

On Hamiltonian Formulation of Non-Conservative Systems

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Received 16.02.2004

Abstract

Fractional derivatives are used to construct the Lagrangian and the Hamiltonian formulation for non-conservative systems. To clarify the theory of Riewe two interesting examples are given. The potentials are obtained using the Laplace transform operator for fractional derivatives and the Lagrangian and Hamiltonian formulations are constructed for the two systems. Besides, it is shown that the Hamilton equations of motion are in agreement with the Euler-Lagrange equations for these systems.

Key Words: Fractional derivatives, Hamiltonian Systems, Non-Conservative systems, Laplace Transform.

1. Introduction

It is well known that the use of the Euler-Lagrange equation to set up the equations of motion for certain physical systems is more convenient and useful than the use of Newtonian mechanics. The important benefit is that when the Lagrangian and the momenta for a certain system are known, then the Hamiltonian function can be written, and once the Hamiltonian is known, then the system becomes amenable to the techniques of quantum mechanics which cannot be done using Newtonian mechanics. But although the mechanics developed by Newton can be applied to both conservative and non-conservative systems, traditional Lagrangian and Hamiltonian mechanics cannot be used with non-conservative systems.

Several methods have been proposed and devised in order to introduce dissipative effects such as friction into classical Lagrangian and Hamiltonian mechanics. One of these methods is the Rayleigh dissipation function, which can be used when the frictional forces are proportional to the velocity [1, 2]. Whereas in this method another scalar function is needed in addition to the Lagrangian to specify the equations of motion, this function cannot appear in the Hamiltonian function, so it is of no use when attempting to quantize friction.

Another method [3, 4] introduces an auxiliary coordinate in the Lagrangian that describes a reverse-time system with negative friction. This method leads to the desired equations of motion but the Hamiltonian leads to extraneous solutions that must be rejected, and the physical meaning of the momenta is unclear.

A good and realistic method is to include the microscopic details of the dissipation directly in the Lagrangian or the Hamiltonian [5]. This method represents a valuable tool in the study of quantum dissipation, but it is not intended to be a general method of introducing the force of friction into Lagrangian mechanics.

Thus, we see that none of the above techniques exhibits the same directness and simplicity found in the mechanics of conservative systems. The problem of developing a generalized mechanics to deal with both conservative and non-conservative systems has been solved when Riewe [6, 7] used fractional derivatives. He derived a generalized Euler-Lagrange equation following the same pattern as in the conventional calculus of variations used in classical mechanics. His generalized equation allows fractional derivatives to appear in the Lagrangian and the Hamiltonian, whereas traditional Lagrangian mechanics often deals with first order derivatives. In this generalization the fractional derivatives appear in both the Lagrangian and the equations of motion. In other words, non-conservative forces can be calculated from potentials that contain fractional derivatives. Also, this generalization motivated him to define a generalized Hamiltonian, which leads to the generalized Hamilton's equations of motion.

An important application of the generalized mechanics, Riewe [6] showed that a frictional force proportional to the velocity would appear in the equations of motion using a Lagrangian that contains a potential proportional to $[\frac{d^{1/2}x}{dt^{1/2}}]^2$. In addition, if the Lagrangian contains a term proportional to $[\frac{d^n x}{dt^n}]^2$, then the equation of motion will contain a term proportional to $[\frac{d^{2n}x}{dt^{2n}}]$, where n is any positive integer. Based on this observation, it was easy for Riewe to guess that a frictional force of the form $\gamma \frac{dx}{dt}$, where γ is constant, should follow directly from a Lagrangian containing a term proportional to $[\frac{d^{1/2}x}{dt^{1/2}}]^2$. This important result motivates the investigation about the potentials for other dissipative forces, which cannot be guessed directly as in the above case. But Riewe didn't give a general method to write potentials corresponding to other dissipative forces such as that proportional to $(\dot{x})^{1/2}$ or $(\dot{x})^2$. Based on Riewe's formalism and the techniques of the Laplace transform of fractional derivatives, a formula was developed to obtain potentials corresponding to any dissipative force of the form $(\dot{x})^p$, $p \geq 0$ [8].

In this work the results obtained in references [8] and the Hamiltonian formulation with fractional derivatives developed by Riewe are used to study the Hamiltonian formulation for some dissipative systems; the Hamiltonian and Hamilton equations of motion for these systems are obtained. Also a general Lagrangian and Hamiltonian are suggested to represent fractionally damped systems.

