

# Multiple Scale and Hamilton-Jacobi Analysis of Extended Mathieu Equation

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## Abstract

In this study, we use perturbation approximations and semiclassical methods to investigate the boundary solutions of non-linear vibrating systems. The extended Mathieu Equation, related to the perturbed Van der Pol oscillator with periodic coefficients, is solved using multiple time scales. Then, using the Von Zeipel Method, which is based on the Hamilton-Jacobi theory, stability conditions are presented. It is shown that the stability boundaries are the same with those obtained by both methods.

**Key Words:** Van der Pol, multiple scale, Von Zeipel, hamiltonian.

## 1. Introduction

The study of non-linear oscillators has been important in the development of the theory of dynamical systems. Van der Pol and Van der Mark [1] studied a simple non-linear electronic circuit (using a neon tube as the non-linear element) and experimentally found—but were not much interested in—noisy behavior that can be identified as chaos. There have been many recent studies about the Van der Pol equation [2–4]. The mechanism for the action of time delay in a non-autonomous system such as the VdP-Duffing oscillator with excitation is investigated by Xu and Chung [5]. Dynamics of relaxation oscillations of VdP equation is investigated using an analytical method requiring only connection at a point of interface between the two dynamic fast and slow regions [6]. We study a variation of the Van der Pol Oscillator equation in the form

$$\ddot{y} - \varepsilon(1 - x^2)\dot{y}(t) + (1 + 2\varepsilon\dot{x}(t))y = 0. \quad (1)$$

Straightforward application of perturbation theory to non-linear equations of motion in classical mechanics gives rise to secular terms that increase with no bound with time, even for periodic motion; and unphysical terms also appear in the application of time dependent perturbation theory to quantum mechanical systems [7]. Although there has been considerable research in non-linear vibrating systems, in general, exact analytic solutions to non-linear differential equations are possible for only a limited number of classes of systems. However, in order to analyze many of the real systems, we must resort to approximate methods. Multiple-Scale Perturbation Theory (MSPT) is one effective technique among approximate methods that can be applied to many problems in physics and natural sciences [8]. Reformulating the perturbation series to get through secular growth, MSPT is an extremely useful method for solving such perturbation physical

problems with a small parameter,  $\varepsilon$ . Developed by Sturrock, Frieman and Nayfeh, MSPT is applicable to both linear and non-linear differential equations through classical and quantum mechanical problems [9–13]. The main goal underlying MSPT is that dynamical systems have distinct characteristic physical behavior at different lengths or slow/ fast time scales [14]. Eliminating the secular terms in the fast time variable  $T_0 = t$  leads to the well-known solvability conditions. For two-scale problems, the solutions oscillate on a time scale of order  $t$  with an amplitude and phase which drifts on a time scale of order  $\varepsilon t$ .

This paper is organized as follows. In section 2, we discuss the first variational equation of the VdP and eliminate the first order derivative in the equations describing the perturbed motion, developing a second order non-linear differential equation with periodic coefficients. The perturbed VdP equation is solved in section 3, without secular term growth, using MSPT as one of the effective singular perturbation methods. Stability boundaries and steady states are obtained. In addition, results with greater accuracy are obtained in order to find the stability boundary to second order and the steady states. Using Von Zeipel’s method, new momenta coordinates for perturbed VdP equation are obtained in section 4; then it is shown that both methods produce the same stability boundaries. Conclusions are presented in Section 5.

## 2. Perturbed Van der Pol Equation

Before introducing MSPT, we introduce in this section the Van der Pol (VdP) equation using Saaty’s discussion [15]. Consider the well-known VdP equation:

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x}(t) + x(t) = 0. \quad (2)$$

Applying the variational method to eq. (2) we get

$$\delta(\ddot{x} - \varepsilon(1 - x^2)\dot{x}(t) + x(t)) = 0, \quad (3)$$

then setting  $\delta x = y$ ,  $\delta \dot{x} = \dot{y}$  and  $\delta \ddot{x} = \ddot{y}$ , we get the perturbed motion in the form [15]

$$\ddot{y} - \varepsilon(1 - x^2)\dot{y}(t) + (1 + 2\varepsilon x(t))y = 0. \quad (4)$$

In order to study the stability of the approximate periodic solution, we set  $x = a \sin t$  and  $y = e^{v(t)}u(t)$ , with  $v(t) = -\frac{\varepsilon}{2} \left[ \left( \frac{a^2}{2} - 1 \right) t - \frac{a^2}{4} \sin 2t \right]$  and substitute these relations into eq. (4). We then see that  $\dot{y}$  vanishes (as per the discussion in [15]) and (4) becomes

$$\ddot{u}(t) + \left( 1 + \frac{1}{2}\varepsilon a^2 \sin(2t) - \frac{3}{4}\varepsilon^2 \left( \frac{1 - a^2 \sin^2 t}{2} \right)^2 \right) u(t) = 0. \quad (5)$$

If the terms in  $\varepsilon^2$  are neglected and  $t$  is transformed into  $t = t + \frac{\pi}{4}$ , (5) reduces to the well-known Mathieu equation:

$$\ddot{u} + \left( 1 + \frac{\varepsilon a^2}{2} \cos 2t \right) u(t) = 0. \quad (6)$$

The Mathieu equation is an example of a differential equation with periodic coefficients [8, 15–17].

