Turk J Phys
26 (2002) , 225 – 228.

A Quadratic Boundary Element Formulation for Neutron Diffusion Equation

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Received 01.07.2000

Abstract

The boundary element method (BEM) is a technique based on the principle of the conversion of the governing differential equation to the boundary integral equation with consequent reduction by one in the dimension of the problem. In this work, quadratic boundary elements and thus BEM have been applied to neutron diffusion calculations to render the treatment of curved boundaries possible. The results of the developed computer program BENDQ has been compared to the analytical solutions and validated.

Key Words: Boundary Element Method, Neutron Diffusion Equation

1. Introduction

The boundary element method (BEM), a method used in the solution of boundary value problems in mathematical physics, is capable of reducing differential equations to boundary integral equations and, thus, restricting the unknowns solely to the system boundary. BEM, which have been used for obtaining numerical solutions in various branches of science since 1980's, have also been utilized in neutron diffusion calculations of reactor physics [1]. BEM is based on the principle of subdivision of the system boundary into a finite number of boundary elements and on the assumption that the unknown function is a low degree interpolatory Lagrangian polynomial within each element. If the unknown function is assumed to be constant or linear, the approximation is taken as a constant or linear BEM, respectively. If the system boundary is not a polygon and is curved (e.g. circle, ellipse), it is not possible to model the system boundary geometrically in an efficient way, using constant or linear BEM. Since BEM restricts the unknowns only to systems boundaries, the number of unknowns and the dimension of the resulting linear system is reduced dramatically compared to alternative methods of solution, i.e. finite difference and finite element methods in which unknowns must be defined throughout the nodes inside the system volume. This reduction in linear system size and the capability of representing curved boundaries exactly via quadratic elements represent the superiorities of the proposed method compared to its alternatives. In this work, using the isoparametric quadratic BEM, a formulation has been developed which can solve the neutron diffusion equation efficiently, even in the presence of curved boundaries. The formulation is implemented in the developed computer program BENDQ from which quadratic BEM solutions are obtained. Via comparisons with analytic solutions, the proposed formulation is validated.

2. Formulation

For a homogeneous nuclear system with volume V, the neutron diffusion equation is:

$$\nabla^2\Phi(\vec{r}) - k^2\Phi(\vec{r}) = -\frac{s(\vec{r})}{D}, \vec{r}\in V. \tag{1}$$

Here, Φ, s, D and k represent the neutron flux, the neutron source, the diffusion constant and the inverse diffusion length, respectively. The outer boundary of the system $S = S_r \cup S_V$ and is subject to the following boundary conditions:

$$\Phi(\vec{r}) = 0, \vec{r}\in S_V$$

$$\frac{\partial\Phi}{\partial n}(\vec{r}) = 0, \vec{r}\in S_r. \tag{2}$$

On the other hand, the fundamental solution $G(\vec{r},\vec{\rho})$ is determined by

$$\nabla^2 G(\vec{r},\vec{\rho}) - k^2 G(\vec{r},\vec{\rho}) = -\delta(\vec{r}-\vec{\rho}) \tag{3}$$

If Eqn.(1) is multiplied by $G(\vec{r},\vec{\rho})$ and integrated over the volume of the system, Green's second identity and Eqn.(3) could be utilized along with the integration property of Dirac delta functions and the definition of the normal component of the neutron current,

$$J = -D\frac{\partial\Phi}{\partial n}, \tag{4}$$

to obtain the boundary integral equation [1]:

$$c(\vec{\rho})\Phi(\vec{\rho}) + \int_{S_r}\Phi(\vec{r})\frac{\partial G}{\partial n}(\vec{r},\vec{\rho})dS + \int_{S_V}\frac{G(\vec{r},\vec{\rho})}{D}J(\vec{r})dS = \int_V \frac{s(\vec{r})}{D}G(\vec{r},\vec{\rho})dV. \tag{5}$$

If $\vec{\rho}$ is within the system volume, $c(\vec{\rho}) = 1$; and if $\vec{\rho}$ is on the system boundary with a continuous tangent, $c(\vec{\rho}) = 1/2$. For quadratic BEM application, the boundary of the system, which is assumed to be two-dimensional in this work, is divided into N_e boundary elements and the neutron flux and current are assumed to be continuous functions on the system boundary and are restricted to be quadratic interpolatory Lagrangian polynomials within each boundary element. Each of the N_e boundary elements possesses three nodes, one at each end and one at the geometric center; thus, the number of nodes at the system boundary is $N = 2N_e$. If $\Psi_j(\vec{r})$ denotes the basis function of the j'th node, and is a quadratic interpolatory Lagrangian polynomial within the boundary elements to which node j belongs and zero elsewhere; and Φ_j and J_j represent the value of the flux and current at node j, respectively, we can write

