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Realizations of the $osp(2,1)$ Superalgebra and Related Physical Systems

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Abstract

Eigenvalues and eigenfunction of two-boson 2×2 Hamiltonians in the framework of the superalgebra $osp(2,1)$ are determined by presenting a similarity transformation. The Hamiltonians include two bosons and one fermion have been transformed in the form of the one variable differential equations and the conditions for its solvability have been discussed. It is observed that the Hamiltonians of the various physical systems can be written in terms of the generators of the $osp(2,1)$ superalgebra and under some certain conditions their eigenstates can exactly be obtained. In particular, the procedure given here is useful in determining eigenstates of the Jaynes-Cummings Hamiltonians.

1. Introduction

The Lie (super)algebras have played important role in the study of quantum physics in particular they are associated with the symmetry properties of physical systems and improve the understanding of physical structures [1, 2, 3, 4]. In the last decade a lot of effort, has been attracted on the quasi-exactly solvable(QES) equations whose finite number of eigenvector can be obtained by solving an algebraic equation [5, 6, 7, 8, 9]. Yet, even today, new contributions to this problem are being made. They appear, however, not to have been fully exploited in the analysis of QES equations. These systems have found application in the different fields of the physics. Recently, great attention is being paid to examine different quantum optical models [10, 11, 12, 13, 14] with Hamiltonians given by nonlinear functions of the bosonic and/or fermionic operators since they enable to reveal new effects and phenomena.

The natural step to relate the quantum optical systems and Lie (super)algebras is to express the generators of the Lie (super)algebra in terms of bosons and/or fermions. The Hamiltonians can be written as combinations of the generators of a relevant symmetry group. Hence, one can able to compute (a part of) the spectrum by performing a suitable transformation of the generators.

One major symmetry group candidates for system of two differential equation and 2×2 matrix Hamiltonians is the supergroup $osp(2,1)$. It is well known that unlike a purely bosonic algebra, the superalgebra admits different Weyl inequivalent choices of simple root systems, which corresponds to inequivalent Dynkin-diagrams. In the case of $osp(2,1)$ one has two choices of simple roots which are unrelated by Weyl transformations: a system of fermionic and bosonic simple roots, or a purely fermionic simple roots [15]. Our aim, in this paper, is to construct bosonic and fermionic realization of the $osp(2,1)$ algebra and to obtain solution of the some quantum mechanical problems by performing a suitable transformation operation for the bosons and fermions.

The paper is organized as follows: In section 2 the construction of the boson-fermion realization of the $osp(2,1)$ algebra briefly reviewed and two different realization have been given. Section 3 includes the general method to transform the boson and fermion operators in the form of the QES. In section 4 we present the solution of the Jaynes-Cummings Hamiltonian with Kerr nonlinearity and modified-Jaynes-Cummings Hamiltonian as an application of the method. The paper ends with a brief conclusion.

2. Construction of the two-boson one fermion $osp(2,1)$ superalgebra

A convenient way to construct a spectrum generating superalgebra for systems with a finite number of bound states is by introducing a set of boson and fermion operators. We introduce two boson operator, a_1 and a_2, which obey the usual commutation relations

$$[a_1, a_1] = \left[a_1, a_2^+\right] = \left[a_2, a_1^+\right] = 0, \quad \left[a_1, a_1^+\right] = \left[a_2, a_2^+\right] = 1. \tag{1}$$

The bilinear combinations $a_1^+ a_1$, $a_1^+ a_2$, $a_2^+ a_1$ and $a_2^+ a_2$ generate the group $su(2)$ and $a_1^+ a_1$, $a_1^+ a_2^+$, $a_2 a_1$ and $a_2^+ a_2$ generate the group $su(1,1)$. Let us start by introducing three generators of $su(2)$,

$$J_+ = a_1^+ a_2, \quad J_- = a_2^+ a_1, \quad J_0 = \frac{1}{2}\left(a_1^+ a_1 - a_2^+ a_2\right). \tag{2}$$