2. Hamiltonian Formulation with Fractional Derivatives

The generalized Euler-Lagrange equations developed by Riewe [1, 2] read as

$$\sum_{n=0}^N (-1)^{s(n)} \frac{d^{s(n)}}{d(t-a)^{s(n)}} \frac{\partial L}{\partial q_{r,s(n)}} = 0, \tag{1}$$

where $r = 1, 2, 3, \dots, R$ indicates the particular coordinate, $s(n)$ indicates the order of the n^{th} derivative in the Lagrangian, which is assumed to contain N different derivatives of the coordinates $x_r = q_r$ with respect to t (including fractional derivatives). Therefore, $n = 1, 2, 3, \dots, N$. In other words the Lagrangian is a function of the coordinates x_r , the derivatives $q_{r,s(n),b}$ and the variable t . For generalized mechanics $s(n)$ can be any non negative real number and for complementation $s(0)$ is defined to be zero, such that $q_{r,s(0)}$ denotes the coordinate x_r . The constants a, b in eq. (1) denote the limits of the time interval $t = a$ to $t = b$.

In references [8] a general formula was developed to obtain potentials corresponding to arbitrary forces, conservative and non-conservative. This formula reads

$$U = (-1)^{-(\alpha+1)} \int \left[\mathcal{L}^{-1} \left(\frac{1}{S^\alpha} \mathcal{L} [F(q_\beta)] \right) \right] dq_\alpha \tag{2}$$

Here, $\alpha = \frac{\beta}{2}$, \mathcal{L} denotes the Laplace transform operator and q_β represents the time derivative of $q_0 = x$ of order β , i.e. $q_1 = \dot{x}$, $q_{\frac{1}{2}} = \frac{d^{\frac{1}{2}}x}{d(t-b)^{\frac{1}{2}}}$, $q_2 = \ddot{x}$. Also, U is the potential function that gives $F(q_\beta)$ in the equation of motion where the Lagrangian reads as $L = T - U$.

Following Riewe, the generalized momenta takes the form

$$p_{r,s(n),b} = \sum_{k=0}^{N-n-1} (-1)^{s(k+n+1)-s(n+1)} \frac{d^{s(k+n+1)-s(n+1)}}{d(t-a)^{s(k+n+1)-s(n+1)}} \left[\frac{\partial L}{\partial q_{r,s(k+n+1)}} \right], \quad (3)$$

where $n = 0, 1, 2, \dots, N - 1$. Then the Hamiltonian will take the form

$$H = \sum_{n=1}^N q_{r,s(n)} p_{r,s(n-1)} - L. \quad (4)$$

The generalized Hamilton equations of motion following the same steps described in many books in classical mechanics are obtained as

$$\frac{\partial H}{\partial q_{r,s(n)}} = (-1)^{s(n+1)-s(n)} \frac{d^{s(n+1)-s(n)}}{d(t-a)^{s(n+1)-s(n)}} p_{r,s(n)} \quad (5)$$

$$\frac{\partial H}{\partial p_{r,s(n)}} = q_{r,s(n+1)} \quad (6)$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (7)$$

Here n takes the values $n = 0, 1, 2, \dots, N - 1$.

These equations are equivalent to the Euler-Lagrange equations and valid for conservative and non-conservative systems. In addition, these equations are generalizations for Hamilton equations, which deal only with integer order derivatives.

Before concluding this section, it is important to note that the law of exponents is not generally satisfied for the fractional differential operators, that is $D^\alpha D^\beta \neq D^{\alpha+\beta}$ [9]. However there are several types of functions for which such a law is satisfied for the fractional differential operators. The one-dimensional case is considered in the following examples. Therefore the subscript r can be omitted in the above equations.

3. Examples

As an example of the applications of fractional derivatives in Lagrangian and Hamiltonian mechanics, consider the following general damping force:

$$F(q_\beta) = -\gamma D^\beta q_0, \quad \beta > 0 \quad (8)$$

Here $q_0 = x$, $D^\beta q_0 = \frac{d^\beta x}{d(t-b)^\beta}$, γ is a positive constant. Using eq. (2), the potential corresponding to this force reads

$$U = (-1)^{-(\alpha+1)} \int [\mathcal{L}^{-1}[\frac{1}{S^\alpha} \mathcal{L}(-\gamma D^\beta q_0)]] dq_\alpha. \quad (9)$$

