Turk J Phys
27 (2003) , 91 – 99.

Interaction Integrals of Extended Bodies and Convolution/Folding Operation*

Şule KEHNEMİ
Physics Department, Boğaziçi University,
İstanbul, TR-34342, TURKEY
R. Ömür AKYÜZ
Physics Department, Yeditepe University,
İstanbul, TR-81120 TURKEY

Received 20.08.2002

Abstract

The convolution, or folding, theorem related to Fourier transforms is applied to the evaluation of interaction integrals of certain extended bodies for the specific purpose of nuclear applications The integrals give analytical results for a class of functions when chosen from among uniform sharp and leptodermous spheres, delta-points, Yukawa, and Coulomb shapes. Such shapes are convenient to use as models for source density and point-point interaction.

1. Introduction

The main purpose of this work is to demonstrate a property of the so-called convolution/folding operation as a method to evaluate interaction integrals in physics. Many problems in physics involve the construction of the expression for the interaction between two extended charge (not necessarily electrical), or rather source distributions which generate potentials as per their functional forms and related two-body/point interactions (see, e.g., [1]). We shall apply these results to obtain a "realistic-looking" surface-interaction potential for two nuclei: a nucleon-nucleus potential [2, 3], and the potential energy contribution to the total energy of a nucleus in a further publication. We exhibit here the forms of the results in a series of figures (Figures 2-4), using some representetive parameters appropriate to two medium-sized nuclei, shown in Table.

The convolution/folding operation is defined as

$$(f * g)_{\mathbf{r}} \equiv (2\pi)^{-3/2} \iiint f(\mathbf{r}-\mathbf{r}')g(\mathbf{r}')\, dv' = (g * f)_{\mathbf{r}} \tag{1}$$

and appears in the study of Fourier and similar transformations. This has the following important and very useful property that [4]

$$(\mathcal{F}^{-1}(Ff \times Fg)_{\mathbf{k}})_{\mathbf{r}} = (f * g)_{\mathbf{r}}, \tag{2}$$

where $\mathcal{F}$ denotes the Fourier or similar transform operator. The integrals involved in these can easily be converted to take the same form as the very same folding process:

*Part of the Ph.D. thesis, "Folding-model Calculation of Nuclear Potential Energy" by Ş. K., unpublished, Boğaziçi University Physics Department, 2003.

$$U(\mathbf{r}) = (\rho_1 * V_{12} * \rho_2)_{\mathbf{r}}, \tag{3}$$

where $\mathbf{r}$ is the displacement between the centers of two extended body distributions, ρ_1 and ρ_2, and V_{12} is the two-body or point-point interaction potential responsible for the interaction. As mentioned above, in this work, we use the form in (2) to evaluate integrals such as (3). Conventionally, however, such problems have been usually treated by the method of multipole expansion (see, e.g., [4]). And though this method is certainly not without good purpose-it is an approximation method that has found success in more general distribtions-it has limitations wheras the distributions and interactions chosen in this work results give exact results.

2. Interactions and Distributions

We consider contact or point-delta, Yukawa and Coulomb-like potentials as two-body interactions as illustrative examples. For extended sahapes, on the other hand, we use the density distributions "uniform hard" and so-called "leptodermous" spheres. The latter are uniform hard spheres "softened" by folding-in a Yukawa shape which results in a shape quite similar to that of so-called Fermi distribution [4]: $f(r) = 1/\{1 + \exp[(r - c)/a]\}$. Such a shape has been in wide use as a model for leptodermous nuclear density distributions (Figure 1). (For similar works related to spherical distributions see, e.g., [5]). The reason why we have chosen the distributions and interactions in equations (4−9) below is because all the integrals involved in evaluating (3) by means of (2) result in *exact* analytical expressions.

Table.

R_1= 3.00 fm	A_1= 40	$\kappa_1 = 1.45$ fm^{-1}	
R_2= 4.20 fm	A_2= 60	$\kappa_2 = 1.50$ fm^{-1}	
$C = \frac{e^2}{4\pi\varepsilon_0} = 1.44$ MeV-fm		$Z_i = A_i/2$	
$w/\kappa = -58.33$ MeV-fm		$\kappa = 0.8333$ fm^{-1}	$J = -1519$ MeV-fm

*The parameters used for Figures 2-4 are comparable to those relevant to some medium-size atomic nuclei. Note that $A_{1,2}$ and $Z_{1,2}$ enter only to the normalization of ρ). (All figures are produced through Mathematica® 4.0) *dashed line*: A leptodermous sphere. *Vertical line* corresponds to R_2.