## 3. Stability Boundary of the Perturbed Van der Pol Equation Using Multiple Scales

One can solve the perturbed VdP eq. (5) using MSPT; so we seek a perturbative solution to (5) having two variables: the short-time scale  $t$ , and the long-time scale  $\tau = \varepsilon t$  when  $\varepsilon$  is sufficiently small. We presently

investigate the solutions of (5) via the perturbation of this equation. To find stable solutions of (5), time variable should be changed as  $t \rightarrow \frac{t}{2} + \frac{\pi}{4}$  and (5) becomes

$$\ddot{u} + \left( \frac{1}{4} + \frac{1}{8}\varepsilon a^2 \cos t - \frac{3}{16}\varepsilon^2 \left( 1 - \frac{a^2}{2}(1 + \sin t) \right)^2 \right) u(t) = 0. \quad (7)$$

Applying the multiple-scale perturbation theory, we find the boundaries between the regions in the  $(\Delta, \varepsilon)$  plane for which all solutions to the perturbed VdP equation are stable. Putting parameter  $\Delta$  in (7), we get

$$\ddot{u} + \left( \frac{\Delta}{4} + \frac{1}{8}\varepsilon a^2 \cos t - \frac{3}{16}\varepsilon^2 \left( 1 - \frac{a^2}{2}(1 + \sin t) \right)^2 \right) u(t) = 0, \quad (8)$$

and expand it as a power series in  $\varepsilon$  [16]. The transition curves in the  $(\Delta, \varepsilon)$  plane separate stable solutions from unstable solutions corresponding to periodic solutions of (8). Some of these curves are determined by expanding both  $\delta$  and  $u$  as functions of  $\varepsilon$ :

$$\Delta = 1 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \dots \quad (9)$$

Then (8) can be rewritten as

$$\ddot{u} + \left( \frac{1}{4} + \varepsilon \left( \frac{\delta_1}{4} + \frac{1}{8}a^2 \cos t \right) + \varepsilon^2 \left( \frac{\delta_2}{4} - \frac{3}{16} \left( 1 - \frac{a^2}{2}(1 + \sin t) \right)^2 \right) \right) u(t) = 0, \quad (10)$$

which is the general equation. An uniform expansion can be written in the form

$$u(t) = U_0(t, \tau) + \varepsilon U_1(t, \tau) + \varepsilon^2 U_2(t, \tau) + O(\varepsilon^3). \quad (11)$$

Applying the chain rule for partial differentiation to compute derivatives of  $u(t)$ , we get

$$\begin{aligned} \ddot{u} &= \frac{\partial^2 U_0(t, \tau)}{\partial t^2} + \varepsilon \left( 2 \frac{\partial^2 U_0(t, \tau)}{\partial \tau \partial t} + \frac{\partial^2 U_1(t, \tau)}{\partial t^2} \right) \\ &+ \varepsilon^2 \left( \frac{\partial^2 U_2}{\partial t^2} + 2 \frac{\partial^2 U_1}{\partial \tau \partial t} + \frac{\partial^2 U_0}{\partial \tau^2} \right) + O(\varepsilon^3), \end{aligned} \quad (12)$$

where we have employed the relation  $\frac{d\tau}{dt} = \varepsilon$ .

Substituting (11) and (12) into (10), we obtain partial differential equations for the dependent variables  $U_0(t, \tau)$ ,  $U_1(t, \tau)$  . . . . So, in this solution there are less secular terms to order in the perturbation series. The first three terms of the series are now given by

$$\frac{\partial^2 U_0(t, \tau)}{\partial t^2} + \frac{1}{4} U_0(t, \tau) = 0, \quad (13)$$

$$\frac{\partial^2 U_1(t, \tau)}{\partial t^2} + \frac{1}{4} U_1(t, \tau) = -2 \frac{\partial^2 U_0(t, \tau)}{\partial \tau \partial t} - \left( \frac{\delta_1}{4} + \frac{1}{8} a^2 \cos t \right) U_0(t, \tau), \quad (14)$$

$$\begin{aligned} \frac{\partial^2 U_2(t, \tau)}{\partial t^2} + \frac{1}{4} U_2(t, \tau) &= -2 \frac{\partial^2 U_1(t, \tau)}{\partial \tau \partial t} - \left( \frac{\delta_1}{4} + \frac{1}{8} a^2 \cos t \right) U_1(t, \tau) - \frac{\partial^2 U_0}{\partial \tau^2} \\ &- \left( \frac{\delta_2}{4} - \frac{3}{16} \left( 1 - \frac{a^2}{2}(1 + \sin t) \right)^2 \right) U_0(t, \tau). \end{aligned} \quad (15)$$