But $\alpha = \frac{\beta}{2}$; therefore

$$U = \frac{\gamma}{(-1)^{\frac{\beta}{2}}} \int [\mathcal{L}^{-1}(\frac{1}{S^{\frac{\beta}{2}}} \mathcal{L}[D^\beta q_0])] dq_{\frac{\beta}{2}}. \quad (10)$$

The Laplace transform of the fractional derivative, with zero boundary conditions, is given as [9]

$$\mathcal{L}[D^\delta f(t)] = S^\delta f(S), \quad (11)$$

where $f(S) = \mathcal{L}[f(t)]$. Taking the inverse transform of both sides of the above equation we get,

$$D^\delta f(t) = \mathcal{L}^{-1}[S^\delta f(S)] = \mathcal{L}^{-1}(S^\delta \mathcal{L}[f(t)]). \quad (12)$$

Using eq. (11), eq. (10) takes the form

$$U = \frac{\gamma}{(-1)^{\frac{\beta}{2}}} \int [\mathcal{L}^{-1}[S^{\frac{\beta}{2}}\mathcal{L}(q_0)]] dq_{\frac{\beta}{2}}. \quad (13)$$

Making use of eq. (12) we have

$$D^{\frac{\beta}{2}}q_0 = \mathcal{L}^{-1}[S^{\frac{\beta}{2}}f(s)] = \mathcal{L}^{-1}[S^{\frac{\beta}{2}}\mathcal{L}(q_0)]. \quad (14)$$

Therefore U becomes

$$U = \frac{\gamma}{(-1)^{\frac{\beta}{2}}} \int [D^{\frac{\beta}{2}}(q_0)] dq_{\frac{\beta}{2}} = \frac{\gamma}{(-1)^{\frac{\beta}{2}}} \int [q_{\frac{\beta}{2}}] dq_{\frac{\beta}{2}},$$

$$U = \frac{\gamma}{2(-1)^{\frac{\beta}{2}}} [q_{\frac{\beta}{2}}]^2 = \frac{\gamma}{2(-1)^{\frac{\beta}{2}}} \left[\frac{d^{\frac{\beta}{2}}q_0}{d(t-b)^{\frac{\beta}{2}}} \right]^2. \quad (15)$$

This result represents a generalization of that suggested by Riewe. Substituting $\beta = 1$, one gets the special case of a frictional force proportional to the velocity.

The Lagrangian for this system takes the form

$$L = \frac{1}{2}mq_1^2 - \frac{\gamma}{2(-1)^{\frac{\beta}{2}}}q_{\frac{\beta}{2}}^2 - V(q_0), \quad (16)$$

where β is any positive number except 0 and 2, and γ is constant. Noting that $N = 2$, $s(0) = 0$, $s(1) = \frac{\beta}{2}$, $s(2) = 1$ and using eq. (1), the equations of motion corresponding to this Lagrangian can be obtained as

$$\frac{\partial L}{\partial q_0} + (-1)^{\frac{\beta}{2}} \frac{d^{\frac{\beta}{2}}}{d(t-b)^{\frac{\beta}{2}}} \frac{\partial L}{\partial q_{\frac{\beta}{2}}} - \frac{d}{dt} \frac{\partial L}{\partial q_1} = 0.$$

Substituting the Lagrangian given in eq. (16), we get

$$-\frac{\partial V}{\partial q_0} - \gamma \frac{d^{\frac{\beta}{2}}}{d(t-b)^{\frac{\beta}{2}}}(q_{\frac{\beta}{2}}) - m \frac{d}{dt}(q_1) = 0,$$

which leads to the following equations of motion:

$$m\ddot{x} = -\frac{\partial V(x)}{\partial x} - \gamma D^{\frac{\beta}{2}}D^{\frac{\beta}{2}}x. \quad (17)$$

If $x(t)$ is such that the law of exponents is satisfied for the fractional differential operators in the second term of the right hand side of eq. (17), then this equation can be rewritten as

$$m\ddot{x} = -\frac{\partial V(x)}{\partial x} - \gamma \frac{d^{\beta}x}{d(t-b)^{\beta}}.$$

For simplicity we will consider the limiting case in which $a \rightarrow b$ while keeping $a < b$. Hence all fractional derivatives we encounter in this work can be approximated by derivatives of the form $\frac{d^u}{d(t-b)^u}$. It is clear from the above equation that for $\beta = 1$, the second term in the right hand side of this equation represents a dissipative force proportional to velocity (viscous force), while the corresponding potential that leads to this force in the equation of motion is

$$U = \frac{\gamma}{2(-1)^{\frac{1}{2}}} \left[\frac{d^{\frac{1}{2}}x}{d(t-b)^{\frac{1}{2}}} \right]^2, \quad (18)$$

which is the result obtained by Riewe. From eqs. (16) and (17), we see that by using fractional derivatives of various orders, it is possible to construct Lagrangians that lead to a wide range of dissipative equations of motion. Hence we can say that these Lagrangians describe non-conservative forces, rather than the functions more commonly used to describe dissipation. Equations having the form given in eq. (17) are called fractional differential equations, which are widely used now to describe systems with damping materials [10, 11]. In general, fractional derivatives are used in viscoelastic representation [12].