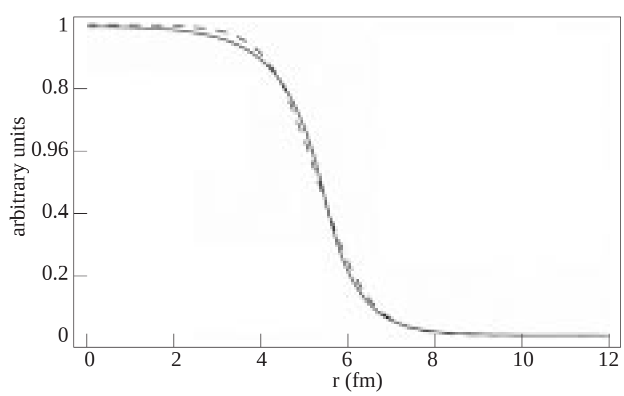

Figure 1. Two equivalent leptodermous distributions [3]. The best fit between a Yukawa-folded uniform sphere (*solid line*) and a "Fermi distribution" (*dashed line*).

The expressions for the above-mentioned spherically symmetric particle distributions, or density functions, are as follows:

A- The uniform sphere,

$$\rho_R(r) = \rho_0 \Theta_R(r) \tag{4}$$

where,

$$\begin{aligned} \Theta_R(r) = 1: &\quad 0 \leq r \leq R; \\ 0: &\quad \quad\; r > R, \end{aligned}$$

and,

B- The leptodermous sphere,

$$\rho_\kappa(r) = \rho_0 \frac{\kappa^2}{4\pi} (\Theta_R * Y_\kappa)_r \equiv \rho_0 F_\kappa(R, r), \tag{5}$$

where,

$$\begin{aligned} F_\kappa(R,r) &\equiv 1 - (1+\kappa R) e^{-\kappa R} \tfrac{S_\kappa(r)}{\kappa}: \quad r \leq R \\ &\equiv \Gamma(\kappa R) \tfrac{Y_\kappa(r)}{\kappa}: \quad\quad\quad\quad\quad r \geq R \\ S_\kappa(r) &\equiv \frac{\sinh(\kappa r)}{r}, \Gamma(x) \equiv x \cosh x - \sinh x, \end{aligned}$$

and

$$Y_\kappa(r) \equiv e^{-\kappa r}/r.$$

C- Point-delta distribution, to complete the list, to represent a particle,

$$\rho_p(r) = \delta(r). \tag{6}$$

One very interesting and useful feature of (5) is that its complete space integral is independent of κ and equal to that of (4), namely of a uniform sphere: $\rho_0 4\pi R^3/3$. This is quite advantageous in evaluating the normalization factors, i.e., ρ_0, which, for almost all of the conventional distributions (see, e.g., [6]), can only be calculated numerically and/or approximately.

The model two-body interactions we consider are:

a- Contact interactions,

$$V_P = J\delta(\vec{\mathbf{r}}), \tag{7}$$

b- Yukawa-like interactions,

$$V_Y = \frac{w}{\kappa} Y_\kappa(r), \tag{8}$$

c- Coulomb-like interactions,

$$V_C = C\frac{1}{r}. \tag{9}$$

(Here, it is good to remember that the Fourier transform of (9) is usually obtained as the long-range, i.e., the limit $\kappa \to 0$ of the Fourier transform of $Y_\kappa(r)$ (see, e.g., [7]).) We give below results for some model combinations of these interaction forms.

3. Details and Results

Using the relation (2) we derive the interaction of a uniform sphere and leptodermous sphere, individually, in contact with Yukawa and Coulomb-like potentials (Eqns. (7-9)). We then apply relation (3) to two uniform hard spheres (i.e., Eqn. (4)) and to two leptodermous spheres (i.e., Eqn. (5)), each pair interacting through on of the two-body interactions described in Eqn.s (9), (8) and (9).

Two uniform hard spheres would result in interaction potentials for pairs of two similar extended distributions, and the point-delta and leptodermous sphere would result in interaction potentials for a particle and an extended distribution. Furthermore, for two identical distributions, one half of the interaction potential at $r = 0$ would obviously give the internal potential energy of that distribution. The results of items 1 to 3 above are listed below (and depicted in Figures 2-4, where for comparison curves for uniform and leptodermous spheres are shown together).

It is, however, important to mention here that the integrations leading to the results below, although straightforward using residue theorem, are neither trivial nor short –tens of hand written pages. So, in order to save space we have not included the details of the integrations.

Note that in all the relevant expressions below, $R_2 \geq R_1$ and $G(x) \equiv \cosh x - x \sinh x$.

