Turk J Phys
28 (2004) , 73 – 79.

The Born Scattering Amplitude of a Non-relativistic Spin One-half particle in an Aharonov-Bohm Potential

Y. ROMIA, M. SHIKAKHWA
Department of Physics, University of Jordan,
111942 Amman-JORDAN

Received 01.10.2003

Abstract

The Born scattering amplitude of a non-relativistic spin one-half particle in an Aharonov-Bohm potential is calculated up to second order. It is demonstrated that perturbation theory works well for this model, in contrast with the case of scalar particles, thanks to the spin-magnetic moment interaction term. The first order amplitude is shown to coincide with the exact amplitude when expanded to the same order; the second order amplitude is finite and null. The polarized scattering cross section is found to be different from the unpolarized one only if the incident particle has a spin component perpendicular to the flux tube.

1. Introduction

The Aharonov-Bohm (AB) effect, a purely quantum mechanical phenomenon with no classical analogue, was introduced into physics about half a century ago by Aharonov and Bohm in a memorial paper [1]. Charged scalar particles moving in a region with non-vanishing vector potential, but a vanishing magnetic field except inside an infinitesimally thin (δ-function) infinitely long flux tube that is inaccessible to the particle undergo scattering so that their scattering amplitude depends on the flux inside the tube, which is inaccessible. The AB effect was—and is actually still—a field of intense research, both theoretical and experimental (see the reviews [2] and [3] and references therein).

The first attempts [4, 5] to calculate the AB scattering amplitude perturbatively, however, revealed the existence of a problem in the application of perturbation theory to this model. It was found that the first order Born approximation gives rise to an amplitude that is different from the exact amplitude when the latter is expanded to the same order in powers of the coupling constant of the theory. The second order amplitude, which should be finite and null, was found to be divergent. Many approaches to remedy this problem were suggested in the literature [6, 7, 8]. One of these approaches was the introduction of a δ-function interaction into the Hamiltonian [9, 10]. This interaction was shown to lead to the correct first order amplitude, and a finite and null second order amplitude. Some authors [9, 10] attributed this δ-function term—that is introduced "by hand"—to a spin-magnetic moment interaction. However, there's no published work, as far as we know, that develops a theory in which this term appears naturally as a result of the inclusion of the spin degree of freedom into the Hamiltonian of non-relativistic particles. This is one of the goals of the present work. The other goal is to demonstrate that the inclusion of the spin degree of freedom into the non-relativistic Hamiltonian leads to a first order cross section with spin dependence identical to that in the exact cross section reported by Hagen for non-relativistic spin-$\frac{1}{2}$ particles [7].

The paper is organized as follows: In section 2 we review the perturbative calculations for scalar non-relativistic particles and demonstrate the failure of the Born approximation. In section 3, we introduce the spin degree of freedom into theory starting from the Pauli equation, and show how the spin-magnetic moment interaction resolves the perturbative difficulties. We also calculate the differential cross section and report its spin dependence. We sum up our conclusions in section 4.

2. The Failure of the Born Approximation for Non-Relativistic Scalar Particles in an AB Potential

The dynamics of a scalar non-relativistic particle in a vector potential $\mathbf{A}(\mathbf{r})$ is governed by the Hamiltonian (we set $\hbar = c = 1$ throughout this paper):

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2. \tag{1}$$

The AB potential $\mathbf{A}$ is given by [1]

$$\mathbf{A} = \frac{\Phi}{2\pi}\frac{\hat{\varepsilon}_3 \times \mathbf{r}}{\left|\hat{\varepsilon}_3 \times \mathbf{r}\right|^2} = \frac{\Phi}{2\pi\rho}\hat{\varepsilon}_\varphi, \tag{2}$$

where $\hat{\varepsilon}_\mathbf{3}$ and $\hat{\varepsilon}_\varphi$ are unit vectors along the z- and φ-directions, respectively, $\rho = \sqrt{x^2 + y^2}$, and Φ is the flux through the tube. For this vector potential, the Hamiltonian of a scalar particle becomes

$$H = H_0 + V; \tag{3}$$

where

$$H_0 = \frac{\mathbf{p}^2}{2m}, \tag{4}$$

and

$$V = \frac{ie}{mc}\frac{\Phi}{2\pi\rho^2}\frac{\partial}{\partial\varphi} + \frac{e^2}{2m}\left(\frac{\Phi}{2\pi}\right)^2\frac{1}{\rho^2}. \tag{5}$$

