

Dirac Sextic Oscillator in the Constant Magnetic Field

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Abstract

We introduce a Dirac equation which reproduces the usual radial sextic oscillator potential in the non-relativistic limit. We determine its energy spectrum in the presence of the magnetic field. It is shown that the equation is solved in the context of quasi-exactly-solvable problems. The equation possesses hidden sl_2 -algebra and the destroyed symmetry of the equation can be recovered for specific values of the magnetic field which leads to exact determination of the eigenvalues.

Key Words: Dirac sextic oscillator, quasi-exactly solvable systems

1. Introduction

A Dirac equation with an interaction linear in coordinates was considered long ago [1] and recently rediscovered in the context of the relativistic many body theories [2]. The equation is known as the *Dirac oscillator*, since in the non-relativistic limit it becomes a harmonic oscillator with a very strong spin-orbit coupling term. Dirac oscillator has attracted much attention and the concept gave rise to a large number of papers concerned with its various aspects [3–10].

In recent years two-dimensional electron systems have become an active research subject due to the rapid growth in nanofabrication technology, which has made possible the production of low dimensional structures such as quantum wells, quantum wires, quantum dots, etc. [11–13]. In the non-relativistic case, the two dimensional parabolic potential has often been used to describe the spectrum of the electron in confined two-dimensional systems. For the relativistic case, the spectrum and properties of the such systems can be determined by using two dimensional Dirac oscillator [12, 14, 15]. Despite their simplicity, both relativistic and non-relativistic oscillator potentials appear to be a good approximation to complicated low dimensional nanostructures.

In order to present a more realistic model we construct a deformed Dirac oscillator by including the term qr^3 . With this term, the Dirac oscillator becomes a radial sextic oscillator potential in the non-relativistic limit. The non-relativistic sextic oscillator potential is quasi-exactly-solvable (QES) for which it is possible to determine algebraically a part of spectrum, but not the whole spectrum [16-19]. It will be shown that the solution of the relativistic sextic oscillator can also be treated in the context of the QES problem.

We also investigate the effect of the magnetic field on the Dirac sextic oscillator. It will be shown that for specific values of the magnetic field the Dirac sextic oscillator problem is exactly solvable.

This paper is organized as follows. In section 2 we discuss the construction of the Dirac sextic oscillator in polar coordinate in the presence of a magnetic field. In section 3, we solve the Dirac sextic oscillator in

the context the QES problem. We show that the problem possesses hidden sl_2 -algebra. In section 4 we briefly discuss our method and results.

2. Construction of the Dirac sextic oscillator

The $(2 + 1)$ -dimensional Dirac equation for free particle of mass M in terms of two-component spinors ψ can be written as

$$E\psi = \left[\sum_{i=1}^2 c\beta\gamma_i p_i + \beta M c^2 \right] \psi. \quad (1)$$

Since we are using only two component spinors, the matrices β and $\beta\gamma_i$ are conveniently defined in terms of the Pauli spin matrices and satisfy the relation $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$, given by

$$\beta\gamma_1 = \sigma_1; \quad \beta\gamma_2 = \sigma_2; \quad \beta = \sigma_3. \quad (2)$$

In $(2 + 1)$ -dimensions, the momentum operator p_i is a two component differential operator, i.e. $\mathbf{p} = -i\hbar(\partial_x, \partial_y)$ for a free particle. In the presence of the magnetic field it is replaced by $\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}$, where \mathbf{A} is the vector potential, and the 2D Dirac oscillator can be constructed by changing the momentum $\mathbf{p} \rightarrow \mathbf{p} - im\omega\beta\mathbf{r}$. We are now looking for some expressions on the right hand side of (1) that can be interpreted as radial sextic oscillator Hamiltonian in the non-relativistic limit. For this purpose we introduce the following momentum operator

$$\mathbf{p} \rightarrow \mathbf{p} - i\sigma_3(m\omega - qr^2)\mathbf{r}, \quad (3)$$

where q is a constant and the Dirac equation takes the form

$$\begin{aligned} [E - \sigma_0 m c^2] \psi &= c [\sigma_+ (p_x - ip_y - i(M\omega - qr^2)(x - iy))] \psi + \\ & c [\sigma_- (p_x + ip_y - i(M\omega - qr^2)(x + iy))] \psi. \end{aligned} \quad (4)$$

