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A Solution to Symmetric Teleparallel Gravity

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Abstract

Teleparallel gravity models, in which the curvature and the nonmetricity of spacetime are both set zero, are widely studied in the literature. We work a different teleparallel theory, in which the curvature and the torsion of spacetime are both constrained to zero, but the nonmetricity is nonzero. After reformulating the general relativity in this spacetime we find a solution and investigate its singularity structure.

1. Introduction

Einstein's general relativity provides an elegant (pseudo-) Riemannian formulation of gravitation in the absence of matter. In the variational approach, Einstein's field equations are obtained by considering variations of the Einstein-Hilbert action with respect to the metric and its associated Levi-Civita connection of spacetime. That is, the absence of matter means that the connection is metric compatible and torsion free, a situation which is natural but not always convenient. A number of developments in physics in recent years suggest the possibility that the treatment of spacetime might involve more than a Riemannian structure [1].

Theories of gravity based on the geometry of distant parallelism [2]–[6] are commonly considered as the closest alternative to general relativity (GR) theory. Teleparallel gravity models possess a number of attractive features both from geometrical and physical viewpoints. Teleparallelism naturally arises within the framework of the gauge theory of the group of general coordinate transformations which underlies GR. Accordingly, the energy-momentum current represents the matter source in the field equations of the teleparallel gravity.

Since gauge theories seem important for the description of fundamental interactions it appears natural to exploit any gauge structure present in theories of gravity. Different authors, however, adopt different criteria in order to determine what properties a theory should possess in order for it to qualify as a gauge theory. We take the gravitational gauge group to be the local Lorentz group [7].

In this paper we will study a gravity model in a spacetime whose curvature and torsion are both zero, but the nonmetricity is nonzero. There is a few works in the literature about gravity models in this kind of spacetimes: the so-called symmetric teleparallel gravity [8].

2. Mathematical Preliminaries

Spacetime is denoted by the triple $\{M, g, \nabla\}$ where M is a 4-dimensional differentiable manifold, equipped with a Lorentzian metric g, which is a second rank, covariant, symmetric, non-degenerate tensor; and ∇ is a linear connection which defines parallel transport of vectors (or more generally tensors and spinors). With an orthonormal basis $\{X_a\}$,

$$g = \eta_{ab} e^a \otimes e^b \quad , \quad a, b, \dots = 0, 1, 2, 3, \tag{1}$$

where $\eta_{ab} = (-, +, +, +)$ is the Minkowski metric and $\{e^a\}$ is the orthonormal co-frame. The local orthonormal frame $\{X_a\}$ is dual to the co-frame $\{e^a\}$, where

$$e^b(X_a) = \delta^b_a \ . \tag{2}$$

The manifold M is oriented with the volume 4-form

$${}^*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3, \tag{3}$$

where $*$ denotes the Hodge map. It is convenient to employ in the following the graded interior operator $\imath_{X_a} \equiv \imath_a$:

$$\imath_a e^b = \delta^b_a \ . \tag{4}$$

In addition, the connection ∇ is specified by a set of connection 1-forms $\Lambda^a{}_b$. In the gauge approach to gravity η_{ab}, e^a, $\Lambda^a{}_b$ are interpreted as the generalized gauge potentials, while the corresponding field strengths; the nonmetricity 1-forms, torsion 2-forms and curvature 2-forms are defined through the Cartan structure equations

$$\begin{aligned} 2Q_{ab} &:= -D\eta_{ab} = \wedge_{ab} + \wedge_{ba} \ , & (5)\\ T^a &:= De^a = de^a + \Lambda^a{}_b \wedge e^b \ , & (6)\\ R^a{}_b &:= D\Lambda^a{}_b := d\Lambda^a{}_b + \Lambda^a{}_c \wedge \Lambda^c{}_b, & (7) \end{aligned}$$

where d and D denote the exterior derivative and the covariant exterior derivative, respectively. These field strengths satisfy the Bianchi identities*

$$\begin{aligned} DQ_{ab} &= \frac{1}{2}(R_{ab} + R_{ba}) & (8)\\ DT^a &= R^a{}_b \wedge e^b & (9)\\ DR^a{}_b &= 0 \ . & (10) \end{aligned}$$