The Schrödinger equation in cylindrical coordinates thus reads

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\left[\frac{\partial}{\partial\varphi} + i\alpha\right]^2 + \frac{\partial^2}{\partial z^2} + k^2\right]\Psi(\mathbf{r}) = 0. \tag{6}$$

Here α, $(0 < \alpha < 1)$ is defined as $\alpha = -e\Phi/2\pi$.

The scattering solutions for the above equation were first calculated in [1] and corrected in [11]. The exact scattering amplitude reported in the latter work reads (when α is taken to satisfy $0 < \alpha < 1$)

$$f(\theta) = \frac{-i}{\sqrt{2\pi k}}\sin(\pi\alpha)\frac{e^{i\frac{\theta}{2}}}{\cos\frac{\theta}{2}}, \tag{7}$$

which for small α reads

$$f(\theta) = \sqrt{\frac{\pi}{2ik}}\left(-\alpha\tan\frac{\theta}{2} - i\alpha\right) + O\left(\alpha^3\right). \tag{8}$$

Note that the above amplitude has no $O\left(\alpha^2\right)$ terms.

As we have mentioned in the introduction, attempts to get the amplitude in Eqn. (8) by employing the Born approximation failed [4, 5]. To see this, recall that generally, the first order Born scattering amplitude reads [5, 12]

$$f_B^{(1)}(\theta) = \frac{-i}{2\sqrt{2\pi i k}} \int d^2x \left(e^{-i\mathbf{k}'\cdot\mathbf{x}} U(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\right), \tag{9}$$

where $\mathbf{k}$and $\mathbf{k}'$ are, respectively, the wave vectors of the incident and scattered waves, with $|\mathbf{k}| = |\mathbf{k}'|$, and θ is the scattering angle. $U(\mathbf{x})$ is related to $V(\mathbf{x})$ defined in Eqn. (5) as $U(\mathbf{x}) = 2mV(\mathbf{x})$. For the AB potential Eqn. (2), $f_B^{(1)}$ thus reads [5]

$$\begin{aligned} f_B^{(1)}(\theta) &= \frac{-i}{2\sqrt{2\pi i k}} \int \rho d\rho d\varphi \left(e^{-i\mathbf{k}'\cdot\mathbf{x}} \left(\frac{-2i\alpha}{\rho^2}\frac{\partial}{\partial\varphi}\right) e^{i\mathbf{k}\cdot\mathbf{x}}\right) \\ &= -\alpha\sqrt{\frac{\pi}{2ik}}\tan\frac{\theta}{2}, \quad \theta \neq 0. \end{aligned} \tag{10}$$

It is obvious that the above amplitude misses the $i\alpha$ term of Eqn. (8). This discrepancy is a result of the fact that the first order Born amplitude-being first order in α-misses the contribution of the s-partial wave, as was first noted in [5] and demonstrated in details through a partial wave analysis in [13].

Turning to the second order Born amplitude, we see that this receives two contributions: one from considering the α^2-term in V, Eqn. (5), and the second from the iteration of the first order term. The first of these reads

$$\begin{aligned} f_B^{(2,a)}(\theta) &= \frac{-i\pi\alpha^2}{\sqrt{2\pi i k}} \int d^2x \frac{e^{i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}}{|\mathbf{x}|^2} \\ &= \frac{-i\pi\alpha^2}{\sqrt{2\pi i k}} \int_0^\infty d\rho \frac{J_0(q\rho)}{\rho}, \end{aligned} \tag{11}$$

where

$$q = |\mathbf{k} - \mathbf{k}'| = 2k\cos\left(\frac{\theta}{2}\right). \tag{12}$$