Using eq. (3) with $N = 2$, the momenta reads

$$\begin{aligned}
 p_0 &= \sum_{k=0}^1 (-1)^{s(k+1)-s(1)} \frac{d^{s(k+1)-s(1)}}{d(t-b)^{s(k+1)-s(1)}} \frac{\partial L}{\partial q_{s(k+1)}} \\
 &= \frac{\partial L}{\partial q_{s(1)}} + (-1)^{s(2)-s(1)} \frac{d^{s(2)-s(1)}}{d(t-b)^{s(2)-s(1)}} \frac{\partial L}{\partial q_{s(2)}} \\
 &= \frac{\partial L}{\partial q_{\frac{\beta}{2}}} + (-1)^{1-\frac{\beta}{2}} \frac{d^{1-\frac{\beta}{2}}}{d(t-b)^{1-\frac{\beta}{2}}} \frac{\partial L}{\partial q_1} \\
 &= -\frac{\gamma}{(-1)^{\frac{\beta}{2}}} q_{\frac{\beta}{2}} + m(-1)^{1-\frac{\beta}{2}} \frac{d^{1-\frac{\beta}{2}}}{d(t-b)^{1-\frac{\beta}{2}}} [q_1] \\
 p_0 &= -\gamma(-1)^{-\frac{\beta}{2}} q_{\frac{\beta}{2}} - m(-1)^{-\frac{\beta}{2}} D^{1-\frac{\beta}{2}} [q_1].
 \end{aligned} \tag{19}$$

Also

$$p_{\frac{\beta}{2}} = (-1)^{s(2)-s(2)} \frac{d^{s(2)-s(2)}}{d(t-b)^{s(2)-s(2)}} \left[\frac{\partial L}{\partial q_{s(2)}} \right] = \frac{\partial L}{\partial q_1} = mq_1. \tag{20}$$

Thus, the Hamiltonian reads

$$\begin{aligned}
 H &= \sum_{n=1}^2 q_{s(n)} p_{s(n-1)} - L \\
 &= q_{s(1)} p_{s(0)} + q_{s(2)} p_{s(1)} - L \\
 &= q_{\frac{\beta}{2}} p_0 + q_1 p_{\frac{\beta}{2}} - \frac{1}{2} m q_1^2 + \frac{\gamma}{2(-1)^{\frac{\beta}{2}}} q_{\frac{\beta}{2}}^2 + V(q_0).
 \end{aligned} \tag{21}$$

Using eq. (20), eq. (21) can be rewritten as

$$H = \frac{p_{\frac{\beta}{2}}^2}{2m} + q_{\frac{\beta}{2}} p_0 + \frac{\gamma}{2(-1)^{\frac{\beta}{2}}} q_{\frac{\beta}{2}}^2 + V(q_0). \tag{22}$$

Then the Hamilton equations of motion are

$$\frac{\partial H}{\partial q_{s(0)}} = (-1)^{s(1)-s(0)} \frac{d^{s(1)-s(0)}}{d(t-b)^{s(1)-s(0)}} p_{s(0)},$$

which can be written as

$$\frac{\partial H}{\partial q_0} = (-1)^{\frac{\beta}{2}} \frac{d^{\frac{\beta}{2}}}{d(t-b)^{\frac{\beta}{2}}} p_0 = (-1)^{\frac{\beta}{2}} D^{\frac{\beta}{2}} p_0 \tag{23}$$

and

$$\frac{\partial H}{\partial q_{s(1)}} = (-1)^{s(2)-s(1)} \frac{d^{s(2)-s(1)}}{d(t-b)^{s(2)-s(1)}} p_{s(1)},$$

which leads to

$$\frac{\partial H}{\partial q_{\frac{\beta}{2}}} = -(-1)^{-\frac{\beta}{2}} D^{1-\frac{\beta}{2}} [p_{\frac{\beta}{2}}]. \quad (24)$$

Using also eq. (6), we get

$$\frac{\partial H}{\partial p_0} = q_{\frac{\beta}{2}} \quad (25)$$

and

$$\frac{\partial H}{\partial p_{\frac{\beta}{2}}} = q_1. \quad (26)$$

From the above Hamilton equations we deduce that eq. (23) yields the Euler-Lagrange equation, eqs.(24) and (26) are equivalent to the definition of the momenta, while eq. (25) is an identity.