A general solution to (13) can be chosen as

$$U_0(t, \tau) = A_0(\tau) e^{it/2} + A_0^*(\tau) e^{-it/2}. \quad (16)$$

In order to determine  $\tau(\tau = \epsilon t)$ -dependent coefficients  $A_0(\tau)$  and  $A_0^*(\tau)$ , we substitute  $U_0(t, \tau)$  into the right hand side of (14). One can see that  $e^{it/2}$  and  $e^{-it/2}$  are the solutions of the left-hand side equation, i.e., homogeneous equation  $\frac{\partial^2 U_1(t, \tau)}{\partial t^2} + \frac{1}{4}U_1(t, \tau) = 0$ . Therefore, the right-hand side contains terms that produce secular terms in  $U_1$ . For a uniform expansion, these terms must be eliminated; thus we proceed by setting coefficients of  $e^{it/2}$  and  $e^{-it/2}$  equal to zero. So,  $A_0(\tau)$  satisfy:

$$-i \frac{\partial A_0}{\partial \tau} - \frac{a^2}{16} A_0^* - \frac{\delta_1}{4} A_0 = 0, \quad (17)$$

$$i \frac{\partial A_0^*}{\partial \tau} - \frac{a^2}{16} A_0 - \frac{\delta_1}{4} A_0^* = 0. \quad (18)$$

Setting  $A_0(\tau) = w_0(\tau) + iv_0(\tau)$ , we find:

$$-w_0'(\tau) - \frac{\delta_1}{4} v_0(\tau) + \frac{a^2}{16} v_0(\tau) = 0, \quad (19)$$

$$v_0'(\tau) - \frac{\delta_1}{4} w_0(\tau) - \frac{a^2}{16} w_0(\tau) = 0, \quad (20)$$

and from (19) and (20),

$$v_0''(\tau) + \frac{1}{4}(\delta_1^2 - \frac{a^4}{16})v_0 = 0, \quad (21)$$

and the solutions of  $w_0(\tau)$ ,  $v_0(\tau)$  are thus:

$$v_0(\tau) = C_1 \cos \lambda \tau + C_2 \sin \lambda \tau, \quad (22)$$

$$w_0(\tau) = \frac{\lambda}{\delta_1 + \frac{a^2}{4}} (-C_1 \sin \lambda \tau + C_2 \cos \lambda \tau), \quad (23)$$

where

$$\lambda = \frac{1}{2} \sqrt{\delta_1^2 - \frac{a^4}{16}}. \quad (24)$$

Here, for the perturbed VdP equation, we get unstable solutions for  $U_0(t, \tau)$  if  $\delta_1^2 - \frac{a^4}{16}$  is negative. That is,  $|\delta_1| > \frac{a^2}{4}$  gives stable solutions and  $|\delta_1| < \frac{a^2}{4}$  gives unstable solutions. Near  $\epsilon = 0$ , the stability boundary for perturbed VdP oscillator is given as:

$$\Delta = 1 \pm \frac{a^2}{4} \epsilon + O(\epsilon^2), \quad \epsilon \rightarrow 0. \quad (25)$$

If the initial conditions are specified as  $U(0, 0) = 1$ ,  $\dot{U}(0, 0) = 0$  and  $U_0(0, 0) = 1$ ,  $\dot{U}_0(0, 0) = 0$ , then  $C_1 = 0$ ,  $C_2 = \frac{\delta_1 + \frac{a^2}{4}}{2\lambda}$  and

$$U_0(t, \tau) = \left( \cos \lambda \tau \cos \frac{t}{2} - \frac{\delta_1 + \frac{a^2}{4}}{\lambda} \sin \lambda \tau \sin \frac{t}{2} \right). \quad (26)$$

Equation (26) is a solution of the Mathieu equation; and it is obvious that, with the appropriate parameters, the result is the same as obtained by Bender [8]. The differential equation in the present case is the simple

harmonic oscillator with an external periodic force and amplitude damping factor for  $\varepsilon > 0$ . Furthermore, one can find the periodic solutions of (14) as

$$U_1(t, \tau) = A_1(\tau)e^{\frac{it}{2}} + A_1^*(\tau)e^{-\frac{it}{2}} - \frac{a^2}{512\lambda} \left[ (a^2 + 4\delta_1) \sin \lambda\tau \sin \frac{3t}{2} + 4\lambda \cos \lambda\tau \cos \frac{3t}{2} \right], \quad (27)$$

where  $A_1(\tau)$  and  $A_1^*(\tau)$  are unknown integral coefficients. To find these coefficients, (15) is used. However, inserting  $U_0(t, \tau)$  and  $U_1(t, \tau)$  into (15) could give rise to another resonance case. At this step, let the coefficients of  $\cos \frac{t}{2}$  and  $\sin \frac{t}{2}$  be zero. Substitute  $U_1(t, \tau)$ ,  $U_0(t, \tau)$  and trigonometric formulas into the right hand-side of (15) and set  $A_1 = w_1 + iv_1$ . We get

$$-16384 \frac{\partial v_1}{\partial \tau} - \eta_1 \sin \lambda\tau - \eta_2 \cos \lambda\tau = 0, \quad (28)$$

$$1024\lambda (4\delta_1 + a^2) w_1 + \eta_3 \sin \lambda\tau + \eta_4 \cos \lambda\tau = 0. \quad (29)$$