These are the Schwinger representation of $su(2)$ algebra. The fourth generator is the total boson number operator

$$N = a_1^+ a_1 + a_2^+ a_2 \tag{3}$$

which commutes with the $su(2)$ generators. The superalgebra $osp(2,1)$ might be constructed by extending $su(2)$ algebra with the fermionic generators

$$V_+ = f^+ a_2, V_- = -f^+ a_1, W_+ = f a_1^+, W_- = f a_2^+ \tag{4a}$$

$$V_+ = f a_2, V_- = -f a_1, W_+ = f^+ a_1^+, W_- = f^+ a_2^+, \tag{4b}$$

where f^+ and f are fermions and they satisfy the anticommutation relation

$$\left\{f, f^+\right\} = 1. \tag{5}$$

The superalgebra $osp(2,1)$ can be constructed with the generators (2) and (4a) or (4b). It is easily seen that the generators given in (4a) and (4b) can be mapped on to each others by a change of the fermionic creation and annihilation operators. As discussed in [16, 17], the generators of the $osp(2,1)$ algebra are written as follows:

$$\{J_\pm, J_0, J \in osp(2,1)_{\bar{0}} \quad | \quad V_\pm, W_\pm \in osp(2,1)_{\bar{1}}\}, \tag{6}$$

where J is the total number operator of the system and is given by

$$J = \frac{1}{2}N + f^+ f \tag{7a}$$

$$J = \frac{1}{2}N + f f^+ \tag{7b}$$

for the generators (2, 4a) and (2, 4b), respectively. The generators of the $osp(2,1)$ superalgebra satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
[J_+, J_-] &= 2J_0, \quad [J_0, J_\pm] = \pm J_\pm, \quad [J, J_{\pm,0}] = 0 \\
[J_0, V_\pm] &= \pm\frac{1}{2}V_\pm, \quad [J_0, W_\pm] = \pm\frac{1}{2}W_\pm, \quad [J, V_\pm] = \frac{1}{2}V_\pm \\
[J_\pm, V_\mp] &= V_\pm, \quad [J_\pm, W_\mp] = W_\pm, \quad [J, W_\pm] = -\frac{1}{2}W_\pm \\
[J_\pm, V_\pm] &= 0, \quad [J_\pm, W_\pm] = 0 \\
\{V_\pm, W_\pm\} &= \pm Q_\pm, \quad \{V_\pm, W_\mp\} = -J_0 \pm J \\
\{V_\pm, V_\pm\} &= \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0.
\end{aligned} \tag{8}$$

It is well known that the $osp(2,1)$ algebra has been constructed by extending purely bosonic $su(2)$ algebra with fermionic generators. Meanwhile, we mention here, one can construct $osp(2,2)$ algebra by extending $su(1,1)$ algebra.

In general, the spectrum of the physical system can be calculated in a closed form when the Hamiltonian of the system can be written in terms of number operator and diagonal operator J_0, it can be diagonalized within the representation $[N]$. The abstract boson and/or fermion algebra can be associated with the exactly solvable Schrödinger equations by using the differential operator realizations of boson operators. This connection opens the way to an algebraic treatment of a large class of potentials of practical interest [18].

The Hamiltonian of the some physical systems can be constructed up to quadratic combination of the generators of the $osp(2,1)$ algebra. It will be shown that the associated Hamiltonian includes various physical system Hamiltonians. In order to relate the $osp(2,1)$ algebraic structure with the two dimensional 2×2 matrix Hamiltonians one can use standard matrix representations of the fermions:

$$f = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad f^+ = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{9}$$

$$ff^+ - f^+f = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The bosons can be realized in the Bargmann-Fock space. In order to obtain a (quasi)exactly solvable Hamiltonian, in the next section, we prepare a suitable transformation procedure.

3. Transformation of the operators

As discussed in [18], the Schwinger representation of $su(2)$ and $su(1,1)$ algebra with two boson operator can be mapped onto the generalized Gelfand-Dyson representation by a similarity transformation. The connection between the two representation can be obtained by introducing the operators

$$\Gamma_1 = (a_2)^{a_1^+ a_1} \quad \text{or} \quad \Gamma_2 = (a_2^+)^{a_1^+ a_1}. \tag{10}$$

Since a_1 and a_2 commute, Γ_1 and Γ_2 are well defined and their actions on the state are given by

$$\Gamma_1 |n_1, n_2\rangle = (a_2)^{n_1} |n_1, n_2\rangle = \sqrt{\frac{n_2!}{(n_2 - n_1)!}} |n_1, n_2 - n_1\rangle \tag{11a}$$

$$\Gamma_2 |n_1, n_2\rangle = (a_2^+)^{n_1} |n_1, n_2\rangle = \sqrt{\frac{n_2!}{(n_2 + n_1 + 1)!}} |n_1, n_2 + n_1\rangle . \tag{11b}$$