Example 3.2

In this example a dissipative force proportional to $(q_1)^{\frac{1}{2}}$ is considered. If we assume an object of mass m projected with initial velocity v_0 , and is subject to a resistive force proportional to $(q_1)^{\frac{1}{2}}$, then the differential equation of motion reads [13]

$$F(q_{\beta}) = -c(q_1)^{\frac{1}{2}} = m \frac{dq_1}{dt} = mq_2, \quad (27)$$

where c is a positive constant. Solving the above equation for the time t , we obtain

$$t = \frac{-2m}{c} [(q_1)^{\frac{1}{2}} - (v_0)^{\frac{1}{2}}],$$

which is equivalent to

$$(q_1)^{\frac{1}{2}} = (v_0)^{\frac{1}{2}} - \frac{c}{2m} t. \quad (28)$$

Substituting in eq. (27), we have

$$F(q_{\beta}) = -c[(v_0)^{\frac{1}{2}} - \frac{c}{2m} t] \quad (29)$$

From eq. (2) the potential corresponding to this force reads

$$\begin{aligned} U &= (-1)^{-(\alpha+1)} \int [\mathcal{L}^{-1}[\frac{1}{S^\alpha} \mathcal{L}(-c[v_0^{\frac{1}{2}} - \frac{c}{2m} t])]] dq_\alpha \\ &= c(-1)^{-\alpha} \int [\mathcal{L}^{-1}(\frac{1}{S^\alpha} [\mathcal{L}(v_0^{\frac{1}{2}}) - \frac{c}{2m} \mathcal{L}(t)])] dq_\alpha \\ &= c(-1)^{-\alpha} \int [\mathcal{L}^{-1}(\frac{1}{S^\alpha} [\frac{(v_0)^{\frac{1}{2}}}{S} - \frac{c}{2m} \frac{1}{S^2}])] dq_\alpha, \end{aligned} \quad (30)$$

where

$$\mathcal{L}(t^n) = \frac{n!}{S^{n+1}}, \quad S > 0, n > -1. \quad (31)$$

Equation (30) can be rewritten in the useful form

$$U = c(-1)^{-\alpha} \int [\mathcal{L}^{-1}(S^\alpha \frac{(v_0)^{\frac{1}{2}}}{S^{1+2\alpha}} - \frac{c}{2m} S^\alpha \frac{1}{S^{2+2\alpha}})] dq_\alpha. \quad (32)$$

Since $\beta = 1$ then $\alpha = \frac{1}{2}$; therefore U takes the form

$$U = c(-1)^{-\frac{1}{2}} \int [\mathcal{L}^{-1}(S^{\frac{1}{2}} \frac{(v_0)^{\frac{1}{2}}}{S^2} - \frac{c}{2m} S^{\frac{1}{2}} \frac{1}{S^3})] dq_{\frac{1}{2}}. \quad (33)$$

The inverse transform in the integrand of this equation can be evaluated separately with the help of eqs. (12) and (31) as

$$\mathcal{L}^{-1}[S^{\frac{1}{2}} \frac{(v_0)^{\frac{1}{2}}}{S^2}] = (v_0)^{\frac{1}{2}} \mathcal{L}^{-1}[S^{\frac{1}{2}} \mathcal{L}(t)] = (v_0)^{\frac{1}{2}} D^{\frac{1}{2}}(t), \quad (34)$$

$$\mathcal{L}^{-1}[\frac{c}{2m} S^{\frac{1}{2}} \frac{1}{S^3}] = \frac{c}{2m} \mathcal{L}^{-1}[S^{\frac{1}{2}} \mathcal{L}(\frac{t^2}{2})] = \frac{c}{4m} D^{\frac{1}{2}}(t^2). \quad (35)$$

Using eqs. (34) and (35), U becomes,

$$U = -ic \int [(v_0)^{\frac{1}{2}} D^{\frac{1}{2}}(t) - \frac{c}{4m} D^{\frac{1}{2}}(t^2)] dq_{\frac{1}{2}}. \quad (36)$$