Here,  $A_1(\tau) = -A_1^*(\tau)$ ,  $\eta_1, \eta_2$  are constants and are given respectively as:

$$\eta_1 = (\lambda - 36) a^6 + (96 + 16\lambda\delta_1 - 576\delta_1) a^4 - (6656 - 8192\lambda^2) \delta_1 + (1536\delta_1 + 512\delta_2 - 512\lambda^2 - 96) a^2, \quad (30)$$

$$\eta_2 = 384a^2\lambda (1 - 2a^2), \quad (31)$$

$$\eta_3 = 24a^2 (a^4 - 2a^2 + 16a^2\delta_1 - 32\delta_1), \quad (32)$$

$$\eta_4 = 64\lambda (-24 - 128\lambda^2 + 128\delta_2 + 24a^2 - 9a^4); \quad (33)$$

$$A_1(\tau) = iC_1 + \sin \lambda\tau \left( -\frac{\eta_3}{1024\lambda (a^2 + 4\delta_1)} - \frac{i\eta_2}{16384\lambda^2} \right) + \cos \lambda\tau \left( -\frac{\eta_4}{1024\lambda (a^2 + 4\delta_1)} + \frac{i\eta_1}{16384\lambda^2} \right) \quad (34)$$

$$A_1^*(\tau) = -iC_1 + \sin \lambda\tau \left( -\frac{\eta_3}{1024\lambda (a^2 + 4\delta_1)} + \frac{i\eta_2}{16384\lambda^2} \right) + \cos \lambda\tau \left( -\frac{\eta_4}{1024\lambda (a^2 + 4\delta_1)} - \frac{i\eta_1}{16384\lambda^2} \right). \quad (35)$$

Here,  $C_1$  is an arbitrary constant and we say  $C_1 = 0$ .  $U_1(t, \tau)$  can then be written as

$$U_1(t, \tau) = \alpha(\eta_3 \sin \lambda\tau \cos \frac{t}{2} + \eta_4 \cos \lambda\tau \cos \frac{t}{2}) + \beta(\eta_2 \sin \lambda\tau \sin \frac{t}{2} - \eta_1 \cos \lambda\tau \sin \frac{t}{2}) - \frac{a^2}{512\lambda} \left[ (a^2 + 4\delta_1) \sin \lambda\tau \sin \frac{3t}{2} + 4\lambda \cos \lambda\tau \cos \frac{3t}{2} \right] \quad (36)$$

where  $\alpha = -\frac{1}{512\lambda(a^2+4\delta_1)}$  and  $\beta = \frac{1}{8192\lambda^2}$ .

### 3.1. Higher Order Corrections

We now examine the the higher order terms in order to to determine with greater precision than given by eq. (25) the location of the stability boundary of the perturbed VdP oscillator.

Toward this end, we write the states  $U_0$  and  $U_1$  in a more accurate and simple form than in (26) and (36). We return to (21):

$$v_0''(\tau) + \frac{1}{4} \left( \delta_1^2 - \frac{a^4}{16} \right) v_0(\tau) = 0. \quad (37)$$

Write  $v_0(\tau)$  again, but this time in the form

$$v_0(\tau) = A \exp \left( \pm i \sqrt{\delta_1^2 - \frac{a^4}{16}} \tau \right), \quad (38)$$

where  $A$  is a constant. Thus, a new time scale for the problem occurs: assume that  $\delta_1 = \frac{a^2}{4} + \delta_2 \epsilon$  in (38). Then  $v_0(\tau)$  becomes approximately  $K \exp \left( \pm i \sqrt{\frac{a^2}{2} \epsilon \delta_2 \tau} \right) = K \exp \left( \pm i \sqrt{\frac{a^2}{2} \delta_2 \epsilon^{3/2} t} \right)$ , which advances that a new time scale  $\sigma = \epsilon^{3/2} t$  must be introduced. Therefore substitute  $\Delta$  into (8), and it is noted that  $\sigma = \epsilon^{3/2} t$ :

$$\ddot{U}(t) + \left( \frac{1}{4} + \epsilon \left( \frac{a^2}{4} + \frac{1}{8} a^2 \cos(t) \right) + \epsilon^2 (2\delta_2 - \frac{3}{16} \left( 1 - \frac{a^2}{2} (1 + \sin t) \right)^2) \right) U(t) = 0. \quad (39)$$

Thus we can expand  $U(t)$  in a new series and second order differential operator as

$$U(t) \simeq U_0(t, \sigma) + \epsilon^{1/2} U_1(t, \sigma) + \epsilon U_2(t, \sigma) + \epsilon^{3/2} U_3(t, \sigma) + \epsilon^2 U_4(t, \sigma) + \dots, \quad (40)$$

$$\ddot{U}(t) = \frac{\partial^2 U_0}{\partial t^2} + \epsilon^{1/2} \frac{\partial^2 U_1}{\partial t^2} + \epsilon \frac{\partial^2 U_2}{\partial t^2} + \epsilon^{3/2} \left( \frac{\partial^2 U_3}{\partial t^2} + 2 \frac{\partial^2 U_0}{\partial t \partial \sigma} \right) + \epsilon^2 \left( \frac{\partial^2 U_4}{\partial t^2} + 2 \frac{\partial^2 U_1}{\partial t \partial \sigma} \right) + \dots \quad (41)$$