1) Contact interaction of two uniform hard spheres (see also Figure 2a):

$$
\begin{aligned}
(\rho_{R_1} * V_P * \rho_{R_2})_r &= \rho_{10} J \rho_{20} \tfrac{4\pi R_1^3}{3} : && r \leq R_2 - R_1 \\
&= \rho_{10} J \rho_{20} \pi [\tfrac{1}{12} r^3 - \tfrac{1}{2}(R_1^2 + R_2^2) r - (R_2^2 - R_1^2)^2 \tfrac{1}{4r} \\
&\quad + \tfrac{2}{3}(R_1^3 + R_2^3)] : && R_2 - R_1 \leq r \leq R_1 + R_2; \\
&= 0 : && r \geq R_1 + R_2
\end{aligned}
$$

2) Yukawa interaction of two uniform hard spheres (see also Figure 3a):

$$
\begin{aligned}
(\rho_{R_1} * V_Y * \rho_{R_2})_r &= \rho_{10} \tfrac{w}{\kappa} \rho_{20} \tfrac{8\pi^2}{\kappa^2} \{ \tfrac{2}{3} R_1^3 - \tfrac{(\kappa R_2 + 1)}{\kappa^4} [e^{-\kappa(R_1+R_2)}(\kappa R_1 + 1) \\
&\quad + e^{-\kappa(R_2 - R_1)}(\kappa R_1 - 1)] S_\kappa(r) \} : \qquad r \leq R_2 - R_1; \\
&= \rho_{10} \tfrac{w}{\kappa} \rho_{20} \tfrac{4\pi^2}{\kappa^2} \{ \tfrac{2}{3}(R_1^3 + R_2^3) + \tfrac{1}{r} [\tfrac{2}{\kappa^4} - \tfrac{1}{4}(R_2^2 - R_1^2)^2 - \tfrac{(R_1^2 + R_2^2)}{\kappa^2}] \\
&\quad + [\tfrac{1}{\kappa^2} - \tfrac{1}{2}(R_2^2 + R_1^2)] r + \tfrac{r^3}{12} \\
&\quad + \tfrac{2}{\kappa^4} \{ (\kappa^2 R_1 R_2 \cosh \kappa (R_2 - R_1) \\
&\quad - G[\kappa(R_2 - R_1)] \} Y_\kappa(r) \\
&\quad - \tfrac{2 e^{-\kappa(R_1+R_2)}}{\kappa^4} (\kappa R_1 + 1)(\kappa R_2 + 1) S_\kappa(r) \} : R_2 - R_1 \leq r \leq R_1 + R_2; \\
&= \rho_{10} \tfrac{w}{\kappa} \rho_{20} \tfrac{16\pi^2}{\kappa^6} Y_\kappa(r) \Gamma(\kappa R_1) \Gamma(\kappa R_2) : \qquad r \geq R_2 + R_1.
\end{aligned}
$$

3) Coulomb-like interaction of two uniform hard spheres (this well known form constitutes a check on Yukawa results in the limit: $\kappa \to 0$) (see also Figure 4a):

$$
\begin{aligned}
(\rho_{R_1} * V_C * \rho_{R_2})_r &= \rho_{10} C \rho_{20} \tfrac{8\pi^2}{3}(R_1^3 R_2^2 - \tfrac{1}{5}R_1^5 - -\tfrac{1}{3}R_1^3 r^2) : && r \le R_2 - R_1; \\
&= \rho_{10} C \rho_{20} \tfrac{4\pi^2}{3}[(R_1^3 R_2^2 + R_1^2 R_2^3) - \tfrac{1}{5}(R_1^5 + R_2^5) \\
&\quad -\tfrac{3}{8r}(R_1^2 R_2^4 + R_1^4 R_2^2) + \tfrac{2}{3r} R_1^3 R_2^3 + \tfrac{1}{24r}(R_1^6 + R_2^6) \\
&\quad +\tfrac{3}{8}(R_2^2 - R_1^2)^2 r - \tfrac{1}{3}(R_1^3 + R_2^3) r^2 \\
&\quad +\tfrac{1}{8}(R_1^2 + R_2^2) r^3 - \tfrac{1}{120} r^5] : && R_2 - R_1 \le r \le R_1 + R_2; \\
&= \rho_{10} C \rho_{20} R_1^3 R_2^3 \tfrac{16\pi^2}{9r} : && r \ge R_1 + R_2;
\end{aligned}
$$

4) Contact interaction of two leptodermous spheres (see also Figure 2a):