In polar coordinate, $x = r \cos \phi$, $y = r \sin \phi$, the 2D Dirac equation (4) can be written as

$$\begin{aligned} \varepsilon^2 \psi_1 &= -c^2 \hbar^2 \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{2i(M\omega - qr^2)}{\hbar} \frac{\partial}{\partial \phi} \right] \psi_1 \\ &+ c^2 (q^2 r^6 - 2M\omega q r^4 + (M^2 \omega^2 - 4\hbar q) r^2 + 2\hbar M \omega) \psi_1, \end{aligned} \quad (5)$$

where $\psi_1 = \psi_1(r, \phi)$ is the upper component of the spinor ψ and $\varepsilon^2 = E^2 - M^2 c^4$. The substitution of

$$\psi_1(r, \phi) = \frac{e^{-im\phi}}{\sqrt{r}} f(r) \quad (6)$$

leads to the equation

$$\varepsilon^2 f(r) = -c^2 \hbar^2 \frac{\partial^2 f(r)}{\partial r^2} + c^2 [V(r) + 2\hbar M \omega (1 - m)] f(r), \quad (7)$$

where $V(r)$ is given by

$$V(r) = \frac{\hbar^2 (m^2 - \frac{1}{4})}{r^2} + q^2 r^6 - 2M\omega q r^4 + (M^2 \omega^2 - 2\hbar q (2 - m)) r^2. \quad (8)$$

It is not difficult to see that in the non-relativistic limit (7) corresponds to the Schrödinger equation with radial sextic oscillator potential with a spin-orbit coupling term. In the presence of the symmetric gauge vector potential $A(x, y) = B/2(-y, x, 0)$, the potential (8) takes the form

$$\begin{aligned} V(r) &= \frac{\hbar^2 (m^2 - \frac{1}{4})}{r^2} + \hbar e B (m - 1) + \\ & q^2 r^6 - (2M\omega - eB) q r^4 + \left(\left(M\omega - \frac{eB}{2} \right)^2 - 2\hbar q (2 - m) \right) r^2 \end{aligned} \quad (9)$$

From now on we restrict ourselves to the solution of (7) and (8).

3. Method of Solution

In this section we show that the Dirac sextic oscillator (7) is one of the recently discovered quasi-exactly solvable operators [16, 17]. It is well known that the underlying idea behind the quasi exact solvability is the existence of a hidden algebraic structure. Let us introduce the following expression of the sl_2 -algebra:

$$J_+ = \rho^2 \frac{d}{d\rho} - j\rho, \quad J_- = \frac{d}{d\rho}, \quad J_0 = \rho \frac{d}{d\rho} - \frac{j}{2} \quad (10)$$

The generators satisfy the commutation relations of the sl_2 -algebra for any value of the parameter j . If j is a positive integer the algebra, (10) possesses $j + 1$ -dimensional irreducible representation:

$$P_{j+1} = \langle 1, \rho, \rho^2, \dots, \rho^j \rangle \quad (11)$$

The linear and bilinear combinations of the operators given in (10) are quasi exactly solvable, when the space is defined in (11).

In order to show the Dirac sextic oscillator has a sl_2 -symmetry, let us consider the following combinations of the operators (10):

$$T = -J_0 J_- + \frac{j+2}{2} J_- + 16c^4 \hbar^3 q J_+ + 4c^2 \hbar M \omega J_0. \quad (12)$$

Then, the eigenvalue problem can be written as

$$T P_k(\rho) = (\varepsilon^2 + 2M c^2 \hbar \omega) P_k(\rho), \quad (13)$$

where $P_k(\rho)$ is the k^{th} degree polynomial in ρ .