Putting these equations into (39) and equating powers of  $\epsilon^{1/2}$ , we get

$$\epsilon^0 : \quad \frac{\partial^2 U_0}{\partial t^2} + \frac{U_0}{4} = 0, \quad (42)$$

$$\epsilon^{1/2} : \quad \frac{\partial^2 U_1}{\partial t^2} + \frac{U_1}{4} = 0, \quad (43)$$

$$\epsilon^1 : \quad \frac{\partial^2 U_2}{\partial t^2} + \frac{U_2}{4} = -\frac{a^2}{4} (1 + e^{it} + e^{-it}) U_0, \quad (44)$$

$$\epsilon^{3/2} : \quad \frac{\partial^2 U_3}{\partial t^2} + \frac{U_3}{4} = -2 \frac{\partial^2 U_0}{\partial t \partial \sigma} - \frac{a^2}{4} (1 + e^{it} + e^{-it}) U_1, \quad (45)$$

$$\begin{aligned} \epsilon^2 : \quad & \frac{\partial^2 U_4}{\partial t^2} + \frac{U_4}{4} = -2 \frac{\partial^2 U_1}{\partial t \partial \sigma} - \frac{a^2}{4} (1 + e^{it} + e^{-it}) U_2 \\ & - (2\delta_2 - \frac{3}{16} \left( 1 - \frac{a^2}{2} (1 + \sin t) \right)^2) U_0. \end{aligned} \quad (46)$$

Solutions to (42) and (43) are, respectively,

$$U_0(t, \sigma) = A_0(\sigma) e^{\frac{it}{2}} + A_0^*(\sigma) e^{-\frac{it}{2}}, \quad (47)$$

$$U_1(t, \sigma) = A_1(\sigma)e^{\frac{it}{2}} + A_1^*(\sigma)e^{-\frac{it}{2}}. \quad (48)$$

Following the same procedure, from the right hand side of (43), secularity can be removed and one can find  $A_0(\sigma) = -A_0^*(\sigma)$  and say  $A_0(\sigma) = iB_0(\sigma)$ . Now  $U_2$  can be solved as

$$U_2(t, \sigma) = A_2(\sigma)e^{\frac{it}{2}} + A_2^*(\sigma)e^{-\frac{it}{2}} + \frac{a^2}{8}A_0e^{\frac{3it}{2}} - \frac{a^2}{8}A_0e^{-\frac{3it}{2}}. \quad (49)$$

Vanishing the coefficients of  $e^{\pm \frac{it}{2}}$  in (44) gives

$$\frac{\partial B_0}{\partial \sigma} = \frac{a^2}{4}(A_1(\sigma) + A_1^*(\sigma)), \quad (50)$$

From (42)–(46),

$$-i\frac{d}{d\sigma}(A_1 + A_1) = (4\delta_2 + \frac{3a^2}{8})A_0, \quad (51)$$

and from (49)

$$B_0''(\tau) + \frac{a^2}{4}\left(4\delta_2 + \frac{3a^2}{8}\right)B_0 = 0, \quad (52)$$

the solution of the  $B_0$  is then found in the form

$$B_0(\sigma) = C_4 \cos \mu\sigma + C_5 \sin \mu\sigma, \quad (53)$$

where  $\mu = \sqrt{\delta_2 + \frac{3a^2}{32}}$ . We can choose  $a = \sqrt{2}$  as an arbitrary parameter. If we look at the stability boundaries of perturbed VdP oscillator, stability occurs for the oscillator when  $\delta_2 < \frac{3}{16}$  and instability occurs when  $\delta_2 > \frac{3}{16}$ . Thus, the higher order stability boundary is given by:

$$\Delta = 1 \pm \frac{1}{2}\epsilon - \frac{3}{16}\epsilon^2 + O(\epsilon^3), \quad \epsilon \rightarrow 0. \quad (54)$$

Then  $A_1(\sigma)$  is given as

$$A_1(\sigma) = 4\mu(-C_4 \sin \mu\sigma + C_5 \cos \mu\sigma). \quad (55)$$

Using the initial conditions as  $U_0(0) = 0$ ,  $\dot{U}_0(0) = 1$ ,  $U_1(0, 0) = 0$ ,  $\dot{U}_1(0) = 0$ , one can find  $U_0$  and  $U_1$  as

$$U_0(t, \sigma) = 2 \cos \mu\sigma \sin \frac{t}{2}, \quad (56)$$

$$U_1(t, \sigma) = -8\mu \sin \mu\sigma \cos \frac{t}{2}. \quad (57)$$

With a similar algorithm, higher order can be calculated with a power series in  $\epsilon$ .

## 4. Stability Boundaries by Using Von Zeipel Method

The classical method of generating canonical transformations is called Von Zeipel's method. The problem with this method is the awkward mixture of odd and new variables that has to be unscrambled. To find higher order approximations, Von Zeipel [18] employed a technique in which the main idea is to expand the generating function  $S$  in powers of a small parameter  $\epsilon$ . First of all, by using generalized momentum vector  $\mathbf{p}$  and coordinate vector  $\mathbf{q}$ , one can write the following canonical equations of motion [17]:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad (58)$$

$$\dot{P}_i = -\frac{\partial H}{\partial q_i}. \quad (59)$$

Under a transformation from  $\mathbf{q}$  and  $\mathbf{p}$  to  $\mathbf{Q}(\mathbf{q}, \mathbf{p}, t)$  and  $\mathbf{P}(\mathbf{q}, \mathbf{p}, t)$ , (58) and (59) are transformed into:

$$\dot{Q}_i = f_i(\mathbf{P}, \mathbf{Q}, t), \quad (60)$$

$$\dot{P}_i = g_i(\mathbf{P}, \mathbf{Q}, t). \quad (61)$$

In terms of  $K(\mathbf{P}, \mathbf{Q}, t)$ ,

$$f_i = \frac{\partial K}{\partial P_i}, \quad g_i = -\frac{\partial K}{\partial Q_i}, \quad (62)$$

$$Q_i = \frac{\partial K}{\partial P_i}, \quad P_i = -\frac{\partial K}{\partial Q_i}. \quad (63)$$

