

Quantum Statistical Entropy of Sen Black Hole

Zhao REN^{1,2}, Wu YUEQIN², Zhang LICHUN²

¹*Department of Environment and Safety Engineering, North CHINA
University of Science and Technology, Taiyuan 030051, P.R. CHINA
e-mail: zhao2969@sina.com*

²*Department of Physics, Yanbei Normal Institute, Datong 037000 P.R. CHINA*

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Abstract

In this paper, by using quantum statistical methods, we directly obtain the partition function of Bose field and Fermi field on the background of the axisymmetrical Sen black hole. The difficult-to-solve wave equation is avoided. Then via the improved brick-wall method and membrane model, we calculate the entropy of Bose field and Fermi field of the black hole. We find that the entropy of the black hole is proportional to the area of the horizon. In our results, we do not encounter the left out term nor the divergent logarithmic term that appears in the original brick-wall method. The question that arose in the brick-wall method of whether the entropy of the scalar or Dirac fields outside the event horizon is the entropy of the black hole does not arise in the present method. The influence of spinning degeneracy of particles on entropy of the black hole is also given. We provide a way to study various complicated black holes.

Key Words: Quantum statistics, brick-wall method, membrane model, entropy of black holes.

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1. Introduction

Entropy of the black hole is one of the more important subjects in theoretical physics. Since entropy has statistical meaning, the understanding of entropy involves the sense of the microscopic essence of the black hole. Fully understanding it needs a good quantum gravitation theory. However, the work produced to present not satisfying. The statistical origin of the black hole is not yet solved [1]. On the other hand, many in the literature gave the same result: that the entropy of the black hole is proportional to the area of the horizon [2–9]. The most frequently used method among them is the brick-wall method advanced by G't Hooft [7]. This method is used to study the statistical property of the scalar and Dirac fields in various black holes [10–12] and it is found that the expression of the entropy of the black hole consists of a term which is proportional to the area of the event horizon and the divergent logarithmic term, which is not proportional to the area of event horizon. In the original brick-wall method, there is the question: are the entropies of the scalar or Dirac fields outside the event horizon the entropy of the black hole? To obtain the result, that the entropy of the black hole is proportional to the area of the horizon, we must neglect the logarithmic term and take the L^3 term as contribution to the vacuum which surround the system to large distances. In

the original brick-wall method, it is a complicated task to calculate the wave functions of the scalar or Dirac fields on the background of the black hole by the WKB method of approximation.

In this paper, by using the quantum statistical method [13], we directly obtain the partition function of the Bose and Fermi fields on the background of the axisymmetrical Sen black hole. We also give an integrated expression for the entropy of the system. Then via the improved brick-wall method and the membrane model, we calculate the entropy of the Bose and Fermi fields of the black hole [14]. Because we directly adopt the quantum statistical method, the difficult-to-solve wave equation in the original brick-wall method is avoided. In the whole process, the physical idea is clear, the calculation is simple and the result is reasonable. We provide a way to calculate the entropies of various complicated space-times. In this paper, we take the simplest function form of temperature ($C = \hbar = G = K_B = 1$).

2. Sen Black Hole

The linear element in Sen black hole is given by [15]:

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 - \frac{4\mu r a \cosh^2 \gamma \sin^2 \theta}{\Sigma} dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\Lambda}{\Sigma} d\varphi^2 \quad (1)$$

where

$$\Delta = r^2 - 2\mu r + a, \quad \Sigma = r^2 + a^2 \cos^2 \theta + 2\mu r \sinh^2 \gamma,$$

and

$$\Lambda = (r^2 + a^2)(r^2 + a^2 \cos^2 \theta) + 2\mu r a^2 \sin^2 \theta + 4\mu r (r^2 + a^2) \sinh^2 \gamma + 4\mu^2 r^2 \sinh^4 \gamma.$$

The relation among mass M , charge Q , angular momentum J , location of horizon and μ , γ , a is

$$M = \frac{\mu}{2}(1 + \cosh 2\gamma), \quad Q = \frac{\mu}{\sqrt{2}} \sinh 2\gamma,$$

$$J = \frac{a\mu}{2}(1 + \cosh 2\gamma), \quad r_{\pm} = \mu \pm \sqrt{\mu^2 - a^2}.$$

The surface gravity of the outer horizon of a black hole is

$$\kappa = \frac{\sqrt{(2M^2 - Q^2) - 4J^2}}{2M\{2M^2 - Q^2 + [(2M^2 - Q^2) - 4J^2]^{1/2}\}}, \quad (2)$$

