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Clebsch-Gordan Equalities that imply the Vanishing of Particular $6j$ Symbols

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Abstract

Three Clebsch-Gordan equalities and three individual Clebsh-Gordan relations are obtained by studying the non-accidental vanishing of certain $6j$ symbols.

Key Words: Clebsch-Gordan coefficients, 6j symbols, Regge symmetries, angular momenta.

1. Introduction

In two interesting papers [1, 2] involving nuclear shell model calculations, Robinson and Zamick show that certain $6j$ and $9j$ symbols vanish and that this vanishing is not accidental. In particular, they point out that

$$\begin{Bmatrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{Bmatrix} = 0,$$

for both integer and half integer values of j. By using Regge [3] symmetries they also show this particular $6j$ is isomorphic with other $6j$ symbols which are therefore also zero.

2. Formalism

In the present note we examine what the result

$$\begin{Bmatrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{Bmatrix} = 0$$

implies at the level of Clebsch-Gordan coefficients and show that unexpected equalities, involving these coefficients, arise.

The $6j$ coefficient

$$\begin{Bmatrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{Bmatrix}$$

is related to the sum over four Clebsch-Gordan coefficients as follows:

$$\begin{aligned} \begin{Bmatrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{Bmatrix} &= \frac{1}{\sqrt{(4j-5)(4j-1)}} \\ \times \quad & \sum_{m_a, m_b} (j\ j\ m_a\ m_b \mid 2j{-}3\ m_a{+}m_b)\,(2j{-}3\ j\ m_a{+}m_b\ M{-}ma{-}m_b \mid 3j{-}4\ M) \\ \times \quad & (j\ j\ m_b\ M{-}m_a{-}m_b \mid 2j{-}1\ M{-}m_a)\,(j\ 2j{-}1\ m_a\ M{-}m_a \mid 3j{-}4\ M)\,, \end{aligned} \tag{1}$$

where $-|3j-4| \leq M \leq 3j-4$. Since this expression is independent of M, in what follows we choose $M = 3j-4$. To understand why the double sum expression (1) is zero, at first seems a daunting task. For an arbitrary j, $(2j+1)^2$ terms are involved. Thus, for $j = 19/2$ this amounts to 400 terms.

A careful analysis however shows that, independent of j, only six non-zero terms arise in expression (1). The Clebsch-Gordan coefficient $(2j-3 \;\; j \;\; m_a+m_b \;\; 3j-4-m_a-m_b \mid 3j-4 \;\; 3j-4)$ requires that m_a+m_b must equal $2j-3$ or $2j-4$. Otherwise it is zero. This is only possible if $m_a = j,\ m_b = j-3$; $m_a = j-1,\ m_b = j-2$; $m_a = j-2,\ m_b = j-1$; $m_a = j-3,\ m_b = j$; or $m_a = j,\ m_b = j-4$; $m_a = j-1,\ m_b = j-3$; $m_a = j-2,\ m_b = j-2$; $m_a = j-3, m_b = j-1$; and $m_a = j-4, m_b = j$.

Thus, there are only nine combinations that in principle can contribute to expression (1). Of these, two are accidentally zero, namely $m_a = j-2,\ m_b = j-2$, since $(j \; j \; j-2 \; j-2 \mid 2j-3 \; 2j-4)$ is zero, and $m_a = j-2,\ m_b = j-1$ since $(j \; j \; j-1 \; j-1 \mid 2j-1 \; 2j-2)$ is zero, and the combination $m_a = j-4,\ m_b = j$, is zero since $(j \; j \; j \; j \mid 2j-1 \; 2j)$ involves a projection $2j$ for an angular momentum $2j-1$. Thus, for any j, only six pairs (m_a, m_b) contribute, namely: $(j,\ j-3)$, $(j,\ j-4)$, $(j-1,\ j-2)$, $(j-1,\ j-3)$, $(j-3,\ j)$, $(j-3,\ j-1)$.

An analysis of the first two pairs $(j,\ j-3)$, $(j,\ j-4)$ in the sum of Eqn. (1) leads to the following expression:

$$\begin{aligned}
&[\,(j\;j\;j\;j-3 \mid 2j-3\;2j-3)\,(2j-3\;j\;2j-3\;j-1 \mid 3j-4\;3j-4)\,(j\;j\;j-3\;j-1 \mid 2j-1\;2j-4)\\
&+\;(j\;j\;j\;j-4 \mid 2j-3\;2j-4)\,(2j-3\;j\;2j-4\;j \mid 3j-4\;3j-4)\,(j\;j\;j-4\;j \mid 2j-1\;2j-4)\,]\\
&\times\;(j\;2j-1\;j\;2j-4 \mid 3j-4\;3j-4)\;.
\end{aligned} \tag{2}$$

But it can be shown that the product

$$\begin{aligned}
&(j\;j\;j\;j-3 \mid 2j-3\;2j-3)\,(2j-3\;j\;2j-3\;j-1 \mid 3j-4\;3j-4)\,(j\;j\;j-3\;j-1 \mid 2j-1\;2j-4)\\
=\;& -\frac{j}{(4j-3)}\sqrt{\frac{2j-3}{6(j-1)}}\\
=\;& -(j\;j\;j\;j-4 \mid 2j-3\;2j-4)\,(2j-3\;j\;2j-4\;j \mid 3j-4\;3j-4)\,(j\;j\;j-4\;j \mid 2j-1\;2j-4)\,.
\end{aligned} \tag{3}$$

Hence the sum of the contributions of this pair vanishes.