$Q$  and  $P$  are called canonical variables and the canonical transformations can be generated using  $S(\mathbf{P}, \mathbf{q}, t)$ , which is the generating function [19]:

$$P_i = \frac{\partial S}{\partial q_i}, \quad Q_i = -\frac{\partial S}{\partial P_i}. \quad (64)$$

When the equations above are solved for

$$\mathbf{q} = \mathbf{q}(\mathbf{P}, \mathbf{Q}, t) \quad (65)$$

and

$$\mathbf{p} = \mathbf{p}(\mathbf{P}, \mathbf{Q}, t), \quad (66)$$

$K$  is given as

$$K(\mathbf{P}, \mathbf{Q}, t) = H(\mathbf{p}(\mathbf{P}, \mathbf{Q}, t), \mathbf{q}(\mathbf{P}, \mathbf{Q}, t), t) + \frac{\partial S}{\partial t}. \quad (67)$$

Also  $S$  must satisfy the Hamilton-Jacobi equation:

$$H\left(\frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_N}, q_1, q_2, \dots, q_N, t\right) + \frac{\partial S}{\partial t} = 0. \quad (68)$$

Complete solutions of the equation are not available for general  $H$ . If  $H = H_0 + H_1$ , where  $H_1$  is small compared to  $H_0$ , a complete solution  $S_0(P_1, \dots, P_N, q_1, \dots, q_N, t)$  is available for

$$H_0\left(\frac{\partial S_0}{\partial q_1}, \dots, \frac{\partial S_0}{\partial q_N}, q_1, q_2, \dots, q_N, t\right) + \frac{\partial S_0}{\partial t} = 0. \quad (69)$$

A generating function  $S = S_0(P_1, \dots, P_N, q_1, \dots, q_N, t)$  can be used where  $\dot{P}_i = -\frac{\partial K}{\partial Q_i}$ ,  $\dot{Q}_i = -\frac{\partial K}{\partial P_i}$  and

$$K = H_0 + \tilde{H} + \frac{\partial S_0}{\partial t} = \tilde{H}. \quad (70)$$

The method is the same as the generalized method of averaging [17]. Stern has shown [20] for Hamiltonian systems Kruskal's technique is [21] equivalent to Von Zeipel's technique. The system under discussion is described by the Hamiltonian

$$H(\mathbf{p}, \mathbf{q}, t) = \sum (\varepsilon^n H_n(\mathbf{p}, \mathbf{q}, t)), \quad \varepsilon \ll 1. \quad (71)$$



If  $S_0 = S_0(\mathbf{P}, \mathbf{q}, t)$  is a complete solution of the Hamilton-Jacobi equation and  $\varepsilon \ll 1$ ,

$$H_0 \left[ \frac{\partial S_0}{\partial \mathbf{q}}, \mathbf{q}, t \right] + \frac{\partial S_0}{\partial t} = 0 \quad (72)$$

and the equations  $\mathbf{p} = \mathbf{p}(\mathbf{P}, \mathbf{Q}, t)$ ,  $\mathbf{q} = \mathbf{q}(\mathbf{P}, \mathbf{Q}, t)$  are the solutions of

$$p_i = \frac{\partial S_0}{\partial q_i}, \quad (73)$$

$$Q_i = \frac{\partial S_0}{\partial P_i}. \quad (74)$$

Assume  $\mathbf{P}$  and  $\mathbf{Q}$  to be time varying and the generating function  $S = S_0(\mathbf{P}, \mathbf{q}, t)$ , to transform from the canonical system  $\mathbf{p}$  and  $\mathbf{q}$  to the canonical system  $\mathbf{P}$  and  $\mathbf{Q}$ . The Hamiltonian is then transformed into

$$\tilde{H} = \sum (\varepsilon^n H_n[\mathbf{p}, \mathbf{q}, t]) + \frac{\partial S_0}{\partial t} \sum (\varepsilon^n \tilde{H}_n). \quad (75)$$

Hence  $\mathbf{P}$  and  $\mathbf{Q}$  are stated by the variational equations

$$\dot{\mathbf{P}} = - \sum \left( \varepsilon^n \frac{\partial \tilde{H}_n}{\partial \mathbf{Q}} \right), \quad (76)$$

$$\dot{\mathbf{Q}} = \sum \left( \varepsilon^n \frac{\partial \tilde{H}_n}{\partial \mathbf{P}} \right). \quad (77)$$

Using the generating function  $S$  to determine an approximate solution to (75) and (76) to any order, we introduce a transformation from the canonical system  $\mathbf{P}$  and  $\mathbf{Q}$  to the new canonical system:

$$S = \sum (P_i^* Q_i) + \sum (\varepsilon_n S_n(P^*, \mathbf{Q}, t)). \quad (78)$$