The linear connection 1-forms can be decomposed uniquely as follows [9],[10]:

$$\Lambda^a{}_b = \omega^a{}_b + K^a{}_b + q^a{}_b + Q^a{}_b \ , \tag{11}$$

where $\omega^a{}_b$ are the Levi-Civita connection 1-forms that satisfy

$$de^a + \omega^a{}_b \wedge e^b = 0 \ , \tag{12}$$

$K^a{}_b$ are the contortion 1-forms such that

$$K^a{}_b \wedge e^b = T^a \ , \tag{13}$$

*Since $Q^{ab} = \frac{1}{2}D\eta^{ab} \neq 0$, we pay special attention in lowering and raising an index in front of the covariant exterior derivative.

and $q^a{}_b$ are the anti-symmetric tensor 1-forms defined by

$$q_{ab} = -(\imath_a Q_{bc}) \wedge e^c + (\imath_b Q_{ac}) \wedge e^c \, . \tag{14}$$

In the above decomposition, the symmetric part is

$$\Lambda_{(ab)} = Q_{ab}, \tag{15}$$

while the anti-symmetric part is

$$\Lambda_{[ab]} = \omega_{ab} + K_{ab} + q_{ab} \, . \tag{16}$$

It is cumbersome to take into account all components of nonmetricity in gravitational models. Therefore we will be content with dealing only with certain irreducible parts of it to gain physical insight. The irreducible decompositions of nonmetricity invariant under the Lorentz group are summarily given below [10]. The nonmetricity 1-forms Q_{ab} can be split into their trace-free $\overline{Q}_{ab}$ and the trace parts as

$$Q_{ab} = \overline{Q}_{ab} + \frac{1}{4}\eta_{ab} Q, \tag{17}$$

where the Weyl 1-form $Q = Q^a_a$ and $\eta^{ab}\overline{Q}_{ab} = 0$. Let us define

$$\begin{aligned} \Lambda_b &:= \imath_a \overline{Q}^a{}_b \ , \qquad \Lambda := \Lambda_a e^a, \\ \Theta_b &:= {}^*(\overline{Q}_{ab} \wedge e^a) \ , \quad \Theta := e^b \wedge \Theta_b \ , \quad \Omega_a := \Theta_a - \frac{1}{3}\imath_a \Theta \end{aligned} \tag{18}$$

as to use them in the decomposition of Q_{ab} as

$$Q_{ab} = {}^{(1)}Q_{ab} + {}^{(2)}Q_{ab} + {}^{(3)}Q_{ab} + {}^{(4)}Q_{ab}, \tag{19}$$

where

$$ {}^{(2)}Q_{ab} = \frac{1}{3} {}^*(e_a \wedge \Omega_b + e_b \wedge \Omega_a) \tag{20}$$

$$ {}^{(3)}Q_{ab} = \frac{2}{9}(\Lambda_a e_b + \Lambda_b e_a - \frac{1}{2}\eta_{ab}\Lambda) \tag{21}$$

$$ {}^{(4)}Q_{ab} = \frac{1}{4}\eta_{ab} Q \tag{22}$$

$$ {}^{(1)}Q_{ab} = Q_{ab} - {}^{(2)}Q_{ab} - {}^{(3)}Q_{ab} - {}^{(4)}Q_{ab} \, . \tag{23}$$

We have $\imath_a {}^{(1)}Q^{ab} = \imath_a {}^{(2)}Q^{ab} = 0$, $\eta_{ab} {}^{(1)}Q^{ab} = \eta_{ab} {}^{(2)}Q^{ab} = \eta_{ab} {}^{(3)}Q^{ab} = 0$, $e_a \wedge {}^{(1)}Q^{ab} = 0$ and $\imath_{(a} {}^{(2)}Q_{bc)}$.

