

Null Membranes in Dyonic Black Hole Background

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Abstract

We consider null bosonic p -branes in curved spacetimes. The general solutions of the classical equations of motions and constraints for the null-membrane in four-dimensional dyonic black hole background are found.

Key Words: general solution, null-membrane, curved space-time.

1. Introduction

The null (tensionless) p -branes correspond to usual p -branes with their tension T taken to be zero. This relationship between null p -branes and the tensionful ones may be regarded as a generalization of the massless-massive particles correspondence. Physically, the *limit* $T \rightarrow 0$ corresponds to the energetic scale with $E \gg M_{planck}$ [1]. In other words, the null p -brane is the high temperature phase of p -brane theory which corresponds to the time period of the Early Universe and Big Band [2].

The study of null p -brane dynamics in curved spacetimes, reveals new insights with respect to null p -brane propagation in flat space-time [3]. For example, it has been shown [4] that an energy momentum tensor describing a fluid of null membrane can act as a source for metrics representing Friedman-Robertson-Walker universes in both its matter and radiation dominated epochs.

The classical evolution of a null p -brane in curved background is described by null geodesic equations of general relativity appended by an additional constraints [5]. A very interesting exact solutions of the classical equations of motion and constraints for the null membrane in a general stationary, axially symmetric, four dimensional gravity background were presented by Bozhilov [6]-[8].

Here we investigate the classical evolution of the null p -branes in a curved background. In sec. 2 we give the corresponding Lagrangian formulation. In sec. 3 the general solutions of the classical equations of motion and constraints for the null membrane in four-dimensional dyonic black hole background are found.

2. Lagrange theory of null p -brane

The action for null p -branes in a cosmological background $G_{MN}(x)$ can be written as [2, 4, 9]

$$S = \int d^{p+1}\xi \frac{\det(\partial_\mu x^M G_{MN} \partial_\nu x^N)}{E(\tau, \sigma^n)}, \quad (1)$$

where $M, N = 0, 1, \dots, D-1; \mu, \nu = 0, 1, \dots, p$ are indices of the hyperworldsheet of the null p -brane, $(\xi = (\tau, \sigma_i), \sigma_i = (\sigma_1, \sigma_2, \dots, \sigma_{p-1}))$ and $E(\tau, \sigma_i)$ is a $(p+1)$ -dimensional hyperworldsheet density which plays

the role of Lagrange multiplier analogous to $e(\tau)$ in the action of massless particle [1]. The determinant g of the induced null p -brane metric $g_{\mu\nu}$ is

$$g_{\mu\nu} = \partial_\mu x^M G_{MN} \partial_\nu x^N = \begin{pmatrix} \dot{x}^A G_{AB}(x) \dot{x}^B & \dot{x}^A G_{AB}(x) \partial_n x^B \\ \partial_m x^A G_{AB}(x) \dot{x}^B & \hat{g}_{mn}(x) \end{pmatrix}, \quad (2)$$

where

$$\hat{g}_{mn}(x) = \partial_m x^A G_{AB}(x) \partial_n x^B$$

may be presented in a factorized form as

$$\begin{aligned} g &= \dot{x}^M \tilde{\Pi}_{MN}(x) \dot{x}^N \hat{g}, \\ \hat{g} &= \det \hat{g}_{mn}, \\ \tilde{\Pi}_{MN}(x) &= G_{MN}(x) - G_{MB}(x) \partial_m x^B (\hat{g}^{-1})^{mn} \partial_n x^L G_{LN}(x), \end{aligned} \quad (3)$$

where $\dot{x}^M = \partial x^M / \partial \tau$; $\partial_m x^A = \partial x^A / \partial \sigma$, ($m = 1, 2, \dots, p$). Therefore the action (1) can be written in the form

$$S = \int d^{p+1} \xi \frac{\det \left(\dot{x}^M \tilde{\Pi}_{MN}(x) \dot{x}^N \hat{g} \right)}{E(\tau, \sigma^n)}. \quad (4)$$