The Hawking radiation temperature is

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{\beta} = \frac{\sqrt{(2M^2 - Q^2) - 4J^2}}{4\pi M\{2M^2 - Q^2 + [(2M^2 - Q^2) - 4J^2]^{1/2}\}}. \quad (3)$$

The area of the outer event horizon of the black hole is

$$A_+ = 8\pi M \left\{ M - \frac{Q^2}{2M} + \left[\left(M - \frac{Q^2}{2M} \right)^2 - \frac{J^2}{M^2} \right]^{1/2} \right\}. \quad (4)$$

3. The Bosonic entropy

According to generalized relativity theory, the frequency shift of a particle from the surface a star, traveling infinite distance to an observer at rest is computed as

$$\nu = \nu_0 \lambda^{1/2}, \quad (5)$$

where ν_0 is the natural frequency of an atom on the surface of the star, and ν is the natural frequency of this particle measured by the observer at rest at an infinite distance. The temperature measured by the observer at rest at an infinite distance is [16, 17]

$$T = \frac{T_H}{\sqrt{-\tilde{g}_{tt}}}, \quad (6)$$

where

$$\tilde{g}_{tt} = \frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} = -\frac{(r-r_+)(r-r_-)(r^2 + 2\mu r \sinh^2 \gamma)}{(r^2 + 2\mu r \sinh^2 \gamma + a^2)^2 - (r-r_+)(r-r_-)a^2 \sin^2 \theta}. \quad (7)$$

For Bosonic gas, we calculate the partition function [13]

$$\ln Z = -\sum_i g_i \ln(1 - e^{-\beta \varepsilon_i}), \quad (8)$$

In a unit volume the number of quantum states with energy between ε and $\varepsilon + d\varepsilon$, or the frequency between ν and $\nu + d\nu$ is as follows:

$$g(\nu)d\nu = j4\pi\nu^2 d\nu. \quad (9)$$

where j is the spinning degeneracy of particles. Since in the space-time (1), the area of curved surface at random point r is

$$A(r) = \int dA = \int \sqrt{g} d\theta d\varphi, \quad (10)$$

where $g = \begin{vmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{vmatrix} = g_{\theta\theta}g_{\varphi\varphi}$. Then the volume element of the lamella at random point r outside the horizon is as follows:

$$dV = A(r)\sqrt{g_{rr}}dr. \quad (11)$$

So, the partition function of the system at the lamella with random thickness at point r outside the horizon is

$$\begin{aligned} \ln Z &= \int A(r)\sqrt{g_{rr}}dr \sum_i g_i \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta\varepsilon_i} = j4\pi \int A(r)\sqrt{g_{rr}}dr \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} e^{-\frac{n\hbar\nu}{T}} \nu^2 d\nu \\ &= j\frac{1}{90}\pi^2 \int \frac{A(r)\sqrt{g_{rr}}dr}{\beta^3} = j\frac{\pi^2}{90} \int \frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}dr d\theta d\varphi}{\beta^3}, \end{aligned} \quad (12)$$

where $\frac{1}{\beta} = T$. And using the relation between the entropy and partition functions

$$S = \ln Z - \beta_0 \frac{\partial \ln Z}{\partial \beta_0}, \quad (13)$$

we can obtain

$$\begin{aligned}
 S_b &= j \frac{2}{45} \pi^2 \frac{1}{\beta_0^3} \int \frac{\sqrt{g_{\theta\theta} g_{\varphi\varphi} g_{rr}}}{(-\tilde{g}_{tt})^{3/2}} dr \\
 &= j \frac{2\pi^2}{45\beta_0^3} \int d\theta d\varphi \int \frac{[(r^2 + a^2 + 2\mu r \sinh^2 \gamma)^2 - (r - r_+)(r - r_-)a^2 \sin^2 \theta]^2 \sin \theta}{(r - r_+)^2 (r - r_-)^2 (r^2 + a^2 \cos^2 \theta + 2\mu r \sinh^2 \gamma)} dr, \tag{14}
 \end{aligned}$$

where $\beta_0 = \frac{1}{T_H}, \beta = \beta_0 \sqrt{-\tilde{g}_{tt}}$.