3. Symmetric Teleparallel Gravity

In the symmetric teleparallel gravity (STPG) [8], we have the two geometrical constraints:

$$R^a{}_b = d\Lambda^a{}_b + \Lambda^a{}_c \wedge \Lambda^c{}_b = 0 \tag{24}$$

$$T^a = de^a + \Lambda^a{}_b \wedge e^b = 0 \, . \tag{25}$$

These equations mean that there is a distant parallelism, but the angles and lengths may change during a parallel transport.

In the literature there are many works on teleparallel gravity models [2]–[6] in which constraints are given as

$$R^a{}_b = 0 \quad , \quad Q^a{}_b = 0 \, . \tag{26}$$

One trivial solution to (26) is $\eta_{ab} = (-,+,+,+)$ and $\Lambda^a{}_b = 0$. Then the orthonormal co-frame $\{e^a\}$ is left over as the only dynamical variable. We call such a choice a *Weitzenbök gauge*. This gauge can not be a solution to STPG because of equations (24) and (25), since when we set $\eta_{ab} = (-,+,+,+)$ and $\Lambda^a{}_b = 0$ this gives rise identically to $e^a = dx^{\hat{a}}$: the so-called *Minkowski gauge* [8].

Now we give a brief outline of GR. GR is written in (pseudo-) Riemannian spacetime in which torsion and nonmetricity are both zero, i.e., the connection is Levi-Civita. The Einstein equation can be written in the form

$$G_a := -\frac{1}{2} R^{bc}(\omega) \wedge {}^*(e_a \wedge e_b \wedge e_c) = \kappa \tau_a, \tag{27}$$

or, alternatively,

$$^*G_a := (\mathrm{Ric})_a - \frac{1}{2}\mathcal{R} e_a = \kappa {}^*\tau_a, \tag{28}$$

where G_a is the Einstein tensor 3-form, $R^{ab}(\omega)$ is the Riemannian curvature 2-form, $(\mathrm{Ric})_a = \imath_b R^b_a(\omega)$ is the Ricci curvature 1-form, $\mathcal{R} = \imath_a (\mathrm{Ric})^a$ is scalar curvature, τ_a is energy-momentum 3-form and κ is a coupling constant.

For the symmetric teleparallel equivalent of Einstein equation, we first decompose non-Riemannian curvature 2-form (7) via (11) as follows: with $K^a{}_b = 0$,

$$R^a{}_b(\Lambda) = R^a{}_b(\omega) + D(\omega)(q^a{}_b + Q^a{}_b) + (q^a{}_c + Q^a{}_c) \wedge (q^c{}_b + Q^c{}_b), \tag{29}$$

where $D(\omega)$ is the covariant exterior derivative with the Levi-Civita connection. After setting $R^a{}_b(\Lambda) = 0$ we obtain the symmetric teleparallel equivalent of (27) as

$$G_a := \frac{1}{2}[D(\omega) q^{bc} + q^b_k \wedge q^{kc} + Q^b_k \wedge Q^{kc}] \wedge {}^*(e_a \wedge e_b \wedge e_c) = \kappa \tau_a \; . \tag{30}$$

3.1. Spherical symmetric solution to the model

We now proceed to attempt to find a solution to the STPG model. As usual in the study of exact solutions, we have two steps. The first one is to choose the convenient local coordinates and make corresponding ansatz for the dynamical fields. The second step concerns providing the invariants of the resulting geometry. While the choice of an ansatz helps to solve the field equations easily, the invariant description provides the correct understanding of the physical contents of a solution.