Then

$$P_i = P_i^* + \sum (\varepsilon^n \frac{\partial S_n}{\partial Q_i}), \quad (79)$$

so  $\tilde{H}$  is transformed into

$$K \equiv \sum \varepsilon^n K_n(P^*, Q, t) = \sum \varepsilon^n \tilde{H}_n + \sum \varepsilon^n \frac{\partial S_n}{\partial t}. \quad (80)$$

Expand the terms in the right-hand side of (79) for small  $\varepsilon$  and equate the coefficients of  $\varepsilon$  on both sides, we get

$$K_1 = \tilde{H}_1 + \frac{\partial S_1}{\partial t}, \quad (81)$$

$$K_2 = \tilde{H}_2 + \sum \frac{\partial S_1}{\partial Q_i} \frac{\partial \tilde{H}_1}{\partial P_i} + \frac{\partial S_2}{\partial t}. \quad (82)$$

If this procedure is applied to (9) to get stability boundaries of perturbed VdP oscillator for  $a = \sqrt{2}$ , (9) turns into:

$$\ddot{U} + \left( \frac{\omega^2}{4} + \frac{1}{4} \varepsilon \cos t - \frac{3}{16} \varepsilon^2 \sin^2 t \right) U = 0. \quad (83)$$

If we write  $H_0$  and  $\tilde{H}$  as

$$H_0 = \frac{1}{2} \left( p^2 + \frac{\omega^2}{4} q^2 \right), \quad (84)$$

$$\tilde{H} = \frac{1}{2} \left( \frac{1}{4} \varepsilon \cos t - \frac{3}{16} \varepsilon^2 \sin^2 t \right) q^2, \quad (85)$$

which are hamiltonians for unperturbed and perturbed VdP equation. The Hamilton-Jacobi equation corresponding to the case  $\varepsilon = 0$  is

$$\frac{1}{2} [(S'(q))^2 + \frac{\omega^2}{4} q^2] + \frac{\partial S}{\partial t} = 0. \quad (86)$$

The equation above (86) can be solved by separation of variables:

$$S = S_1(q) + \sigma(t). \quad (87)$$

Equations (87) separates into  $\dot{\sigma} = -\alpha$  and  $\sigma = -\alpha t$ .

We find  $S_1$  generator and  $\beta$  new coordinate and  $q$  as:

$$S_1 = -\alpha t + \int \sqrt{2\alpha - \frac{\omega^2}{4} q^2} dq, \quad (88)$$

$$\beta = -t + \frac{2}{\omega} \arcsin \left( \frac{q\omega}{2\sqrt{2\alpha}} \right), \quad (89)$$

$$q = \frac{2\sqrt{2\alpha}}{\omega} \cos \left( \frac{\omega(t + \beta)}{2} \right). \quad (90)$$

Hence  $\alpha$  and  $\beta$  are canonical variables with respect to

$$\tilde{H} = \frac{4\alpha}{\omega^2} \cos^2 \left( \frac{\omega(t + \beta)}{2} \right) \left[ \frac{1}{4} \varepsilon \cos t - \frac{3}{16} \varepsilon^2 \sin^2 t \right] \quad (91)$$

with  $\dot{\alpha} = -\frac{\partial \tilde{H}}{\partial \beta}$ ,  $\dot{\beta} = \frac{\partial \tilde{H}}{\partial \alpha}$ . To determine an approximate solution to these equations, we introduce a near-identity transformation from  $\alpha$  and  $\beta$  to  $\alpha^*$  and  $\beta^*$  using the generating function

$$S = \alpha^* \beta + \varepsilon S_1(\alpha^*, \beta, t) + \varepsilon^2 S_2(\alpha^*, \beta, t) + \dots \quad (92)$$

Hence

$$\alpha = \alpha^* + \varepsilon \frac{\partial S_1}{\partial \beta} + \dots \quad (93)$$

Using (91) and (92),  $K$  can be written as

$$K = \varepsilon K_1 + \varepsilon^2 K_2 + \dots \quad (94)$$

Hence  $K_1$  and  $K_2$  are

$$K_1 = \frac{\partial S_1}{\partial t} + \frac{4\alpha^*}{\omega^2} \cos^2 \left( \frac{\omega(t + \beta)}{2} \right) \cos t, \quad (95)$$

$$K_2 = \frac{\partial S_2}{\partial t} + \frac{4}{\omega^2} \frac{\partial S_1}{\partial \beta} \cos^2 \left( \frac{\omega(t + \beta)}{2} \right) \cos t - \frac{3\alpha^*}{4\omega^2} \cos^2 \left( \frac{\omega(t + \omega\beta)}{2} \right) \sin^2 t. \quad (96)$$

Using some trigonometric relations, equation (95) above can be written in the form

$$K_1 = \frac{\partial S_1}{\partial t} + \frac{\alpha^*}{\omega^2} (2 \cos t + \cos((\omega + 1)t + \omega\beta) + \cos((\omega - 1)t + \omega\beta)). \quad (97)$$

In the same way one can write  $K_2$ . In the case of  $\omega \neq 1$ , all the terms on the right-hand side of (97) are fast varying. Hence  $K_1 = 0$  and  $S_1$  is

$$S_1 = -\frac{\alpha^*}{\omega^2} \left( 2 \sin t + \frac{1}{\omega + 1} \sin((\omega + 1)t + \omega\beta) + \frac{1}{\omega - 1} \sin((\omega - 1)t + \omega\beta) \right). \quad (98)$$