In (14), for r , we take the integral region $[r_+ + \varsigma, r_+ + N\varsigma]$, where δ is a small nonnegative quantity, N is a constant larger than one. Then

$$\begin{aligned}
 S_b &= j \frac{4\pi^3}{45\beta_0^3} \int_0^\pi d\theta \int_{r_+ + \varsigma}^{r_+ + N\varsigma} \frac{[(r^2 + a^2 + 2\mu r \sinh^2 \gamma)^2 - (r - r_+)(r - r_-)a^2 \sin^2 \theta]^2 \sin \theta}{(r - r_+)^2 (r - r_-)^2 (r^2 + a^2 \cos^2 \theta + 2\mu r \sinh^2 \gamma)} dr \\
 &= j \frac{4\pi^3}{45\beta_0^3} \int_0^\pi \frac{(r_+^2 + a^2 + 2\mu r_+ \sinh^2 \gamma)^4}{(r_+ - r_-)^2 (r_+^2 + a^2 \cos^2 \theta + 2\mu r_+ \sinh^2 \gamma)} \left[\frac{N-1}{N\varsigma} \right] \sin \theta d\theta + G(r_+, N, \varsigma) \\
 &= j \frac{8\pi^3}{45\beta_0^3} \frac{(r_+^2 + a^2 + 2\mu r_+ \sinh^2 \gamma)^4}{(r_+ - r_-)^2} \frac{\alpha}{\sin \alpha \cos \alpha} \left[\frac{N-1}{N\varsigma} \right] + G(r_+, N, \varsigma) + f(r_+, \theta) \\
 &= j \frac{8\pi^3}{45\beta_0^3} \frac{(2\mu r_+ \cosh^2 \gamma)^3}{(r_+ - r_-)^2} \frac{\alpha}{\sin \alpha \cos \alpha} \left[\frac{N-1}{N\varsigma} \right] + G(r_+, N, \varsigma) + f(r_+, \theta) \\
 &= j \frac{8\pi^3}{45\beta_0^3} \frac{(2Mr_+)^3}{(r_+ - r_-)^2} \frac{\alpha}{\sin \alpha \cos \alpha} \left[\frac{N-1}{N\varsigma} \right] + G(r_+, N, \varsigma) + f(r_+, \theta), \tag{15}
 \end{aligned}$$

where

$$f(r_+, \theta) = \left[\frac{dy_1}{dr} \Big|_{r=r_+} - y_2(r_+) \right] \ln N, \alpha = \arctan \frac{a}{\sqrt{r_+^2 + 2\mu r_+ \sinh^2 \gamma}},$$

$$G(r_+, N, \varsigma) = j \frac{4\pi^3}{45\beta_0^3} \int_0^\pi \int_{r_+ + \varsigma}^{r_+ + N\varsigma} [f_1(r, \theta) + f_2(r, \theta) + f_3(r, \theta)] \sin \theta d\theta dr,$$

$$f_1(r, \theta) = \sum_{n=2} \frac{1}{n!} y_1^{(n)}(r_+) (r - r_+)^{n-2} dr, f_2(r, \theta) = \sum_{n=1} \frac{1}{n!} y_2^{(n)}(r_+) (r - r_+)^{n-2} dr$$

$$f_3(r, \theta) = \sum_{n=0} \frac{1}{n!} y_3^{(n)}(r_+) (r - r_+)^{n-2} dr,$$

$$y_1 = \frac{(r^2 + a^2 + 2\mu \sinh^2 \gamma)^2}{(r - r_-)^2 (r^2 + a^2 \cos^2 \theta + 2\mu r \sinh^2 \gamma)},$$

$$y_2 = -\frac{2(r^2 + a^2 + 2\mu \sinh^2 \gamma)^2 a^2 \sin^2 \theta}{(r - r_-)(r^2 + a^2 \cos^2 \theta + 2\mu r \sinh^2 \gamma)},$$

and

$$y_3 = \frac{a^4 \sin^4 \theta}{(r^2 + a^2 \cos^2 \theta + 2\mu r \sinh^2 \gamma)} \quad y_1^{(n)} = \frac{d^n y_1}{dr^n}, \quad y_2^{(n)} = \frac{d^n y_2}{dr^n}, \quad y_3^{(n)} = \frac{d^n y_3}{dr^n}.$$

From (3.17) in the Ref. [7], we know when $N\varsigma = L \gg r_+$, if we take

$$\varsigma = \frac{T_H}{90}, \quad (16)$$

we obtain that the leading term of the entropy of the black hole is proportional to the area of its horizon.