Since metric and connection are independent quantities in non-Riemannian spacetimes, we have to predict separately appropriate candidates for them. Therefore we first write a line element in order to determine the metric. We naturally start dealing with the case of spherical symmetry for realistic simplicity:

$$g = -F^2 dt^2 + G^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \; , \tag{31}$$

where $F = F(r)$ and $G = G(r)$. A convenient choice for a tetrad reads

$$e^0 = F dt, \quad e^1 = G dr, \quad e^2 = r d\theta, \quad e^3 = r \sin\theta d\varphi \; . \tag{32}$$

In addition, for the non-Riemannian connection, we choose

$$\begin{aligned}
\Lambda_{12} &= -\Lambda_{21} = -\frac{1}{r} e^2, \quad \Lambda_{13} = -\Lambda_{31} = -\frac{1}{r} e^3, \quad \Lambda_{23} = -\Lambda_{32} = -\frac{\cot\theta}{r} e^3, \\
\Lambda_{00} &= \frac{F'}{FG} e^1, \quad \Lambda_{11} = \frac{1}{r}(1 - \frac{1}{G}) e^1, \quad \Lambda_{22} = \frac{1}{r}(1 - \frac{1}{G}) e^1, \\
\Lambda_{33} &= \frac{1}{r}(1 - \frac{1}{G}) e^1, \quad \text{others} = 0 \; .
\end{aligned} \tag{33}$$

These gauge configurations (32) and (33) satisfy the constraint equations $R^a{}_b(\Lambda) = 0$, $T^a(\Lambda) = 0$. One can certainly perform a local Lorentz transformation

$$e^a \to L^a{}_b e^b \quad , \quad \Lambda^a{}_b \to L^a{}_c \Lambda^c_d {L^{-1}}^d{}_b + L^a{}_c d {L^{-1}}^c{}_b \ , \tag{34}$$

which yields the Minkowski gauge $\Lambda^a{}_b = 0$. This may mean that we propose a set of connection components in a special frame and coordinate which seems contrary to the spirit of relativity theory. However in physically natural situations we can choose a reference and coordinate system at our best convenience.

We deduce from equations (32)–(33)

$$\begin{aligned}
\omega_{01} &= -\frac{F'}{FG}e^0, \quad \omega_{12} = -\frac{1}{rG}e^2, \quad \omega_{13} = -\frac{1}{rG}e^3, \quad \omega_{23} = -\frac{\cot\theta}{r}e^3 \\
Q_{00} &= \frac{F'}{FG}e^1, \quad Q_{11} = \frac{1}{r}(1-\frac{1}{G})e^1, \quad Q_{22} = \frac{1}{r}(1-\frac{1}{G})e^1, \quad Q_{33} = \frac{1}{r}(1-\frac{1}{G})e^1 \\
q_{01} &= \frac{F'}{FG}e^0, \quad q_{12} = \frac{1}{r}(\frac{1}{G}-1)e^2, \quad q_{13} = \frac{1}{r}(\frac{1}{G}-1)e^3, \quad \text{others} = 0 \ .
\end{aligned} \tag{35}$$

When we put (35) into (30) we obtain, with $\tau_a = 0$,

$$\left(dq^{bc} + 2\omega^b{}_f \wedge q^{fc} + q^b{}_f \wedge q^{fc}\right) \wedge {}^*(e_a \wedge e_b \wedge e_c) = 0 \ , \tag{36}$$

whose components read explicitly

$$\textit{Zeroth component} \qquad \left[\tfrac{2(G^{-1})'}{rG} - \tfrac{G^2-1}{r^2G^2}\right] e^1 \wedge e^2 \wedge e^3 = 0 \tag{37}$$

$$\textit{First component} \qquad -\left[\tfrac{2F'}{rFG^2} - \tfrac{G^2-1}{r^2G^2}\right] e^0 \wedge e^2 \wedge e^3 = 0 \tag{38}$$

$$\textit{Second component} \qquad \left[\tfrac{(F'G^{-1})'}{FG} + \tfrac{F'}{rFG^2} + \tfrac{(G^{-1})'}{rG}\right] e^0 \wedge e^1 \wedge e^3 = 0 \tag{39}$$

$$\textit{Third component} \qquad -\left[\tfrac{(F'G^{-1})'}{FG} + \tfrac{F'}{rFG^2} + \tfrac{(G^{-1})'}{rG}\right] e^0 \wedge e^1 \wedge e^2 = 0 \ . \tag{40}$$

Then from (37) and (38)

$$G(r) = 1/F(r) \ , \tag{41}$$

and from (39) and (40)

$$F^2(r) = 1 - \frac{C}{r} \ , \tag{42}$$

where C is a constant.