The above integral diverges. However, for later convenience, we isolate this divergence by introducing a cut off R at the lower limit of the integral. Integrating now, gives:

$$\begin{aligned} f_B^{(2,a)}(\theta) &= \frac{-i\pi\alpha^2}{\sqrt{2\pi i k}} \left[-\gamma - \ln\left(\frac{1}{2}qR\right) + O(R)\right] \\ &= \frac{-i\pi\alpha^2}{\sqrt{2\pi i k}} \left[-\gamma - \ln\left(\frac{1}{2}kR\right) - \ln\left(2\cos\frac{\theta}{2}\right) + O(R)\right], \end{aligned} \tag{13}$$

γ being the Euler-Mascharoni constant. The second contribution to $f_B^{(2)}(\theta)$ reads

$$f_B^{(2,b)}(\theta) = \frac{i}{2\sqrt{2\pi i k}} \int d^2x e^{-i\mathbf{k}'\cdot\mathbf{x}} \left(\frac{-2i\alpha}{\rho^2}\frac{\partial}{\partial\varphi}\right) \int d^2x' G_{k,+}(\mathbf{x}-\mathbf{x}') \left(\frac{-2i\alpha}{\rho'^2}\frac{\partial}{\partial\varphi'}\right) e^{i\mathbf{k}.\mathbf{x}'}, \tag{14}$$

where $G_{k,+}(\vec{x} - \vec{x}')$ is the Green's function for the Lippmann-Schwinger equation that behaves asymptotically as an outgoing wave

$$G_{k,+}\left(\mathbf{x}-\mathbf{x}'\right)=\tfrac{i}{4}H_0^{(1)}\left(k\left|\mathbf{x}-\mathbf{x}'\right|\right), \tag{15}$$

and $H_0^{(1)}\left(k\left|\mathbf{x}-\mathbf{x}'\right|\right)$ is the zeroth-order Hankel function of the first type. The calculation of $f_B^{(2,b)}$ is most convenient in momentum space, where an expansion of the integrand in terms of the Gegenbauer polynomials [9] can be involved. The procedure was sketched in [9], where the result is:

$$f_B^{(2,b)}\left(\theta\right)=\frac{-i\pi\alpha^2}{\sqrt{2\pi ik}}\left[\ln\left(2\cos\frac{\theta}{2}\right)+\frac{i\pi}{2}\right], \tag{16}$$

which is finite. Therefore,

$$\begin{aligned} f_B^{(2)}\left(\theta\right)&=f_B^{(2,a)}+f_B^{(2,b)}\\ &=\frac{-i\pi\alpha^2}{\sqrt{2\pi ik}}\left[-\gamma+\frac{i\pi}{2}-\ln\left(\frac{1}{2}kR\right)\right], \end{aligned} \tag{17}$$

and is divergent.

As we mentioned earlier, we expect $f_B^{(2)}\left(\theta\right)$ (which is of $O\left(\alpha^2\right)$) to give a null result, to be consistent with the expansion (8). Obviously, this is not the case here. So, indeed the Born amplitude up to second order fails in the case of scalar particles. Below we will show that, if the spin degree of freedom is taken into account, the Born approximation works well as in the relativistic case [14, 15].

3. The Born Approximation for Non-Relativistic Spin-$\frac{1}{2}$ Particles

The correct inclusion of the spin degree of freedom into the Schrödinger equation of a spin-$\frac{1}{2}$ particle in an AB potential was considered by Hagen [7]. It was demonstrated that the exact differential cross section for the scattering of a spin-$\frac{1}{2}$ particle differs from that of scalar particles only when the incident particle's spin has a component perpendicular to the AB solenoid. This result was also verified for the first order Born amplitude of relativistic spin-$\frac{1}{2}$ particles [14]. In this section, we will verify that the first order Born approximation works well for non-relativistic spin-$\frac{1}{2}$ particles. First, we will show that the inclusion of spin leads to an amplitude that is consistent with the expansion of the exact amplitude in powers of α, Eqn. (8), in that it reproduces the "correct" $O\left(\alpha\right)$ term, and a finite and null $O\left(\alpha^2\right)$ term, the reason being the additional spin-magnetic moment interaction in the Hamiltonian. Second, we will show that this particular term reproduces to this order the spin-dependence of the AB amplitude reported in [11, 14].