$$
\begin{aligned}
(\rho_{R_1} * V_p * \rho_{R_2})_r &= \rho_{10} J \rho_{20} \kappa_1^2 \kappa_2^2 2\pi \{ \tfrac{2R_1^3}{3\kappa_1^2\kappa_2^2} - \tfrac{(\kappa_1 R_2 + 1)}{\kappa_1^6(\kappa_2^2 - \kappa_1^2)} S_{\kappa_1}(r) [e^{-\kappa_1(R_2+R_1)}(\kappa_1 R_1 + 1) \\
&\quad + e^{-\kappa_1(R_2 - R_1)}(\kappa_1 R_1 - 1)] - \tfrac{(\kappa_2 R_2 + 1)}{\kappa_2^6(\kappa_1^2 - \kappa_2^2)} S_{\kappa_2}(r) [e^{-\kappa_2(R_2+R_1)}(\kappa_2 R_1 + 1) \\
&\quad + e^{-\kappa_2(R_2 - R_1)}(\kappa_2 R_1 - 1)]\} : \qquad r \le R_2 - R_1, \\
&= \rho_{10} J \rho_{20} \kappa_1^2 \kappa_2^2 \pi \{ \tfrac{2(R_1^3 + R_2^3)}{3\kappa_1^2\kappa_2^2} - \tfrac{(R_1^2+R_2^2)(\kappa_1^2+\kappa_2^2)}{\kappa_1^4\kappa_2^4 r} - \tfrac{(R_2^2 - R_1^2)^2}{4\kappa_1^2\kappa_2^2 r} \\
&\quad + \tfrac{2(\kappa_1^2+\kappa_2^2)^2}{\kappa_1^6\kappa_2^6 r} - \tfrac{2}{\kappa_1^4\kappa_2^4 r} - \tfrac{(R_1^2+R_2^2) r}{2\kappa_1^2\kappa_2^2} + \tfrac{(\kappa_1^2+\kappa_2^2) r}{\kappa_1^4\kappa_2^4} + \tfrac{r^3}{12\kappa_1^2\kappa_2^2} \\
&\quad + \tfrac{2}{\kappa_1^6(\kappa_2^2 - \kappa_1^2)} [Y_{\kappa_1}(r) \{ (R_1 R_2 \kappa_1^2 \cosh \kappa_1 (R_2 - R_1) \\
&\qquad - G[\kappa_1(R_2 - R_1)] \\
&\qquad - e^{-\kappa_1(R_1+R_2)}(\kappa_1 R_1 + 1)(\kappa_1 R_2 + 1) S_{\kappa_1}(r)] \\
&\quad + \tfrac{2}{\kappa_2^6(\kappa_1^2 - \kappa_2^2)} [Y_{\kappa_2}(r) \{ (R_1 R_2 \kappa_2^2 \cosh \kappa_2 (R_2 - R_1) \\
&\qquad - G[\kappa_2(R_2 - R_1)] \\
&\qquad - e^{-\kappa_2(R_1+R_2)}(\kappa_2 R_1 + 1)(\kappa_2 R_2 + 1) S_{\kappa_2}(r)]\} : R_2 - R_1 \le r \le R_1 + R_2
\end{aligned}
$$

5) Yukawa interaction of two leptodermous spheres (see also Figure 3a):

$$
\begin{aligned}
(\rho_{R_1} * V_p * \rho_{R_2})_r &= \rho_{10} \tfrac{w}{\kappa} \rho_{20} \kappa_1^2 \kappa_2^2 8\pi^2 \\
&\quad \times \{ \tfrac{2R_1^3}{3\kappa^2\kappa_1^2\kappa_2^2} - \tfrac{(\kappa R_2 + 1)}{\kappa^6(\kappa_1^2 - \kappa^2)(\kappa_2^2 - \kappa^2)} S_\kappa(r) \\
&\quad \times [e^{-\kappa(R_2+R_1)}(\kappa R_1 + 1) + e^{-\kappa(R_2 - R_1)}(\kappa R_1 - 1)] \\
&\quad - \tfrac{(\kappa_1 R_2 + 1)}{\kappa_1^6(\kappa^2 - \kappa_1^2)(\kappa_2^2 - \kappa_1^2)} S_{\kappa 1}(r) \\
&\quad \times [e^{-\kappa_1(R_2+R_1)}(\kappa_1 R_1 + 1) + e^{-\kappa_1(R_2 - R_1)}(\kappa_1 R_1 - 1)] \\
&\quad - \tfrac{(\kappa_2 R_2 + 1)}{\kappa_1^6(\kappa^2 - \kappa_1^2)(\kappa_2^2 - \kappa_1^2)} S_{\kappa 2}(r) \\
&\quad \times [e^{-\kappa_2(R_2+R_1)}(\kappa_2 R_1 + 1) + e^{-\kappa_2(R_2 - R_1)}(\kappa_2 R_1 - 1)]\} : r \le R_2 - R_1,
\end{aligned}
$$