Let us return our attention to the Dirac sextic oscillator (7). Introducing a new function

$$f(r) = r^{m-\frac{1}{2}} e^{-\frac{m\omega}{2\hbar} r^2 - \frac{q}{4\hbar} r} F(r), \quad (14)$$

and then changing the variable $r = 2c\hbar\sqrt{\rho}$, we obtain the following expression:

$$\begin{aligned} \varepsilon^2 F(\rho) = & \\ -\rho \frac{\partial^2 F(\rho)}{\partial \rho^2} + (m-1 + 4c^2 \hbar \omega M \rho - 16qc^4 \hbar^3 \rho^2) \frac{\partial F(\rho)}{\partial \rho} & \\ -c^2 (4M \hbar \omega (m-1) - 16qc^2 \hbar^3 (m-2)\rho) F(\rho). & \end{aligned} \quad (15)$$

We can show that the eigenvalue equation (7) and (13) are identical, when the following holds:

$$m = j + 2, \quad F(\rho) = P_k(\rho). \quad (16)$$

When the generators act on the polynomial (11), we can obtain the recurrence relation

$$\begin{aligned} 16qc^4 \hbar^3 (k-j) P_{k+1}(\varepsilon) + (\varepsilon^2 + 4M c^2 \hbar \omega (j-k+1)) P_k(\varepsilon) - \\ k(j-k+2) P_{k-1}(\lambda) = 0, \end{aligned} \quad (17)$$

with the initial condition $P_0(1) = 1$. If ε_i is a root of the polynomial $P_{k+1}(\varepsilon)$, the wavefunction is truncated at $k = j$ and belongs to the spectrum of the Hamiltonian T . This property implies that the wavefunction is itself the generating function of the energy polynomials. The roots of the recurrence relation (17) can be computed and the first few of them are given by

$$\begin{aligned} P_1(\varepsilon) &= \varepsilon^2 + 4M c^2 \hbar \omega \\ P_2(\varepsilon) &= (\varepsilon^2 + 4M c^2 \hbar \omega) (\varepsilon^2 + 8M c^2 \hbar \omega) - 32qc^4 \hbar^3 \\ P_3(\varepsilon) &= \varepsilon^6 + 8c^2 \hbar (3\varepsilon^4 M \omega + 48c^4 \hbar^2 M \omega (M^2 \omega^2 - 3q\hbar)) + \\ & \quad 2c^2 \hbar \varepsilon^2 (11M^2 \omega^2 - 10q\hbar) \\ P_4(\varepsilon) &= \varepsilon^8 + 40\varepsilon^6 M c^2 \hbar \omega + 80\varepsilon^4 c^4 \hbar^2 (7M^2 \omega^2 - 6\hbar q) + \\ & \quad 128\varepsilon^2 c^6 \hbar^3 M \omega (25M^2 \omega^2 - 69\hbar q) + 6144c^8 \hbar^4 (3\hbar^2 q^2 - 6M^2 \omega^2 \hbar q + M^4 \omega^4). \end{aligned} \quad (18)$$

The function $P_j(\rho)$ forms a basis for sl_2 -algebra and can be written in the form

$$P_j(\rho) = \sum_{k=0}^j c_k P_k(\varepsilon) \rho^k. \quad (19)$$

In the presence of the magnetic field the Hamiltonian can be solved by the same procedure given above. When the magnetic field $B = 2M\omega/e$, the Hamiltonian (7) takes the form

$$\begin{aligned} \varepsilon^2 f(r) = & -c^2 \hbar^2 \frac{\partial^2 f(r)}{\partial r^2} + \\ & c^2 \left[\frac{\hbar^2 (m^2 - \frac{1}{4})}{r^2} + q^2 r^6 - 2\hbar q(2-m)r^2 + 4\hbar M\omega(1-m) \right] f(r). \end{aligned} \quad (20)$$