The starting point is the Pauli equation

$$\frac{1}{2m}\left[\sigma\cdot\left(\mathbf{p}-e\mathbf{A}\right)\right]^2\Psi\left(\mathbf{r}\right)=E\Psi\left(\mathbf{r}\right), \tag{18}$$

where the σ_i's $(i=1,2,3)$, are the Pauli spin matrices, and $\mathbf{A}$ is the AB potential, equation (2). Expanding the Hamiltonian in Eqn. (18), we get

$$H=H_0+V, \tag{19}$$

where

$$H_0=\frac{\mathbf{p}^2}{2m}, \tag{20}$$

and

$$V=\frac{-i\alpha}{m}\frac{1}{\rho^2}\frac{\partial}{\partial\varphi}+\frac{\alpha^2}{2m\rho^2}+\frac{\pi}{m}\alpha\sigma_3\delta^{(2)}\left(\mathbf{r}\right), \tag{21}$$

where α was defined earlier.

Note the appearance of a new interaction term in (21) that results from the coupling of the spin-magnetic moment of the particle to the magnetic field of the solenoid, which is a δ-function at the origin. The Schrödinger equation in cylindrical coordinates now reads

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\left(\frac{\partial}{\partial\varphi}+i\alpha\right)^2-2\pi\alpha\delta^2\left(\mathbf{r}\right)+k^2\right]\Psi\left(\mathbf{r}\right)=0. \tag{22}$$

The first order Born amplitude will obviously now receive a new contribution from the new $O\left(\alpha\right)$ term in V. So, we have now

$$f_B^{(1)}\left(\theta\right)=f_B^{(1,a)}\left(\theta\right)+f_B^{(1,b)}\left(\theta\right), \tag{23}$$

where $f_B^{(1,a)}\left(\theta\right)$ is just the one given by Eqn. (10), except that we have to multiply the incident and outgoing waves with the Pauli spinors; thus it reads

$$f_B^{(1,a)}\left(\theta\right)=\chi^+\left(-\alpha\sqrt{\frac{\pi}{2ik}}\tan\frac{\theta}{2}\right)\chi,\quad \theta\neq 0. \tag{24}$$

As for $f_B^{(1,b)}\left(\theta\right)$, it reads

$$\begin{aligned} f_B^{(1,b)}\left(\theta\right)&=\frac{i}{2\sqrt{2\pi ik}}\int d^2r\left(e^{-i\mathbf{k}'\cdot\mathbf{r}}\chi^{+(s')}\left(-2\pi\alpha\sigma_3\delta^{(2)}\left(\mathbf{r}\right)\right)\chi^{(s)}e^{i\mathbf{k}\cdot\mathbf{r}}\right)\\ &=\frac{-i}{2\sqrt{2\pi ik}}\int d^2r\left(e^{i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{r}'}\frac{\alpha}{R}\delta\left(r-R\right)\right), \end{aligned} \tag{25}$$

where we have applied the regularization scheme suggested in [9], to deal with the δ-function through the introduction of a regulator R:

$$2\pi\alpha\delta^2\left(\mathbf{r}\right)=\frac{\alpha}{R}\delta\left(r-R\right). \tag{26}$$

The regulator R will be set to zero at the end of the calculations. This regularization scheme is powerful, and will serve to isolate the divergences in the second order amplitude, as will be seen shortly, in a manner similar to the regularization schemes used in field theory [15]. Integrating, we end up with

$$f_B^{(1,b)}\left(\theta\right)=-i\alpha\sqrt{\frac{\pi}{2ik}}\sigma_3+O\left(R\right), \tag{27}$$

which is finite.