In order to have a correct understanding of the resulting solution, we need to construct invariants of the *Riemannian* curvature and nonmetricity. Although the total curvature is identically zero in the teleparallel gravity, the Riemannian curvature of the Levi-Civita connection is nontrivial:

$$\begin{aligned}
R^{01}(\omega) &= \frac{(F'G^{-1})'}{FG}e^{10} \quad , \quad R^{02}(\omega) = \frac{F'}{rFG^2}e^{20} \quad , \quad R^{03}(\omega) = \frac{F'}{rFG^2}e^{30} \quad , \\
R^{12}(\omega) &= \frac{(G^{-1})'}{rG}e^{21} \quad , \quad R^{13}(\omega) = \frac{(G^{-1})'}{rG}e^{31} \quad , \quad R^{23}(\omega) = \frac{1}{r^2}(1-\frac{1}{G^2})e^{32} \ .
\end{aligned} \tag{43}$$

Thus the quadratic invariant of the Riemannian curvature reads

$$\begin{aligned}
R_{ab}(\omega) \wedge {}^*R^{ab}(\omega) &= \left\{ 2\left[\frac{(F'G^{-1})'}{FG}\right]^2 + 4\left(\frac{F'}{rFG^2}\right)^2 + 4\left[\frac{(G^{-1})'}{rG}\right]^2 + 2\left[\frac{1}{r^2}\left(1-\frac{1}{G^2}\right)\right]^2 \right\} {}^*1 \\
&= \frac{6C^2}{r^6}{}^*1
\end{aligned} \tag{44}$$

and the spacetime geometry is naturally characterized by the quadratic invariant of the nonmetricity

$$\begin{aligned} Q_{ab} \wedge {}^*Q^{ab} &= \left\{ \left(\frac{F'}{FG} \right)^2 + 3 \left[\frac{1}{r} \left(1 - \frac{1}{G} \right) \right]^2 \right\} {}^*1 \\ &= \left\{ \frac{C^2}{4r^3(r-C)} - \frac{3C}{r^3} - \frac{6}{r^2} \left[1 - \left(1 - \frac{C}{r} \right) \right]^{1/2} \right\}^* 1 \, . \end{aligned} \tag{45}$$

These two quadratic invariants provide the sufficient tools for understanding the contents of the classical solutions.

4. Discussion

In this paper we have studied a gravity model in the spacetime only with nonmetricty. A similar analysis, the so-called symmetric teleparallel gravity (STPG), was performed in [8]. The main motivation for studying STPG is to determine the place and significance of the symmetric teleparallel GR-equivalent model since the GR-equivalent models are satisfactorily supported by observations. Thus we hope to gain physical insights to nonmetricity. Important observation is that the Riemannian curvature invariant (44) is singular at $r = 0$, but regular at the zero ($r = C$) of the metric function $F(r)$, which means that we have a horizon here. The resulting geometry then describes the well known Schwarzschild black hole at $r = 0$ with the horizon at $r = C$. Since we are dealing with symmetric teleparallel gravity, it is necessary also to analyze the behavior of nonmetricity. As seen from (45), the nonmetricity invariant diverges not only at the origin $r = 0$, but also at the Schwarzschild horizon $r = C$. The horizon is a regular surface from the viewpoint of the Riemannian geometry, but it is singular from the viewpoint of symmetric teleparallel gravity. We intend to clarify the geometrical and physical meaning of the singularities in STPG by investigating matter coupling to STPG in a separate paper.

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