Our task is now to develop a transformation procedure to obtain generalized Gelfand-Dyson *like* representation of the $osp(2,1)$ superalgebra. This can be done by introducing the following similarity transformation induced by the metric:

$$‘S = (a_2^+)^{a_1^+ a_1 + \alpha\sigma_+\sigma_-}, \tag{12}$$

where α takes the values ± 1 . Since a_1, and a_2 commute and $\sigma_{\pm,0}$ also commute with the bosonic operators, the transformation of a_1 and a_1^+ under S can be obtained by writing $a_2^+ = e^b$, with $[a_1, b] = [a_1^+, b] = 0$,

$$\begin{aligned} S a_1 S^{-1} &= a_1 (a_2^+)^{-1} \\ S a_1^+ S^{-1} &= a_1^+ a_2^+ ; \end{aligned} \tag{13}$$

the transformations of a_2 and a_2^+ are

$$\begin{aligned} S a_2 S^{-1} &= a_2 - n (a_2^+)^{-1} \\ S a_2^+ S^{-1} &= a_2^+ , \end{aligned} \tag{14}$$

and the transformations of the $\sigma_\pm$ are given by

$$S \sigma_\pm S^{-1} = \sigma_\pm (a_2^+)^{\pm\alpha} \tag{15}$$

where n is given by

$$n = a_1^+ a_1 + \alpha\sigma_+\sigma_- \,. \tag{16}$$

The transformations of the bosons and fermions (13) through (16) play a key role in the construction of QES one-variable 2×2 matrix Hamiltonians. For two different values of $\alpha = \pm 1$, the two component polynomial spinors form a basis function for the generators of the $osp(2,1)$ algebra. Consequently, we obtain two classes of Hamiltonians which can be solved (quasi)exactly under some certain conditions.

3.1. Case: $S = (a_2^+)^{a_1^+ a_1 + \alpha\sigma_+\sigma_-}$; $\alpha = 1$

In the case of $\alpha = 1$, under the transformations (13–16), the generators (2), (4a) and (7a) of the $osp(2,1)$ algebra take the form

$$\begin{aligned} J'_+ &= S J_+ S^{-1} = -a_1^+ a_1^+ a_1 + a_1^+ (a_2^+ a_2 - \sigma_+\sigma_-) \\ J'_- &= S J_- S^{-1} = a_1 \\ J'_0 &= S J_0 S^{-1} = \frac{1}{2}\left(2a_1^+ a_1 - a_2^+ a_2 + \sigma_+\sigma_-\right) \\ J' &= S J S^{-1} = \frac{1}{2}\left(a_2^+ a_2 + \sigma_+\sigma_-\right) \\ V'_+ &= S V_+ S^{-1} = \sigma_+ (a_2^+ a_2 - a_1^+ a_1 - \sigma_+\sigma_-) \\ V'_- &= S V_- S^{-1} = -\sigma_+ a_1 \\ W'_+ &= S W_+ S^{-1} = \sigma_- a_1^+ \\ W'_- &= S W_- S^{-1} = \sigma_- \,. \end{aligned} \tag{17}$$

The difference between the representations of $osp(2,1)$ given in section 1 and primed representations (17) is that, while in the first the total number of a_1, a_2 bosons and f, f^+ fermions characterize the the system, in the later it is only the number of a_2 bosons that characterize the system. When we characterize the algebra by a fixed number $a_2^+a_2 = j$ in the primed representation, the generators can be expressed in terms of one-boson operator a_1 and yield the following realization:

$$\begin{aligned}
J'_+ &= SJ_+S^{-1} = -a_1^+a_1^+a_1 + a_1^+(j - \sigma_+\sigma_-) \\
J'_- &= SJ_-S^{-1} = a_1 \\
J'_0 &= SJ_0S^{-1} = \frac{1}{2}\left(2a_1^+a_1 - j + \sigma_+\sigma_-\right) \\
J' &= SJS^{-1} = \frac{1}{2}\left(j + \sigma_+\sigma_-\right) \\
V'_+ &= SV_+S^{-1} = -\sigma_+(a_1^+a_1 - j) \\
V'_- &= SV_-S^{-1} = -\sigma_+a_1 \\
W'_+ &= SW_+S^{-1} = \sigma_-a_1^+ \\
W'_- &= SW_-S^{-1} = \sigma_- \,.
\end{aligned} \tag{18}$$