The variation of the action (4) with respect to $E(\tau, \sigma_i)$ generates the degeneracy condition for the induced metric $g_{\mu\nu}$

$$g \equiv \det g_{\mu\nu}. \quad (5)$$

which separates the class of $(p + 1)$ -dimensional isotropic geodesic hypersurface characterized by the null volume. In the gauge

$$\dot{x}^M G_{MN}(x) \partial_m x^N = 0, \quad \left(\frac{\hat{g}}{E(\tau, \sigma_i)} \right) = 0, \quad (6)$$

we find the equations of motion and constraints in the form

$$\ddot{x}^M + \Gamma_{PQ}^M \dot{x}^P \dot{x}^Q = 0, \quad (7)$$

$$\dot{x}^M G_{MN} \dot{x}^N = 0, \quad \dot{x}^M G_{MN} \partial_m x^N = 0. \quad (8)$$

3. Dyonic black hole background and null membrane motion

We now move on towards solving the null membrane equation of motion and constraints in four-dimensional dyonic black hole background.

The family of exact, four-dimensional dyonic black holes in string theory are constructed as a tensor product of electrically charged two dimensional black holes with the angular magnetic monopole CFT obtained by quotienting a $SU(2)$ WZW model by the discrete subgroup $Z(m)$, where m is an integer. The level of the corresponding WZW model is denoted as K_{SU} . The two dimensional electrically charged black hole part is obtained by a Kaluza-Klein reduction of the string analogue of $2 + 1$ dimensional rotating black hole solution. This tensor product leads to a solution describing the throat limit of a four-dimensional black hole with electric and magnetic charge. The corresponding metric is given by [10]:

$$dS^2 = -a(r) dt^2 + \frac{1}{a(r)} dr^2 + \frac{1}{4} K_{SU} (d\theta^2 + \sin^2(\theta) d\varphi^2), \quad (9)$$

where $a(r) = -M + \frac{r^2}{7^2} + \frac{J^2}{4r^2}$, K_{SU} is the level of $SU(2)$ WZW model as discussed before, M is the mass of the black hole, and J is the angular momentum and the cosmological constant is proportional to l^2 . We restrict ourselves to the case when $J = 0$.

The null membrane equations of motion and constraints in this background turn out to be:

$$t_{,\tau\tau} + \frac{a'(r)}{a(r)} r_{,\tau} t_{,\tau} = 0, \quad (10)$$

$$r_{,\tau\tau} - \frac{a'}{2a(r)} r_{,\tau}^2 + \frac{1}{2} a(r) a'(r) t_{,\tau}^2 = 0, \quad (11)$$

$$\theta_{,\tau\tau} - \frac{1}{2} \sin(2\theta) \varphi_{,\tau}^2 = 0, \quad (12)$$

$$\varphi_{,\tau\tau} + 2ctg(\theta) \theta_{,\tau} \varphi_{,\tau} = 0, \quad (13)$$

$$a(r) t_{,\tau}^2 - \frac{1}{a(r)} r_{,\tau}^2 - B [\theta_{,\tau}^2 + \sin^2(\theta) \varphi_{,\tau}^2] = 0, \quad (14)$$

$$a(r) t_{,\tau} t_{,\sigma} - \frac{1}{a(r)} r_{,\tau} r_{,\sigma} - B [\theta_{,\tau} \theta_{,\sigma} + \sin^2(\theta) \varphi_{,\tau} \varphi_{,\sigma}] = 0, \quad (15)$$

where: $(\dots)_{,\tau} = \partial(\dots)/\partial\tau$, $(\dots)_{,\sigma} = \partial(\dots)/\partial\sigma$, $a'(r) = \partial a(r)/\partial r$, $\sigma_i = (\sigma_1, \sigma_2)$, $B = \frac{1}{4} K_{SU}$, $a(r) = -M + r^2/l^2$.

Equations (10) and (13) easily integrate. The only difference from that of the general relativistic point particle case is that now the ‘‘constants of motion’’ must depend on the membrane coordinate σ_1, σ_2 : i.e.

$$t_{,\tau} = \frac{P_t}{a(r)}, \quad (16)$$

$$\varphi_{,\tau} = \frac{P_\varphi}{B \sin^2(\theta)}. \quad (17)$$

Combining Eq. (12) with Eq. (17) we obtain

$$\theta_{,\tau}^2 = Q^2 - \frac{P_\varphi^2}{B^2 \sin^2(\theta)}. \quad (18)$$

The standard potential equation for the radial coordinate is obtained by integrating Eq. (11):

$$r_{,\tau}^2 = P_t^2 - a(r) B Q^2. \quad (19)$$

where: $P_t = P_t(\sigma_1, \sigma_2)$, $P_\varphi = P_\varphi(\sigma_1, \sigma_2)$, $Q = Q(\sigma_1, \sigma_2)$.