$$
\begin{aligned}
&= \rho_{10} J \rho_{20} \kappa_1^2 \kappa_2^2 4\pi [\tfrac{\Gamma(\kappa_1 R_1)\Gamma(\kappa_1 R_2)}{\kappa_1^6(\kappa_2^2-\kappa_1^2)} Y_{\kappa_1}(r) + \tfrac{\Gamma(\kappa_2 R_1)\Gamma(\kappa_2 R_2)}{\kappa_2^6(\kappa_1^2-\kappa_2^2)} Y_{\kappa_2}(r)] : r \geq R_1 + R_2 \\
&= \rho_{10} \tfrac{w}{\kappa} \rho_{20} 4\pi^2 \kappa_1^2 \kappa_2^2 \{ \tfrac{2(R_1^3+R_2^3)}{3\kappa^2\kappa_1^2\kappa_2^2} + \tfrac{r^3}{12\kappa^2\kappa_1^2\kappa_2^2} - \tfrac{(R_1^2+R_2^2)r}{2\kappa^2\kappa_1^2\kappa_2^2} \\
&\quad + \tfrac{(\kappa^2\kappa_1^2+\kappa_2^2\kappa_1^2+\kappa^2\kappa_2^2)}{\kappa^4\kappa_1^4\kappa_2^4}[r - \tfrac{(R_1^2+R_2^2)}{r}] - \tfrac{(R_2^2-R_1^2)^2}{4\kappa^2\kappa_1^2\kappa_2^2 r} \\
&\quad + \tfrac{2(\kappa^2\kappa_1^2+\kappa_2^2\kappa_1^2+\kappa^2\kappa_2^2)^2}{\kappa^6\kappa_1^6\kappa_2^6 r} - \tfrac{2(\kappa^2+\kappa_1^2+\kappa_2^2)}{\kappa^4\kappa_1^4\kappa_2^4 r} \\
&\quad - \tfrac{2(\kappa R_1+1)(\kappa R_2+1)e^{-\kappa(R_1+R_2)}}{\kappa^6(\kappa_1^2-\kappa^2)(\kappa_2^2-\kappa^2)} S_\kappa(r) \\
&\quad - \tfrac{2(\kappa_1 R_1+1)(\kappa_1 R_2+1)e^{-\kappa_1(R_1+R_2)}}{\kappa_1^6(\kappa^2-\kappa_1^2)(\kappa_2^2-\kappa_1^2)} S_{\kappa_1}(r) \\
&\quad - \tfrac{2(\kappa_2 R_1+1)(\kappa_2 R_2+1)e^{-\kappa_2(R_1+R_2)}}{\kappa_2^6(\kappa^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)} S_{\kappa_2}(r) \\
&\quad + \tfrac{2[\kappa^2 R_1 R_2 \cosh\kappa(R_2-R_1) - G[\kappa(R_2-R_1)]}{\kappa^6(\kappa_1^2-\kappa^2)(\kappa_2^2-\kappa^2)} Y_\kappa(r) \\
&\quad + \tfrac{2[\kappa_1^2 R_1 R_2 \cosh\kappa_1(R_2-R_1) - G[\kappa_1(R_2-R_1)]}{\kappa_1^6(\kappa^2-\kappa_1^2)(\kappa_2^2-\kappa_1^2)} Y_{\kappa_1}(r) \\
&\quad + \tfrac{2[\kappa_2^2 R_1 R_2 \cosh\kappa_2(R_2-R_1) - G[\kappa_2(R_2-R_1)]}{\kappa_2^6(\kappa^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)} Y_{\kappa_2}(r) \} : \qquad R_2 - R_1 \leq r \leq R_1 + R_2 \\
&= \rho_{10} \tfrac{w}{\kappa} \rho_{20} 16\pi^2 \kappa_1^2 \kappa_2^2 [\tfrac{\Gamma(\kappa R_2)\Gamma(\kappa R_1)}{\kappa^6(\kappa_1^2-\kappa^2)(\kappa_2^2-\kappa^2)} Y_\kappa(r) \\
&\quad + \tfrac{\Gamma(\kappa_1 R_2)\Gamma(\kappa_1 R_1)}{\kappa_1^6(\kappa^2-\kappa_1^2)(\kappa_2^2-\kappa_1^2)} Y_{\kappa_1}(r) \\
&\quad + \tfrac{\Gamma(\kappa_2 R_2)\Gamma(\kappa_2 R_1)}{\kappa_2^6(\kappa^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)} Y_{\kappa_2}(r)] : \qquad R_1 + R_2 \leq r
\end{aligned}
$$

6) Coulomb-like interaction of two leptodermous spheres (see also Figure 4a):