These generators play an important role in the quasi-exact solution of the matrix Schrödinger equation. The generators of the $osp(2,1)$ algebra can be expressed as differential equation in the Bargmann-Fock space by defining the bosonic operators:

$$a_1 = \frac{d}{dx}, \quad a_1^+ = x. \tag{19}$$

The two component polynomials of degree j and $j+1$ form a basis function for the generators of the $osp(2,1)$ algebra in the Bargmann-Fock space:

$$P_{n+1,n}(x) = \begin{pmatrix} x^0, x^1, \cdots, x^{n+1} \\ x^0, x^1, \cdots, x^n \end{pmatrix}. \tag{20}$$

The general QES operator can be obtained by linear and bilinear combinations of the generators of the $osp(2,1)$ superalgebra. Action of the QES operator on the basis function (20) gives us a recurrence relation; therefore, the wavefunction is itself the generating function of the energy polynomials. The eigenvalues are then produced by the roots of such polynomials. Before illustrating this application of the procedure given here on the physical examples, let us construct another representations of the $osp(2,1)$ superalgebra.

3.2. Case: $S = (a_2^+)^{a_1^+a_1 + \alpha\sigma_+\sigma_-}$; $\alpha = -1$

Using the same similarity transformation procedure given in section 3, we obtain the second class of the $osp(2,1)$ superalgebra. In this case, the generators $J_{\pm,0}$ remain the same as in (18), while the generators $V_\pm, W_\pm$ and J given in (4b) and (7b), respectively, take the form

$$\begin{aligned}
J' &= SJS^{-1} = \frac{1}{2}\left(j + 1 + \sigma_-\sigma_+\right) \\
V'_+ &= SV_+S^{-1} = -\sigma_-(a_1^+a_1 - j - 1) \\
V'_- &= SV_-S^{-1} = -\sigma_-a_1 \\
W'_+ &= SW_+S^{-1} = \sigma_+a_1^+ \\
W'_- &= SW_-S^{-1} = \sigma_+ \,.
\end{aligned} \tag{21}$$

The basis function of this structure takes the form

$$P_{n,n+1}(x) = \begin{pmatrix} x^0, x^1, \cdots, x^n \\ x^0, x^1, \cdots, x^{n+1} \end{pmatrix}. \tag{22}$$

The other transformation can be done by introducing the following similarity transformation induced by the metric

$$T = (a_2)^{-a_1^+ a_1 + \eta\sigma_+\sigma_-}, \tag{23}$$

where η takes the values ± 1. By using the similar arguments given in the previous section one can easily obtain the following transformations:

$$\begin{aligned} Ta_1T^{-1} &= a_1 a_2^+ \\ Ta_1^+T^{-1} &= a_1^+(a_2^+)^{-1} \\ Ta_2T^{-1} &= a_2 \\ Ta_2^+T^{-1} &= a_2^+ + n(a_2^+)^{-1} \\ T\sigma_\pm T^{-1} &= \sigma_\pm (a_2^+)^{\pm\alpha}. \end{aligned} \tag{24}$$

With this transformation we can construct two more different realizations of the $osp(2,1)$ algebra for $\eta = \pm 1$.

3.3. Case: $T = (a_2)^{-a_1^+ a_1 + \eta\sigma_+\sigma_-}$; $\eta = 1$

In the case of $\eta = 1$, under the transformations (24) the generators (2), (4b) and (7b) of the $osp(2,1)$ algebra take the form

$$\begin{aligned} J'_+ &= TJ_+T^{-1} = a_1^+ \\ J'_- &= TJ_-T^{-1} = a_1^+ a_1 a_1 + (a_2^+ a_2 + \sigma_+\sigma_-) \\ J'_0 &= TJ_0T^{-1} = \frac{1}{2}\left(2a_1^+ a_1 - a_2^+ a_2 - \sigma_+\sigma_-\right) \\ J' &= TJT^{-1} = \frac{1}{2}\left(a_2^+ a_2 + 1 + \sigma_-\sigma_+\right) \\ V'_+ &= TV_+T^{-1} = \sigma_- \\ V'_- &= TV_-T^{-1} = -\sigma_- a_1 \\ W'_+ &= TW_+T^{-1} = \sigma_+ a_1^+ \\ W'_- &= TW_-T^{-1} = \sigma_+(a_2^+ a_2 - a_1^+ a_1 + 1 + \sigma_+\sigma_-). \end{aligned} \tag{25}$$

This realization can also be characterized by $a_2^+ a_2 = j$. The basis function of the realization is given by

$$P_{n+1,n}(x) = \begin{pmatrix} x^0, x^1, \cdots, x^{n+1} \\ x^0, x^1, \cdots, x^n \end{pmatrix}, \tag{26}$$

in the Bargmann-Fock space.