A general solutions of the equations (16) – (19) is

$$r(\tau, \sigma_1, \sigma_2) = A \sin\left(\frac{\sqrt{B}}{l} Q\tau + r_0\right), \quad (20)$$

$$\cos(\theta(\tau, \sigma_1, \sigma_2)) = F \sin(Q\tau + \theta_0), \quad (21)$$

$$\varphi(\tau, \sigma_1, \sigma_2) = \left\{ \begin{array}{l} \varphi_0 + \tan^{-1} \left[\frac{P_\varphi}{BQ} tg(Q\tau + \theta_0) \right], \quad \text{for } 0 < F < 1 \\ \varphi_0, \quad \text{for } F = 0, 1 \end{array} \right\}, \quad (22)$$

$$t(\tau, \sigma_1, \sigma_2) = t_0 - \frac{P_T}{M} \int \frac{d\tau}{1 - F^2 \sin^2(\frac{\sqrt{B}}{l} Q\tau + r_0)}, \quad (23)$$

where

$$A = A(\sigma_1, \sigma_2) = \frac{l}{Q} \sqrt{\frac{P_t^2 + MBQ^2}{B}}, \quad (24)$$

$$F = F(\sigma_1, \sigma_2) = \sqrt{1 - \frac{P_\varphi^2}{B^2 Q^2}}, \quad (25)$$

$$f = f(\sigma_1, \sigma_2) = \sqrt{1 + \frac{P_t^2}{BMQ^2}} \geq 1. \quad (26)$$

The explicit form of the solutions (20)–(23) allows to transform the constraints (15) into those for the cauchy initial data:

$$P_t t_{0,\sigma_i} - P_\varphi \varphi_{0,\sigma_i} + Q \left[l\sqrt{B} r_{0,\sigma_i} - B\theta_{0,\sigma_i} \right] = 0, \quad i = 1, 2. \quad (27)$$

For example, we can consider the initial configuration in the form:

$$\begin{aligned} r_0 = \theta_0 = 0, & & P_t = P_\varphi = \cos(\sigma_1) \sin(\sigma_2), \\ l = B = M = Q = 1, & & t_0 = \varphi_0 = \sin(\sigma_1) \cos(\sigma_2). \end{aligned} \quad (28)$$

Combining Eq. (28) with Eqs. (20) – (23) we obtain:

$$\tau = \tan^{-1} \left\{ \frac{g(t, \sigma_1, \sigma_2)}{\cos(\sigma_1) \sin(\sigma_2)} \right\}, \quad (29)$$

$$\varphi = \cos(\sigma_1) \sin(\sigma_2) + \tan^{-1} \{g(t, \sigma_1, \sigma_2)\}, \quad (30)$$

$$\theta = \arccos \left(\sqrt{1 - \cos^2(\sigma_1) \sin^2(\sigma_2)} \sin \left(\tan^{-1} \left\{ \frac{g(t, \sigma_1, \sigma_2)}{\cos(\sigma_1) \sin(\sigma_2)} \right\} \right) \right), \quad (31)$$

$$r = \sqrt{1 - \cos^2(\sigma_1) \sin^2(\sigma_2)} \sin \left(\tan^{-1} \left\{ \frac{g(t, \sigma_1, \sigma_2)}{\cos(\sigma_1) \sin(\sigma_2)} \right\} \right), \quad (32)$$

where $g(t, \sigma_1, \sigma_2) = \tan(\sin(\sigma_1) \cos(\sigma_2) - t)$.

The world - sheets of null membrane in this case are shown in fig.1 and fig. 2.