Similarly, the sum of the pairs $(j-1,\ j-2)$ and $(j-1,\ j-3)$ vanishes because of the identity

$$\begin{aligned}
&(j\;j\;j-1\;j-2 \mid 2j-3\;2j-3)\,(2j-3\;j\;2j-3\;j-1 \mid 3j-4\;3j-4)\,(j\;j\;j-2\;j-1 \mid 2j-1\;2j-3)\\
&= \frac{\sqrt{j(2j-3)}}{2(4j-3)}\\
&= -(j\;j\;j-1\;j-3 \mid 2j-3\;2j-4)\,(2j-3\;j\;2j-4\;j \mid 3j-4\;3j-4)\,(j\;j\;j-3\;j \mid 2j-1\;2j-3)\,;
\end{aligned} \tag{4}$$

and the sum of the pairs $(j-3,\ j)$ and $(j-3,\ j-1)$ vanishes because of the identity:

$$\begin{aligned}
&(j\;j\;j-3\;j \mid 2j-3\;2j-3)\,(2j-3\;j\;2j-3\;j-1 \mid 3j-4\;3j-4)\,(j\;j\;j\;j-1 \mid 2j-1\;2j-1)\\
&= -\frac{1}{2}\sqrt{\frac{j(2j-3)}{3(j-1)(4j-3)}}\\
&= -(j\;j\;j-3\;j-1 \mid 2j-3\;2j-4)\,(2j-3\;j\,2j-4\;j \mid 3j-4\;3j-4)\,(j\;j\;j-1\;j \mid 2j-1\;2j-1)\,.
\end{aligned} \tag{5}$$

General results, involving relations between single Clebsch-Gordan Coefficients, follow by combining expressions (3), (4), and (5). In particular, combining Eqns. (3) and (5), one obtains

$$(j\ j\ j{-}3\ j{-}1 \mid 2j{-}1\ 2j{-}4) = -\sqrt{\frac{2j}{(4j-3)}}(j\ j\ j\ j{-}1 \mid 2j{-}1\ 2j{-}1), \tag{6}$$

while combining Eqns. (4) and (5) yields

$$(j\ j\ j{-}3\ j \mid 2j{-}1\ 2j{-}3) = -\sqrt{\frac{3(j-1)}{(4j-3)}}(j\ j\ j\ j{-}1 \mid 2j{-}1\ 2j{-}1). \tag{7}$$

Combining the results of Eqns. (6) and (7) one obtains

$$(j\ j\ j{-}3\ j{-}1 \mid 2j{-}1\ 2j{-}4) = \sqrt{\frac{2j}{3(j-1)}}(j\ j\ j{-}3\ j \mid 2j{-}1\ 2j{-}3). \tag{8}$$

It is of interest to compare these results with those obtained using the symmetries of Regge's elegant expression [4].

Written in terms of $3-j$'s Eqn. (8) becomes:

$$\begin{pmatrix} j & j & 2j-1 \\ j-3 & j-1 & -2j+4 \end{pmatrix} = \sqrt{\frac{2j}{3(j-1)}} \begin{pmatrix} j & j & 2j-1 \\ j-3 & j & -2j+3 \end{pmatrix}.$$

From Regge's symmetries (aside from the standard relations one obtains if one interchanges rows, or changes the signs of the projections), one obtains additionally

$$\begin{pmatrix} j & j & 2j-1 \\ j-3 & j-1 & -2j+4 \end{pmatrix} = \begin{pmatrix} 2j-2 & 2j-1 & 2 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} j & 2j-2 & j+1 \\ 1-j & 2j-3 & 2-j \end{pmatrix}.$$

These involve different angular momenta, whereas Eqns. (6)–(8) involve different projections for the same j's. The three relations, Eqns. (6)–(8), can be verified by construction, starting with the state $\psi^{2j}_{2j} = \phi^j_j \eta^j_j$, using lowering operators to obtain the state $\psi^{2j}_{2j-1} = \sqrt{\frac{1}{2}}\left(\phi^j_j \eta^j_{j-1} + \phi^j_{j-1}\eta^j_j\right)$, constructing the orthogonal state ψ^{2j-1}_{2j-1}, and finally using lowering operators to obtain ψ^{2j-1}_{2j-2}, ψ^{2j-1}_{2j-3}, and ψ^{2j-1}_{2j-4}.

3. Conclusions

It has been shown that three unexpected Clebsch-Gordan equalities, namely Eqns. (3), (4), (5) lead to the non-accidental vanishing of the $6j$ symbol:

$$\begin{Bmatrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{Bmatrix}$$

that can generally be expressed as a sum of six terms. These three relations are also of some interest by themselves and lead to general relations between individual Clebsch-Gordan coefficients namely Eqns. (6), (7) and (8).

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References

[1] S. J. Q. Robinson and L. Zamick, Degeneracies when T=0 Two Body Interaction Matrix Elements are Set Equal to Zero and Regge's 6j Symmetry Relations, Physical Review C, **63**, (2001), 64316.

[2] S. J. Q. Robinson and L. Zamick, Degeneracies when T=0 Two Body Interaction Matrix Elements are Set Equal to Zero: Talmi's method of calculating coefficients of fractional parentage to states forbidden by the Pauli Principle, Physical Review C, **64**, (2001), 57302.

[3] T. Regge, Il Nuovo Cimento, Vol. XI, N. 1, (1959), 298–299.

[4] T. Regge, Il Nuovo Cimento, Vol. X, N. 3, (1959), 296–297.