3.4. Case: $T = (a_2)^{-a_1^+ a_1 + \eta\sigma_+\sigma_-}$; $\eta = -1$

The generators (2) of the $osp(2,1)$ superalgebra take the same form as in (25) and the remaining generators can be expressed as:

$$\begin{aligned} J' &= TJT^{-1} = \frac{1}{2}\left(a_2^+ a_2 - \sigma_+\sigma_-\right) \\ V'_+ &= TV_+T^{-1} = \sigma_+ \\ V'_- &= TV_-T^{-1} = -\sigma_+ a_1 \\ W'_+ &= TW_+T^{-1} = \sigma_- a_1^+ \\ W'_- &= TW_-T^{-1} = \sigma_-(a_2^+ a_2 - a_1^+ a_1 + 1 - \sigma_+\sigma_-). \end{aligned} \tag{27}$$

The basis function of this structure is given by

$$P_{n,n-1}(x) = \begin{pmatrix} x^0, x^1, \cdots, x^n \\ x^0, x^1, \cdots, x^{n-1} \end{pmatrix}. \tag{28}$$

Consequently we have obtained four classes generators for the $osp(2,1)$ superalgebra by using two transformation operators S and T. These generators can be expressed as one-variable 2×2 matrix differential operators useful in the study of (quasi) exactly solvable systems.

4. Application

This section includes solution of the some physical Hamiltonians by using the procedure given in the previous sections of this article. In particular, our approach is useful for the study of nonlinear optical systems.

4.1. Jaynes-Cummings Hamiltonian with the Kerr nonlinearity

The effective Hamiltonian, which represents the Jaynes-Cummings model with Kerr nonlinearity, has been expressed as [19]

$$H = \omega a^+ a + \frac{1}{2}\omega_0 \sigma_0 + \kappa(a^+ \sigma_- + a\sigma_+) + \lambda a^+ a a^+ a, \tag{29}$$

where a and a^+ are annihilation and creation operators of the radiation mode which, with a frequency $\sigma_{\pm,0}$, are the standard Pauli matrices for the atom, and has a frequency of transition ω_0; κ, λ are the coupling constant of the field and atom and the coupling constant of the field and Kerr medium, respectively. The eigenvalue equation can be written as

$$H\psi = E\psi. \tag{30}$$

Our task is now to express the Hamiltonian (29) in terms of the generators of the $osp(2,1)$ algebra. In terms of the generators given in (2), (4a) and (7a) and number operator N, the Hamiltonian can be written as

$$H' = \omega(2J_0 + N) + \frac{\omega_0}{2}(J - N - J_0 - 1) + \lambda(2J_0 + N)^2 + \kappa(W_+ - V_-). \tag{31}$$

Note that the bosonic operators $a_1 = a$ and $a_1^+ = a^+$. By considering the transformation procedures given in (18) one can obtain the single variable differential equation, which in the Bargmann-Fock space reads:

$$H' = (\omega + \omega_0)(2x\frac{d}{dx} + 1 - j) + (\omega - \omega_0)\sigma_0 + \lambda(2x\frac{d}{dx} + 1 + \sigma_0 - j)^2 + \kappa(x\sigma_- + \frac{d}{dx}\sigma_+). \tag{32}$$

The action of the Hamiltonian on the two component spinor

$$P_{n,m}(x) = \begin{pmatrix} u_n(x) \\ v_m(x) \end{pmatrix} \tag{33}$$

gives us the following recurrence relation:

$$\big(2\omega - (\omega + \omega_0)(j - 2n) + \lambda(j - 2(n+1))^2 - E\big)\, u_n(E) + \kappa m v_{m-1}(E) \;=\; 0 \tag{34a}$$

$$-\,(E + (j - 2m)(\omega - \lambda(j - m) + (j - 2m - 2)\omega_0)\, v_m(E) + \kappa u_{n+1} \;=\; 0. \tag{34b}$$