$$\Phi(\vec{r}) = \sum_{j=1}^{N}\Psi_j(\vec{r})\Phi_j \tag{6}$$

$$J(\vec{r}) = \sum_{j=1}^{N}\Psi_j(\vec{r})J_j. \tag{7}$$

If we require Eqn.(5) to be valid for each node j under the assumptions of (6) and (7), we obtain the equation

$$\sum_{j=1}^{N}(h_{ij}\Phi_j + g_{ij}J_j) = f_i, \tag{8}$$

where

$$h_{ij} = \frac{1}{2}\delta_{ij}\delta_{iR} + \int_{S_r}\Psi_j(\vec{r})\frac{\partial G}{\partial n}(\vec{r},\vec{\rho}_i)dS \tag{9}$$

$$g_{ij} = \int_{S_V}\frac{G(\vec{r},\vec{\rho}_i)}{D}\Psi_j(\vec{r})dS \tag{10}$$

$$f_i = \int_V \frac{s(\vec{r})}{D}G(\vec{r},\vec{\rho}_i)dV. \tag{11}$$

δ_{iR} equals unity if the flux is an unknown at node j, otherwise it vanishes. Due to Eqns.(2) and (4), either flux or current vanishes at each node. If the flux is zero at node j, we take $a_{ij} = g_{ij}$ and $u_j = J_j$, and if the current is zero at node j we take $a_{ij} = h_{ij}$ and $u_j = \Phi_j$ and rewrite (8) as

$$\sum_{j=1}^{N} a_{ij} u_j = f_i, \tag{12}$$

an N-dimensional linear system. After solving (12) and determining the boundary distribution, $\vec{\rho}$ in (5) can be taken as the position vector of any desired internal point and fluxes within the system interior could be determined. This work is restricted to two-dimensional systems and thus the fundamental solution,

$$G(\vec{r}, \vec{\rho}) = \frac{1}{2\pi} K_0(k|\vec{r} - \vec{\rho}|), \tag{13}$$

is expressed as a zeroth order modified Bessel function of the second kind.

3. Application

The proposed formulation is implemented in the FORTRAN computer program BENDQ, which is capable of solving one-group neutron diffusion equations both with external neutron sources and and critical eigenvalue search by fission source iteration. To exemplify the capability of the program in producing solutions for system with curved boundaries, we consider an infinite bare, homogeneous cylindrical system with radius R=30 cm. One-group constants are taken as: D=1.77764 cm, $\nu\Sigma_f = 0.0262573\,\mathrm{cm}^{-1}$ $\Sigma_a = 0.0143676$ cm^{-1}. The analytical effective multiplication factor of this system (k_{eff}) is given by:

$$k_{eff} = \frac{\nu\Sigma_f}{\Sigma_a + D(\frac{2.4048}{R})^2} \tag{14}$$

and is calculated as 1.01657 with the given data. Because of system symmetry, BEM discretization is applied to an octant of the system. In Figure, the system boundary is divided into 8 quadratic boundary elements; and the internal mesh, which consists of 8 isoparametric finite elements, is used only for numerical evaluation of the elements of right hand side vector in Eqn. (12). The system boundary is divided into varying number of quadratic elements and BENDQ runs are carried out. The obtained k_{eff} values and the errors associated with them relative to the analytical value are presented in Table. As the number of boundary elements is increased, the fraction of errors decrease and the convergence to the analytical value is evident. For further development along these lines, the method is in the process of extension for application to multiregion systems.

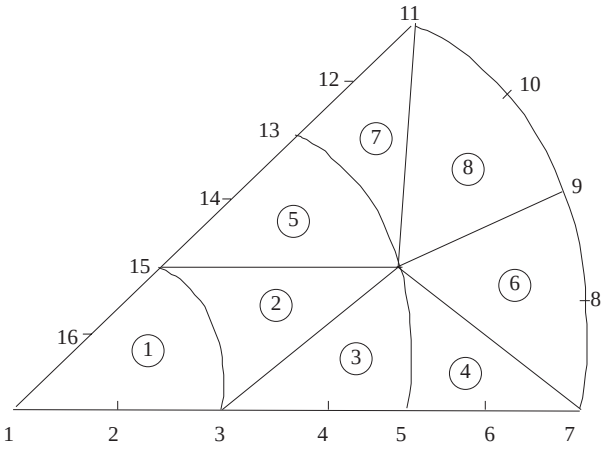

Figure. Sample Mesh

Table. Variation of the k_{eff} With Respect to the Number of Boundary Elements

No.of Boundary Elem.	k_{eff}	Per Cent Error
8	1.01043	0.604
15	1.01313	0.337
23*	1.01494	0.160
46*	1.01597	0.0590

* Only with 1/16 of the system is disretized using system symmetry

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