Delta\nu)]^2}. \tag{43}$$

Obviously, at

$$\Delta \ell = 2mL$$

or

$$\Delta \ell = (2m+1)L$$

the degree of second-order coherence $g^{(2)}(\tau)$ is related to the relative intensity $k(\Delta\nu)$, to the frequency-drift effect $\Delta\nu$ of the longitudinal modes.

Moreover, when

$$\Delta \ell = (2m+1)\frac{L}{2} \quad m = 0, \pm 1, \pm 2, ... \tag{44}$$

the optical path difference is odd multiples of $L/2$, the frequency-dependent $g^{(2)}(\tau)$ of the two-mode laser field is given by

$$g^{(2)}\left[\Delta\nu, (2m+1)\frac{L}{2}\right] = 1 \tag{45}$$

if $\delta\nu_H \to 0$.

When the longitudinal mode is drifted and the relative intensity ratio $k(\Delta\nu)$ varies from 0.1 to 10.0, the frequency-dependent curve of $g^{(2)}(\tau)$ is calculated from Equations (25)-(26) and (28)-(29) for the optical path difference $\Delta\ell = 0, \frac{L}{2}, L, \frac{3}{2}L, ...$ and $2L$ or for $k(\Delta\nu)$=0.1,0.5,1.0,5.0 and 10.0. The variation of $g^{(2)}(\tau)$ against the relative intensity ratio $k(\Delta\nu)$, or the optical path difference $\Delta\ell$ for L =25 cm, $\Delta\nu_q$=600 MHz, $\delta\nu_H$=30 MHz is obtained [14].

For a two-mode He-Ne laser with frequency tuning $\Delta\nu$, optical path difference $\Delta\ell$=L from Equation (35), one has

$$g^{(2)}(\Delta\nu, L) = \frac{\cosh\left(8\ln 2\Delta\nu_q \frac{\Delta\nu}{\Delta\nu_D^2}\right)}{1+\cosh\left(8\ln 2\Delta\nu_q \frac{\Delta\nu}{\Delta\nu_D^2}\right)}. \tag{46}$$

This is the tuning Equation of $g^{(2)}(\tau)$ for a two-mode He-Ne laser field at $\Delta\ell = L$. Similarly, when $\Delta\ell = 0$ or $2L$, Equation (35) gives

$$g^{(2)}(\Delta\nu, 0) = g^{(2)}(\Delta\nu, 2L) = 1 + \frac{1}{1+\cosh\left(8\ln 2\Delta\nu_q \frac{\Delta\nu}{\Delta\nu_D^2}\right)}. \tag{47}$$

Assuming L=25 cm, $\Delta\nu_q$=600 MHz, $\Delta\nu_D$=800, 1000 and 1200 MHz when the tuning amount $\Delta\nu$=±600 MHz, the frequency-tuning curves of $g^{(2)}(\tau)$ is calculated from Equations (46) and (47).

Differentiation of Equation (46) gives the slope of $g^{(2)}(\Delta\nu, L)$ tuning curve at some amount $\Delta\nu$ of frequency tuning:

$$k(\Delta\nu) = \frac{d}{d(\Delta\nu)} g^{(2)}(\Delta\nu, L) = \frac{a \sin k(a\Delta\nu)}{[1 + \cosh(a\Delta\nu)]^2}, \tag{48}$$

where

$$a = 8 \ln 2 \frac{\Delta\nu_q}{\Delta\nu_D^2} \tag{49}$$

3. Results and Discussions

In this study, the general formula of the degree of second-order coherence $g^{(2)}(\tau)$ and of the frequency-dependent relationship are investigated from the quantum theory of the light. The second-order quantum coherence, its frequency-dependent and photon statistical properties of the steady state and of the free running two-mode laser field have been investigated.

The degree of second-order coherence $g^{(2)}(\tau)$ is related to the relative intensity $k(\Delta\nu)$, to the frequency-drift effect $\Delta\nu$ of the longitudinal modes.

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