In the case of  $\omega \approx 1$ ,  $\cos((\omega - 1)t + \omega\beta)$  is slowly varying because  $S_1$  is singular at  $\omega \approx 1$ , as it is seen in eq. (100). By equating  $K_1$  to (97), we have

$$K_1 = \frac{\alpha^*}{\omega^2} \cos((\omega - 1)t + \omega\beta). \quad (99)$$

Substituting  $S_2$  into (96) one can easily get  $K_2$  and equating the  $K_2$  to the long terms in this equation, we have

$$K_2 = -\left( \frac{1}{\omega^3(\omega + 1)} + \frac{3}{16\omega^2} \right) \alpha^* \quad (100)$$

Therefore  $K$  can be written to second order as

$$K = \frac{\alpha^* \varepsilon}{\omega^2} \cos((\omega - 1)t + \omega\beta) - \frac{\alpha^* \varepsilon^2}{\omega^2} \left( \frac{3}{16} + \frac{1}{\omega(\omega + 1)} \right). \quad (101)$$

It is obvious that  $\alpha$  and  $\beta$  in terms of  $\alpha^*$   $\beta^*$ :

$$\alpha = \alpha^* - \frac{\varepsilon \alpha^*}{\omega(\omega + 1)} \cos((\omega + 1)t + \omega\beta), \quad (102)$$

$$\beta = \beta^* - \frac{2\varepsilon}{\omega^2} \left( \sin t + \frac{1}{2(\omega + 1)} \sin((\omega + 1)t + \omega\beta) \right). \quad (103)$$

We remove the dependence of  $K$  on  $t$  by changing from  $\alpha^*$  and  $\beta^*$  to  $\alpha'$  and  $\beta'$  via the assignments

$$S' = \alpha' ((\omega - 1)t + \omega\beta^*) \quad (104)$$

and

$$\alpha^* = \frac{\partial S'}{\partial \beta^*}, \quad (105)$$

$$\beta' = \frac{\partial S'}{\partial \alpha'}. \quad (106)$$

Now  $K$  can be written as  $K' = K + \frac{\partial S'}{\partial t}$ , and

$$K = \frac{\varepsilon \alpha'}{\omega} \cos \beta' - \frac{\varepsilon^2 \alpha'}{\omega} \left( \frac{3}{16} + \frac{1}{\omega} (\omega + 1) \right) + (\omega - 1) \alpha'. \quad (107)$$

Therefore, we can write a couple of differential equations as

$$\dot{\alpha}' = \frac{\varepsilon \alpha'}{\omega} \sin \beta', \quad (108)$$

$$\dot{\beta}' = -\frac{\varepsilon}{\omega} \cos \beta' - \frac{3\varepsilon^2}{16\omega} - \frac{\varepsilon^2}{\omega^2(\omega+1)} + \omega - 1. \quad (109)$$

So, the integration of these equations leads to the equation

$$\ln \alpha' = \ln \left[ \frac{\varepsilon}{\omega} \cos \beta' + \left( \frac{3}{16\omega} + \frac{1}{\omega^2(\omega+1)} \right) \varepsilon^2 + \omega - 1 \right]. \quad (110)$$

The requirement of finite frequency and asymptotics imply

$$\frac{\varepsilon}{\omega} = \left| \omega - 1 + \varepsilon^2 \left( \frac{3}{16\omega} + \frac{1}{\omega^2(\omega+1)} \right) \right|. \quad (111)$$

Finally, we note that the stability boundaries of perturbed VdP oscillator around  $\omega \approx 1$  are in agreement with those obtained in sections(3)–(4) using multiple scales:

$$\Delta = \frac{\omega^2}{4} \rightarrow 1 + \frac{1}{2}\varepsilon - \frac{3}{16}\varepsilon^2. \quad (112)$$

## 5. Conclusions

In this study, we examined the perturbed Van der Pol equation, to which we applied MSPT, turning it into a second order differential equation with time dependent periodic coefficients. For convenient solutions, necessary transforms are applied to this variational equation. We employed  $(\Delta, \epsilon)$  parameter space in the perturbed equation in order to impose the stability conditions through the boundaries. As it has already been shown [9, 10] that the use of MSPT allows for the elimination of all harmonic and subharmonic solution-related secular terms appearing in the evolution of the  $U_0(t, \tau)$  and  $U_1(t, \tau)$ . So the method worked effectively for limiting the growth of the secular terms; and more accurate results have been obtained for sub-harmonics by using different time scales. By comparing the results obtained for two different time scales, stability and instability conditions have been shown to depend on constants, a technique with which one can also see if the motion is strictly periodic or not. Our results show that solutions of the terminated nonlinear structure of the Van der Pol equation are periodic and stable and are similar to solutions of the original VdP equation [17]. To determine a first approximation to our Hamiltonian system for the perturbed VdP equation, we use the Von Zeipel method [17], a perturbation method for classical Hamiltonian systems using an averaging procedure in phase space. Transition curves are in agreement with those obtained by MSPT methods. As a result, we can say that this work contains useful tools of these applications in the case of perturbed motions.

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