We define the wavefunction

$$f(r) = r^{\frac{1}{2}-m} e^{-\frac{qr^4}{4\hbar}} F(r), \quad (21)$$

and changing the variable $r = 2c\hbar\sqrt{\rho}$, we obtain the differential equation

$$\varepsilon^2 F(\rho) = -\rho \frac{d^2}{d\rho^2} + (j+1 - 16qc^4\hbar^3\rho^2) \frac{\partial F(\rho)}{\partial \rho} + 16jqc^4\hbar^3\rho F(\rho).$$

When $F(\rho) = P_k(\rho)$, where $P_k(\rho)$ is k^{th} degree polynomial in ρ , we obtain the following recurrence relation

$$16qc^4\hbar^3 P_{k+1}(\varepsilon) - \varepsilon^2 P_k(\varepsilon) + k(j+2-k)P_{k-1}(\varepsilon) = 0$$

The polynomial $P_k(\varepsilon)$ vanishes for $k = j+1$ and the roots of $P_j(\varepsilon)$ belongs to the spectrum of the (20). This is the first ten $P_k(\varepsilon)$:

$$\begin{aligned} P_1(\varepsilon) &= \varepsilon^2 \\ P_2(\varepsilon) &= \varepsilon^4 - 2\eta^2 \\ P_3(\varepsilon) &= \varepsilon^6 - 10\varepsilon^2\eta^2 \\ P_4(\varepsilon) &= \varepsilon^8 - 30\varepsilon^4\eta^2 + 72\eta^4 \\ P_5(\varepsilon) &= \varepsilon^{10} - 70\varepsilon^6\eta^2 + 712\varepsilon^2\eta^4 \\ P_6(\varepsilon) &= \varepsilon^{12} - 140\varepsilon^8\eta^2 + 3820\varepsilon^4\eta^4 - 10800\eta^6 \\ P_7(\varepsilon) &= \varepsilon^{14} - 252\varepsilon^{10}\eta^2 + 14796\varepsilon^6\eta^4 - 164592\varepsilon^2\eta^6 \\ P_8(\varepsilon) &= \varepsilon^{16} - 420\varepsilon^{12}\eta^2 + 46380\varepsilon^8\eta^4 - 1307600\varepsilon^4\eta^6 + 4233600\eta^8 \\ P_9(\varepsilon) &= \varepsilon^{18} - 660\varepsilon^{14}\eta^2 + 125004\varepsilon^{10}\eta^4 - 7250320\varepsilon^6\eta^6 + 88504707\varepsilon^2\eta^8. \end{aligned} \quad (22)$$

Here, $\eta^2 = 16qc^4\hbar^3$. Therefore we can obtain the eigenfunction of (20) in the closed form. We conclude that anharmonic interaction destroys the general symmetry of the Dirac equation, but the specified magnetic field can restore the symmetries of the Dirac equation. This feature implies that analytical solutions of the roots of the polynomials are available for specific values of the magnetic field $B = 2\hbar\omega/e$.

4. Conclusion

We have constructed Dirac sextic oscillator in two-dimensional space in the presence of a magnetic field. We have given eigenstates of the corresponding equation in terms of the orthogonal polynomials [20]. We have also shown that the Dirac sextic oscillator possesses hidden sl_2 -symmetry.

We note that similar constructions of the other quasi-exactly solvable Dirac equations also seem possible by considering the momentum operator of the form: $\mathbf{p} \rightarrow \mathbf{p} - i\sigma_3(m\omega + v(r))\mathbf{r}$. The method we have introduced can be applied to construct Dirac equations with the inclusion of other non-relativistic potentials such as Pöschle-Teller, Eckart, Scarf, Hultén, among others. We hope that the Dirac sextic oscillator may be used as a model in related areas of physics.

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