Turning now to $f_B^{(2)}\left(\theta\right)$, we see that this receives three contributions. The first, $f_B^{(2,a)}$, comes from the $O\left(\alpha^2\right)$ term in V, Eqn. (21), which is the same as in the scalar case, Eqn. (13). The same holds for the second contribution, namely $f_B^{(2,b)}$ in Eqn. (16). The third contribution, $f_B^{(2,c)}$, which is absent in the scalar case, comes from the iteration of the first order δ-function term in V, and is given by

$$
\begin{aligned}
f_B^{(2,c)}(\theta) &= \frac{i}{2\sqrt{2\pi i k}} \int d^2 r d^2 r' \left(\chi^{+(s')} e^{i\left(\mathbf{k}\cdot\mathbf{r}' - \mathbf{k}'\cdot\mathbf{r}\right)} G_{k,+}(\mathbf{r},\mathbf{r}') \frac{\alpha^2}{R^2} \delta(r'-R)\,\delta(r-R)\,\chi^{(s)} \right) \\
&= \frac{2i\pi^2\alpha^2}{\sqrt{2\pi i k}} \chi^{+(s')} \left[\frac{i}{4} - \frac{1}{2\pi} \ln\left(\frac{1}{2} k e^{\gamma} R \right) + O(R) \right] \chi^{(s)}.
\end{aligned}
\tag{28}
$$

The above expression is clearly divergent in the limit $R \to 0$. Note, however, the way the regulator divided the result into a finite part (independent of R) and a divergent part. Also note that the above expression is equal exactly to the negative of the sum of $f_B^{(2,a)}$ and$f_B^{(2,b)}$, Eqns. (13) and (17), so that the two divergent terms delicately cancel, and we get for the spinor second order Born amplitude

$$
f_B^{(2)}(\theta) = f_B^{(2,a)}(\theta) + f_B^{(2,b)}(\theta) + f_B^{(2,c)}(\theta) = 0,
\tag{29}
$$

a finite and null result, as it should be, thanks to the spin magnetic moment interaction.

One might wonder at this point why we didn't consider the contribution to $f_B^{(2)}(\theta)$ from the crossed iteration of the δ-function and the $-2i\alpha\partial/\partial\varphi$ terms in V, Eqn. (21), which is $O\left(\alpha^2\right)$, too. These contributions simply vanish. The reason being that they act in orthogonal Hilbert subspaces, that is, while the contact (δ-function) interaction only affects the s-waves, the $-2i\alpha\partial/\partial\varphi$ vanishes in that subspace. This fact is most transparent if one considers partial wave analysis of the Born amplitude [13].

So, finally we get for the Born scattering amplitude of spin-$\frac{1}{2}$ particles, up to $O\left(\alpha^2\right)$

$$
f_B(\theta) = \chi^{+(s')} \left(\sqrt{\frac{\pi\alpha^2}{2ik}} \left(-\tan\frac{\theta}{2} - i\sigma_3 \right) + O\left(\alpha^3\right) \right) \chi^{(s)}.
\tag{30}
$$

The above result has no α^2-terms, neither finite nor infinite, as one expects. Moreover, it makes manifest the fact that any transitions in the spin space of the particle are induced by the second term in $f_B^{(1)}(\theta)$, which came up as a result of the inclusion of the spin-magnetic moment interaction into the Hamiltonian, Eqn. (21) and was absent in the scalar case. To see the effect of the inclusion of the spin degree of freedom more explicitly, it is best to look at the scattering cross sections.

The unpolarized particle cross section is given by:

$$
\begin{aligned}
\left(\frac{d\sigma}{d\Omega} \right)_{\text{unpd.}}^{\text{Born}} &= \frac{1}{2} \sum_{s,s'} \left| \chi^{+(s')} \sqrt{\frac{\pi\alpha^2}{2ik}} \left(-\tan\tfrac{\theta}{2} - i\sigma_3 \right) \chi^{(s)} \right|^2 \\
&= \frac{\pi\alpha^2}{2k} \left(1 + \tan^2\frac{\theta}{2} \right) = \frac{e^2\Phi^2}{8\pi k} \left(\cos^2\frac{\theta}{2} \right)^{-1},
\end{aligned}
\tag{31}
$$