$$
\begin{aligned}
(\rho_{R_1} * V_C * \rho_{R_2})_r &= \rho_{10} C \rho_{20} 8\pi^2 \kappa_1^2 \kappa_2^2 \\
&\quad \times \{ \tfrac{R_1^3 R_2^2 - \frac{1}{5} R_1^5}{3\kappa_1^2\kappa_2^2} - \tfrac{2R_1^3(\kappa_1^2+\kappa_2^2)}{3\kappa_1^4\kappa_2^4} - \tfrac{R_1^3 r^2}{9\kappa_1^2\kappa_2^2} \\
&\quad + \tfrac{(\kappa_1 R_2+1)}{\kappa_1^8(\kappa_2^2-\kappa_1^2)}[(\kappa_1 R_1+1)e^{-\kappa_1(R_1+R_2)} + (\kappa_1 R_1 - 1)e^{-\kappa_1(R_2-R_1)}] S_{\kappa_1}(r) \\
&\quad + \tfrac{(\kappa_2 R_2+1)}{\kappa_2^8(\kappa_1^2-\kappa_2^2)}[(\kappa_2 R_1+1)e^{-\kappa_2(R_1+R_2)} + (\kappa_2 R_1 - 1)e^{-\kappa_2(R_2-R_1)}] S_{\kappa_2}(r) \} : \\
&\quad r \leq R_2 - R_1 \\
&= \rho_{10} C \rho_{20} 4\pi^2 \kappa_1^2 \kappa_2^2 \times \\
&\quad \{ \tfrac{1}{\kappa_1^2\kappa_2^2}[\tfrac{1}{3}(R_1^3 R_2^2 + R_1^2 R_2^3) - \tfrac{1}{15}(R_1^5 + R_2^5) - \tfrac{\kappa_1^2+\kappa_2^2}{\kappa_1^2\kappa_2^2}\{\tfrac{2}{3}(R_1^3+R_2^3) + \tfrac{1}{12} r^3 \\
&\quad - \tfrac{1}{2}(R_1^2+R_2^2) r - \tfrac{(R_2^2-R_1^2)^2}{4r}\} - \tfrac{r^5}{360} + \tfrac{1}{24}(R_1^2+R_2^2) r^3 - \tfrac{1}{9}(R_1^3+R_2^3) r^2 \\
&\quad + \tfrac{1}{8}(R_2^2-R_1^2)^2 r + \tfrac{R_1^6+R_2^6}{72r} + \tfrac{2R_1^3R_2^3}{9r} - \tfrac{(R_1^2R_2^4+R_1^4R_2^2)}{8r} \\
&\quad + \tfrac{\kappa_1^4+\kappa_2^4+\kappa_1^2\kappa_2^2}{\kappa_1^4\kappa_2^4}(\tfrac{R_1^2+R_2^2}{r} - r) - \tfrac{\kappa_1^6+\kappa_1^4\kappa_2^2+\kappa_1^2\kappa_2^4+\kappa_2^6}{\kappa_1^6\kappa_2^6}\tfrac{2}{r}] \\
&\quad + 2\tfrac{(\kappa_1 R_1+1)(\kappa_1 R_2+1)e^{-\kappa_1(R_1+R_2)}}{\kappa_1^8(\kappa_2^2-\kappa_1^2)} S_{\kappa_1}(r) + 2\tfrac{(\kappa_2 R_1+1)(\kappa_2 R_2+1)e^{-\kappa_2(R_1+R_2)}}{\kappa_2^8(\kappa_1^2-\kappa_2^2)} S_{\kappa_2}(r)
\end{aligned}
$$

$$
\begin{aligned}
& -2\tfrac{\kappa_1^2 R_1 R_2 \cosh\kappa_1(R_2-R_1)-G[\kappa_1(R_2-R_1)]}{\kappa_1^8(\kappa_2^2-\kappa_1^2)} Y_{\kappa_1}(r) \\
& -2\tfrac{\kappa_2^2 R_1 R_2 \cosh\kappa_2(R_2-R_1)-G[\kappa_2(R_2-R_1)]}{\kappa_2^8(\kappa_1^2-\kappa_2^2)} Y_{\kappa_2}(r)\} : R_2 - R_1 \leq r \leq R_1 + R_2 \\
= \quad & \rho_{10} C \rho_{20} 16\pi^2 \kappa_1^2 \kappa_2^2 [\tfrac{R_1^3 R_2^3}{9\kappa_1^2\kappa_2^2 r} - \tfrac{\Gamma(\kappa_1 R_1)\Gamma(\kappa_1 R_2)}{\kappa_1^8(\kappa_2^2-\kappa_1^2)} Y_{\kappa_1}(r) \\
& -\tfrac{\Gamma(\kappa_2 R_1)\Gamma(\kappa_2 R_2)}{\kappa_2^8(\kappa_1^2-\kappa_2^2)} Y_{\kappa_2}(r)] : R_1 + R_2 \leq r.
\end{aligned}
$$

7) Contact potential of a uniform sphere (see also Figure 2b):

$$
\begin{aligned}
(V_P * \rho_R)_r = J\rho_0 : & \qquad r \leq R \\
= 0 : & \qquad r \geq R.
\end{aligned}
$$