The fractional derivatives in eq. (36) can be easily evaluated. The results are:

$$D^{\frac{1}{2}}(t) = \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}, \quad (37)$$

$$D^{\frac{1}{2}}(t^2) = \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}. \quad (38)$$

Thus U takes the form

$$U = -ic \int \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}} [(v_0)^{\frac{1}{2}} - \frac{c}{3m} t] dq_{\frac{1}{2}}.$$

Evaluating the integral, we obtain

$$U = -\frac{2ic}{\sqrt{\pi}} [t^{\frac{1}{2}} [(v_0)^{\frac{1}{2}} - \frac{c}{3m} t]] q_{\frac{1}{2}}. \quad (39)$$

We see that U depends on the time explicitly. The Lagrangian for this system takes the form

$$L = \frac{1}{2} m q_1^2 + \frac{2ic}{\sqrt{\pi}} [t^{\frac{1}{2}} [(v_0)^{\frac{1}{2}} - \frac{c}{3m} t]] q_{\frac{1}{2}}. \quad (40)$$

Substituting this Lagrangian in the generalized Euler-Lagrange equation, one gets eq. (27) [8]. To determine the momenta, we have $N = 2$, $n = 0, 1$ and $s(0) = 0$, $s(1) = \frac{1}{2}$, $s(2) = 1$, thus eq. (3) gives

$$\begin{aligned} p_0 &= \sum_{k=0}^1 (-1)^{s(k+1)-s(1)} \frac{d^{s(k+1)-s(1)}}{d(t-b)^{s(k+1)-s(1)}} \frac{\partial L}{\partial q_{s(k+1)}} \\ &= \frac{\partial L}{\partial q_{\frac{1}{2}}} + i \frac{d^{\frac{1}{2}}}{d(t-b)^{\frac{1}{2}}} \frac{\partial L}{\partial q_1} \\ &= \frac{2ic}{\sqrt{\pi}} [t^{\frac{1}{2}} [(v_0)^{\frac{1}{2}} - \frac{c}{3m} t]] + im D^{\frac{1}{2}}[q_1] \end{aligned} \quad (41)$$

and

$$p_{\frac{1}{2}} = (-1)^{s(2)-s(2)} \frac{d^{s(2)-s(2)}}{d(t-b)^{s(2)-s(2)}} \frac{\partial L}{\partial q_{s(2)}} = m q_1. \quad (42)$$

The Hamiltonian reads

$$\begin{aligned} H &= \sum_{n=1}^2 q_{s(n)} p_{s(n-1)} - L \\ &= q_{\frac{1}{2}} p_0 - q_1 p_{\frac{1}{2}} - \frac{1}{2} m q_1^2 - \frac{2ic}{\sqrt{\pi}} [t^{\frac{1}{2}} (v_0^{\frac{1}{2}} - \frac{c}{3m} t)] q_{\frac{1}{2}}, \end{aligned} \quad (43)$$

while the Hamilton equations of motion are

$$\frac{\partial H}{\partial q_{s(0)}} = \frac{\partial H}{\partial q_0} = (-1)^{s(1)} \frac{d^{s(1)}}{d(t-b)^{s(1)}} p_0 = iD^{\frac{1}{2}}[p_0], \quad (44)$$

$$\frac{\partial H}{\partial q_{s(1)}} = \frac{\partial H}{\partial q_{\frac{1}{2}}} = (-1)^{s(2)-s(1)} \frac{d^{s(2)-s(1)}}{d(t-b)^{s(2)-s(1)}} p_{s(1)} = iD^{\frac{1}{2}}[p_{\frac{1}{2}}], \quad (45)$$

and

$$\frac{\partial H}{\partial p_{s(0)}} = q_{s(1)},$$

which is equivalent to

$$\frac{\partial H}{\partial p_0} = q_{\frac{1}{2}} \quad (46)$$

and

$$\frac{\partial H}{\partial p_{s(1)}} = q_{s(2)},$$

or

$$\frac{\partial H}{\partial p_{\frac{1}{2}}} = q_1. \quad (47)$$

From the above Hamilton equations we see that eq. (44) leads to Euler-Lagrange equation, eqs.(45) and (47) are equivalent to the definition of the momenta while eq. (46) is an identity.

4. Conclusion

It is observed that the Lagrangian and the Hamiltonian formulation can be constructed for different kinds of non-conservative systems using the definition of fractional derivatives. This motivates us to achieve the quantization of non-conservative systems in the same manner as in the conservative systems.

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