which, as expected by [7], is just the cross section that one gets from the scalar amplitude at the same order: Eqn. (8). As for polarized particles, if we take the incident particle to be polarized along an arbitrary direction given by a unit vector $\hat{n}$, and we consider transitions to a final state with the same polarization, we get

$$
\begin{aligned}
\left(\frac{d\sigma}{d\Omega} \right)_{\text{pol.}}^{\text{Born}} &= \left| \chi^{+}(\hat{n}) \sqrt{\frac{\pi\alpha^2}{2ik}} \left(-\tan\frac{\theta}{2} - i\sigma_3 \right) \chi(\hat{n}) \right|^2 \\
&= \left(1 - (\hat{n}\times\hat{z})^2 \cos^2\frac{\theta}{2} \right) \left(\frac{d\sigma}{d\Omega} \right)_{\text{unpd.}}^{\text{Born}}.
\end{aligned}
\tag{32}
$$

The above cross section is different from the unpolarized one only if the particle's polarization has a component perpendicular to the solenoid. This is in agreement with the results reported earlier for the exact cross section of a non-relativistic particle [7], and the first order born amplitude of a Dirac particle [14]. Therefore, our calculations show that the Born amplitude for non-relativistic spin-$\frac{1}{2}$ particles, not only gives a finite and null result at $O\left(\alpha^2\right)$ [15], but also reproduces the same-dependence of the cross section as in the exact theory.

4. Conclusions

We have shown that the spin-magnetic moment interaction of spin-$\frac{1}{2}$ particles in an AB potential provides a δ-function term that leads to a correct and finite Born scattering amplitude. This term, which appears naturally in the Hamiltonian once one starts from the Pauli equation, renders the second order Born amplitude finite and null. It also catches the contribution of the s-wave to the first order amplitude, thus leading to an amplitude that is consistent with the exact amplitude expanded to the same order. The unpolarized cross section is the same as the one calculated for scalar particles at the same order, while the cross section for polarized particles is found to be different from that for unpolarized ones only if the initial polarization has a component normal to the AB solenoid. The above spin dependence is the same as the one reported earlier for the exact cross section of non-relativistic spin one-half particles [7], and for the first order Born cross section of Dirac particles [14].

References

[1] Y. Aharonov and D. Bohm, *Phys. Rev.*, **115,** (1959), 485.

[2] J. Hamilton, *Aharonov-Bohm and other Cyclic Phenomena*, 1^{st} ed., (Springer-Verlag, Berlin 1997).

[3] M. Peshkin and A. Tonomura, *The Aharonov-Bohm Effect*, Lecture Notes in Physics **340,** (Springer-Verlag, Berlin 1989).

[4] E. Feinberg, *Sov. Phys. Usp.*, **5,** (1963), 753.

[5] E. Corinaldesi and F. Rafeli, *Am. J. Phys.*, **46,** (1978), 1185.

[6] C. R. Hagen*, Phys. Rev.* **D 52,** (1995), 2466.

[7] C. R. Hagen, *Phys. Rev. Lett.* **64,** (1990), 503.

[8] Ph. Gerbert de Sonsa, *Phys. Rev.* **D 40,** (1989), 1346.

[9] C. Manuel and R. Tarrach, *Phys. Lett.* **B 328,** (1994), 113.

[10] S. Ouvry, *Phys. Rev.* **D 52,** (1994), 5296.

[11] C. R. Hagen, *Phys. Rev.* **D 41,** (1990), 2015.

[12] I. R. Lapidus, *Am. J. Phys.* **50,** (1982), 45.

[13] M. S. Shikakhwa and N.K. Pak, *Phys. Rev.* **D 67,** (2003), 105019.

[14] F. Vera and I. Schmidt, *Phys. Rev.* **D 42,** (1990), 3591.

[15] M. Boz and N.K.Pak, Phys.Rev.**D62**, 045022.

[16] P. Gosdzinsky and R. Tarrach, *Am. J. Phys.* **59,** (1991), 70.