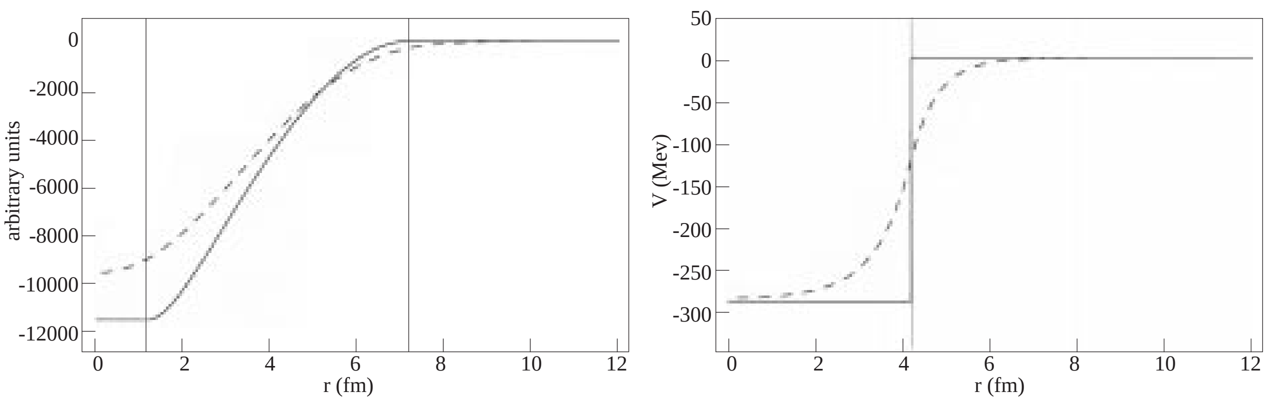

Figure 2. a) Contact interaction between two extended distributions. *Solid line*: Two uniform hard spheres; *dashed line*: Two leptodermous spheres. The vertical line pairs represent the positions corresponding to $R_2 - R_1$ and $R_2 + R_1$. b) Contact potential between an extended distribution and a particle. *Solid line*: A uniform hard sphere; *dashed line*: A leptodermous sphere. *Vertical line* corresponds to R_2.

8) Yukawa potential of a uniform sphere (see also Figure 3b):

$$
(V_\kappa * \rho_R)_r = \frac{w}{\kappa}\rho_0 \frac{4\pi F_\kappa(R,r)}{\kappa^2}.
$$

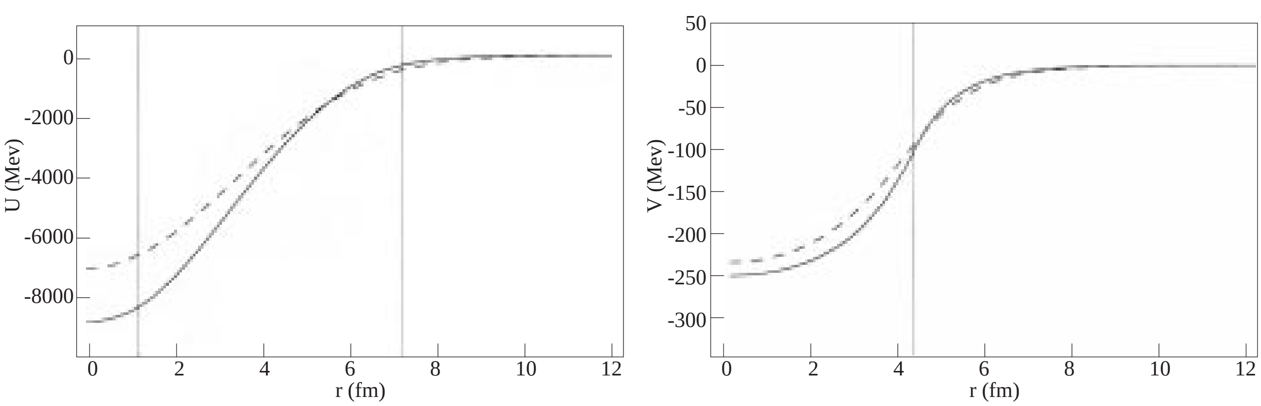

Figure 3. a) Yukawa interaction between two extended distributions. *Solid line*: Two uniform hard spheres; *dashed line*: Two leptodermous spheres. The vertical line pair represent the positions corresponding to $R_2 - R_1$ and $R_2 + R_1$. b) Yukawa potential between a leptodermous sphere and a particle. *Solid line*: A uniform hard sphere; *dashed line*: A leptodermous sphere. *Vertical line* corresponds to R_2.

9) Coulomb potential of a uniform sphere (these well-known results −obtained here by the same method as the others− are included for completeness, comparison, and a check.)