It is known to all that the entropy of the black hole is proportional to the area of horizon and the existence of the horizon is the basic property of the black hole. It has been proved that the existence of the horizon generally results in the Hawking effect [18]. And whether the black hole possesses entropy or not relates to the existence of the horizon [19]. Thus it is a natural disposition that the entropy of the black hole is proportional to the area of horizon, and its value has nothing to do with the radiation field outside the horizon; the horizon only has the property of the two-dimensional membrane in three-dimensional space. To calculate the number of quantum states of the two-dimensional membrane, for the Sen black hole, we take the ultraviolet cutoff as

$$\varsigma = \frac{T_H}{90} \frac{\alpha}{\sin \alpha \cos \alpha} \frac{N-1}{N}, \quad (17)$$

and the infrared cutoff $N\varsigma$ as

$$S_b = j\pi 2Mr_+ + G(r_+, N, \varsigma) + f(r_+, \theta) = j\frac{1}{4}A_+ + G(r_+, N, \varsigma) + f(r_+, \theta). \quad (18)$$

When $N \rightarrow 1$, $\varsigma \rightarrow 0$, $N\varsigma \rightarrow 0$. That is, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole, but the black hole's entropy is

$$S_b = \frac{j}{4}A_+ = j2\pi M \left\{ M - \frac{Q^2}{2M} + \left[\left(M - \frac{Q^2}{2M} \right)^2 - \frac{J^2}{M^2} \right]^{1/2} \right\}. \quad (19)$$

In our calculation, $\lim_{N \rightarrow 1} G(r_+, N, \varsigma) \rightarrow 0$, and $\lim_{N \rightarrow 1} f(r_+, \theta) \rightarrow 0$. When the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole, the black hole's entropy is irrelative to the radiation field outside the black hole. So the entropy given by (29) should be the entropy of the black hole.

4. The Fermionic entropy

For a Fermionic gas, the partition function is

$$\ln Z = \sum_i g_i \ln(1 + e^{-\beta \varepsilon_i}). \quad (20)$$

From (9), we obtain

$$\begin{aligned} \ln Z &= \int A(r) \sqrt{g_{rr}} dr \sum_i g_i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-n\beta \varepsilon_i} \\ &= i4\pi \int A(r) \sqrt{g_{rr}} dr \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^{\infty} e^{-\frac{n\hbar\nu}{T}} \nu^2 d\nu \end{aligned}$$

$$= i \frac{\pi^2}{90} \frac{7}{8} \int \frac{A(r) \sqrt{g_{rr}} dr}{\beta^3} = i \frac{\pi^2}{90} \frac{7}{8} \int \frac{\sqrt{g_{\theta\theta} g_{\varphi\varphi} g_{rr}} dr d\theta d\varphi}{\beta^3}, \quad (21)$$

where ω is the spinning degeneracy of Fermionic particles. Using the result of section three, we can get the Fermionic entropy as

$$S_f = i \frac{7}{8} \frac{1}{4} A_+, \quad (22)$$

where i is the spinning degeneracy of Fermion.

5. Conclusion

By using the quantum statistical method, we have directly obtained the partition function of various fields on the background of the axisymmetrical Sen black hole. We can derive the integrated expressions given in the original brick-wall method, avoiding the difficult-to-solve wave equation, and not using the approximation method in the calculations. Then via the improved brick-wall method and membrane model, we calculate the entropy of various field of the black hole. The problem that the state density is divergent around horizon no longer presents itself. For $N \rightarrow 1$, $\delta \rightarrow 0$, $N\delta \rightarrow 0$, that is, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole. From (19) and (22), the result is irrelative to the radiation field. We know that the left out term and the divergent logarithmic term in the original brick-wall method no longer exist. The obtained entropy is proportional to the area of black hole's horizon, so it can be taken as black hole's entropy.

From the above analysis, we found that by using the statistical and membrane model methods to calculate the entropy of the black hole, the question that arose in the brick-wall method—of whether the entropy of the scalar or Dirac fields outside the event horizon is the entropy of the black hole—did not exist and the complicated approximations in the solution is avoided. Over the whole process, the physical idea is clear, the calculation is simple and the result is reasonable. We also consider the influence of the spinning degeneracy of particles on the entropy. For calculating the entropy in various space-times, we only need to change the red-shift factor, the others remaining the same. Especially for non-spherical symmetrical space-time, we can directly derive the partition function of various space-times without solving the complicated wave equation. A new way to study the entropy of various black holes is given. Furthermore, the entropy of the black hole is a natural property of the black hole. The entropy is irrelevant to the radiation field out of the horizon. It will deepen the understanding of relationship between black hole's entropy and horizon's area.

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