In Figures 3 and 4 we present the world-sheets of null membrane for initial configuration in the form:

$$\begin{aligned} r_0 = \theta_0 = 0, & & P_t = P_\varphi = \cos^2(\sigma_1) \sin(\sigma_2), \\ l = B = M = Q = 1, & & t_0 = \varphi_0 = \sin^2(\sigma_1) \cos(\sigma_2). \end{aligned} \quad (33)$$

In this paper we performed some investigation on the classical dynamics of the null bosonic-branes in curved background. In the second section we gave the action, the equations of motion and constraints. In the third section we considered the dynamics of the null membranes in a four - dimensional dyonic black hole background. The exact solution of the equations of motion and of the constraints was found there. The exact solution obtained in this paper, can be using as the test solutions indispensable at a numerical modeling of motion of null membranes in curved background.

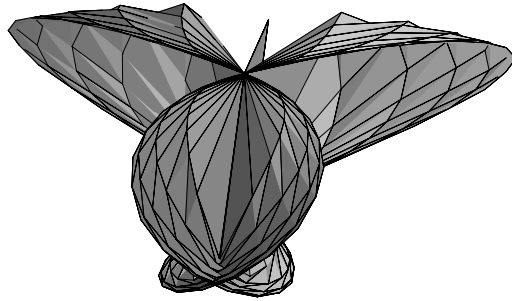


Figure 1. The world-sheet of null membrane for initial configuration (28), $t=0$

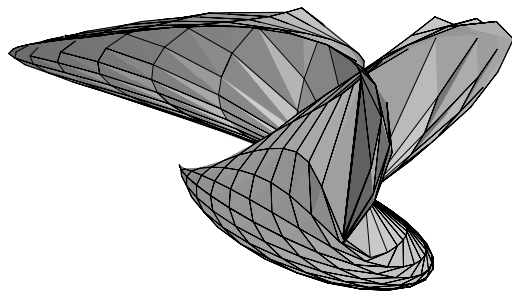


Figure 2. The world-sheet of null membrane for initial configuration (28), $t=0.5$

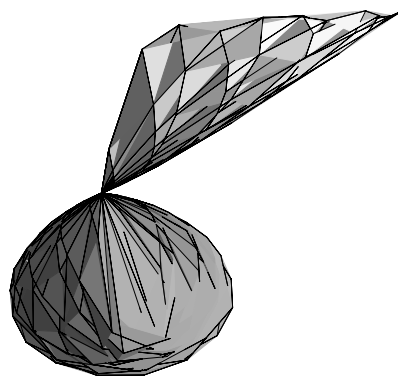


Figure 3. The world-sheet of null membrane for initial configuration (33), $t=0$

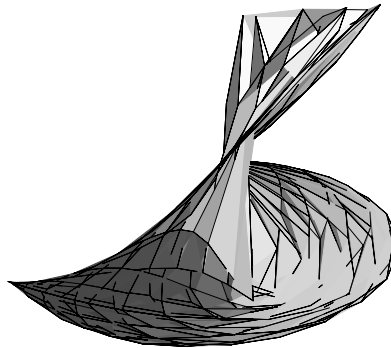


Figure 4. The world-sheet of null membrane for initial configuration (33), $t=0.3$

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References

- [1] I.A. Bandos, and A.A Zheltukhin, *Fortschr. Phys.*, **41** (7), (1993), 619.
- [2] I.A Bandos, and A.A Zheltukhin, *Sov. J. Nucl. Phys.*, **50** (3), (1989), 556; A.A Zheltukhin, *Jetp Letters*, *46* (6), (1987), 262.
- [3] H.J DeVega, and N. Sanches, Lectures on String Theory in Curved Spacetimes, hep - th/9512074.
- [4] S.N. Roshchupkin, and A.A. Zheltukhin, On a Possibility of Membrane Cosmology, hep - th/9607119.
- [5] S.N. Roshchupkin, *Turk. J. Phys.*, **25** (2001), 399.
- [6] P. Bozhilov, *Phys. Rev.*, **D 60** (1999), 125011, hep - th/9904208.
- [7] P. Bozhilov and B. Dimitrov, *Phys. Lett.*, **B 472** (2000), 54, hep - th/9909092.
- [8] P. Bozhilov, *Phys. Rev.*, **D 62**, (2000), 105001, hep - th/9911210.
- [9] S.N. Roshchupkin, and A.A. Zheltukhin, *Nucl. Phys.*, **B 543**, (1999), 365, hep - th/9806054.
- [10] Swapna Mahapatra, *Phys. Rev.*, **D 55**, (1997), 6403, hep - th/9608028.