(Figure 4b):

$$
\begin{aligned}
(V_C * \rho_R)_r &= C\rho_0 \tfrac{4\pi R^2}{3}\tfrac{1}{2}(3 - \tfrac{r^2}{R^2}) : && r \leq R \\
&= C\rho_0 \tfrac{4\pi R^3}{3r} : && r \geq R.
\end{aligned}
$$

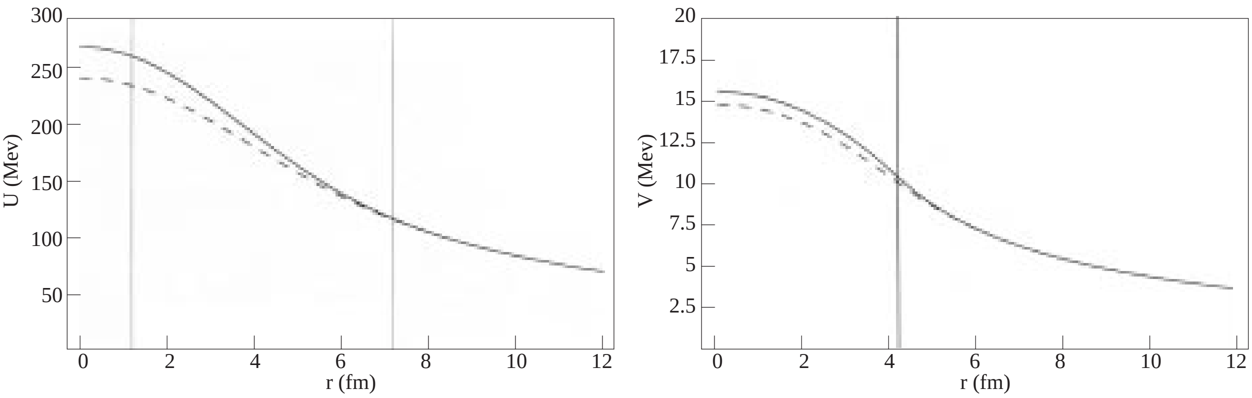

Figure 4. a) Coulomb interaction between two extended distributions. *Solid line*: Two uniform hard spheres; *dashed* line: Two leptodermous spheres. The vertical line pairs represent the positions corresponding to $R_2 - R_1$ and $R_2 + R_1$. b) Coulomb potential between a leptodermous sphere and a particle. *Solid line*: A uniform hard sphere.

10) Contact potential of a leptodermous sphere (Figure 2b):

$$(V_P * \rho_{\kappa 1})_r = J\rho_{10} F_{\kappa_1}(R_1, r).$$

11) Yukawa potential of a leptodermous sphere (Figure 3b):

$$(V_Y * \rho_{\kappa 1})_r = \frac{w}{\kappa}\rho_{10}\frac{4\pi}{(\kappa^2 - \kappa_1^2)}[\frac{F_{\kappa_1}(R_1, r)}{\kappa_1^2} - \frac{F_\kappa(R_2, r)}{\kappa^2}].$$

12) Coulomb potential of a leptodermous sphere (Figure 4b):

$$
\begin{aligned}
(V_C * \rho_{\kappa 1})_r &= C\rho_{10} 4\pi[\tfrac{R_1^2}{2} - \tfrac{r^2}{6} - \tfrac{F_{\kappa_1}(R_1,r)}{\kappa_1^2}] : && r < R_1 \\
&= C\rho_{10} 4\pi[\tfrac{R_1^3}{3r} - \tfrac{F_{\kappa_1}(R_1,r)}{\kappa_1^2}] : && r \geq R_1.
\end{aligned}
$$

Acknowledgement

We appreciate the corrections and suggestions of the referee.

References

[1] See, e.g., Jackson, J.D., *Classical Electrodynamics*, John Wiley & Sons Inc., NY 1966, pp. 8, 211, 141, 198.

[2] Satchler, G.R. and W.G. Love, “Folding Model Potentials from Realistic Interactions for Heavy Ion Scattering”, *Phys. Lett.* **C55,** (1979), 183-254.

[3] Akyüz, R.Ö. and Aa. Winther, “Nuclear Surface-Surface Interaction in the Folding Model”, in *Nuclear Structure and Heavy-Ion Collisions*, ed. R. Broglia & C. Dasso, Soc. Ital. Fisica, Bologna, 1981, pp. 492-508. Note that the equations A.15 & A.15 in page 506 therein are incomplete due to an unwarranted omission.

[4] See, e.g., Margenau, H. and G.M. Murphy, *The Mathematics of Physics and Chemistry*, D. Van Nostrand Co. Inc., NY 1956, 2. ed., pp. 261-263.

[5] C.W. Wong, *Introduction to Mathematical Physics*, Oxford University Press, NY 1991, pp. 158-159.

[6] Gauthier, N., "Yukawa potential approach to the nuclear binding energy formula" *Am. J. Phys.* **58**, (1990), 375-378, and Gauthier, N. and S. Sherrit, "Nuclear potential in Yukawa model", *Am. J. Phys.* **59**, (1991), 1144-1146.

[7] See, e.g., Segre, E., *Nuclei and Particles*, W. A. Benjamin, Inc., NY 1965, pp. 199-201, and Bohr, Aa. and B. Mottelson, *Nuclear Structure*, W. A. Benjamin, Inc., NY 1969, pp. 159-160.

[8] See, e.g., Gottfried, K., *Quantum Mechanics*, W. A. Benjamin, Inc., NY 1966, pp. 108-109.