The recurrence relation implies that the wavefunction is itself the generating function of the energy polynomials. The eigenvalues are then given by the roots of such polynomials. If E_j is a root of the polynomial, the wavefunction is truncated at $n+1 = m = j$ and belongs to the spectrum of the Hamiltonian H. Considering the initial condition $v_{j+1} = u_{j+1} = 0$, we have obtained first few eigenvalues of the Hamiltonian H:

$$\begin{pmatrix} 9\lambda + 3\omega + \omega_0 - E & 0 \\ 0 & \lambda - \omega + \omega_0 - E \end{pmatrix} \begin{pmatrix} u_1 \\ v_0 \end{pmatrix} = 0.$$

Therefore we can obtain

$$E = \{9\lambda + 3\omega + \omega_0, \quad \lambda - \omega + \omega_0\}$$

for $j = 1$. Similarly for $j = 2$, the eigenvalues of the Hamiltonian are given by

$$E = \{2\omega_0, \quad 2(\omega + 2\lambda), \quad \omega + 10\lambda + \omega_0 \pm \sqrt{\kappa^2 + (3\omega + 6\lambda + \omega_0)^2}. \tag{35}$$

Analytical solutions of the recurrence relations (34a) and (34b) are available only for the first few values of j, for large j the solutions become numerical.

4.2. Modified Jaynes-Cummings Hamiltonian

The modified Jaynes-Cummings Hamiltonian has been constructed to investigate single two level atom placed in the common domain of two cavities interacting with two quantized modes. It is given by [20]

$$H = \omega(a_1^+ a_1 + a_2^+ a_2) + \frac{\omega_0}{2}\sigma_0 + \lambda_1(a_1\sigma_+ + a_1^+\sigma_-) + \lambda_2(a_2\sigma_+ + a_2^+\sigma_-). \tag{36}$$

Using the same procedure as in section 4.1, we can obtain the the transformed form of the Hamiltonian (36) in terms of the generators $osp(2,1)$ algebra, given in (2), (4a) and (7a) and with number operator N; then the Hamiltonian can be written as

$$H = \omega N + \frac{\omega_0}{2}\left(J - 1 - \frac{N}{2}\right) + \lambda_1(W_+ - V_-) + \lambda_2(W_- + V_+). \tag{37}$$

The eigenvalues of this Hamiltonian can be obtained by using the transformation procedure given (17) or (27). The transformed Hamiltonian can be expressed as one variable differential operator:

$$H' = \omega(j - 1 - \sigma_0) + \frac{\omega_0}{2}\sigma_0 + \lambda_1\left(x\sigma_- + \frac{d}{dx}\sigma_+\right) + \lambda_2\left(\sigma_- - (x\frac{d}{dx} - j)\sigma_+\right). \tag{38}$$

Therefore two component one variable spinor (33) from a basis function for the Hamiltonian (38)

$$\begin{aligned} \phi_1(x) &= C_1(\lambda_2 + \lambda_1 x)^n(\lambda_1 - \lambda_2 x)^{j-n} \\ \phi_2(x) &= C_2(\lambda_2 + \lambda_1 x)^{n-1}(\lambda_1 - \lambda_2 x)^{j-n}, \end{aligned}$$

where $n = 0, 1, \cdots, j$, and eigenvalues of the Hamiltonian are given by

$$E = \omega(j-1) \pm \sqrt{(\omega_0 - 2\omega)^2 + 4n(\lambda_1^2 + \lambda_2^2)}\,.$$

Consequently we have obtained the exact result for the eigenvalues of the modified Jaynes-Cummings Hamiltonian. The procedure given here can be applied to obtain eigenfunction and eigenvalues of various physical Hamiltonians. In this section we have discussed solution of the some simple physical Hamiltonians by using the transformation of the $osp(2,1)$ superalgebra, without further details.

5. Conclusion

The basic features of our approach is to construct $osp(2,1)$ invariant subspaces. We consider systems whose Hamiltonian can be expressed in terms of two bosons and one fermions. Furthermore, we have presented two different boson-fermion realization of $osp(2,1)$ algebra. The corresponding realizations have been transformed in the form of one dimensional differential equations. Meanwhile, we have shown that our procedure is appropriate to obtain eigenvalues and eigenfunctions of various systems. In particular, we have constructed solutions to the Jaynes-Cummings and Modified Jaynes-Cummings Hamiltonians.

The suggested approach can be generalized in various directions. Invariant subspaces of the multi-boson and multi-fermion systems can be obtained by extending the method given in this paper. In particular, the subspace of the $osp(2,2)$ superalgebra can easily be constructed.

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