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A Treatment of a Higher-Order Singular Lagrangian as Fields System

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Abstract

The higher-order singular Lagrangian system is treated as field system. Euler-Lagrange equations are solved with some constraints. An example is studied and a comparison with Hamiltonian formulation is done.

Key Words: Lagrangian and Hamiltonian approach, Singular Lagrangian systems.

1. Introduction

The study of singular Lagrangian systems was initiated by Dirac [1]. He developed the basic theories of the classical treatment and quantization of such systems. An alternative method is the Hamilton-Jacobi formulation, or the canonical method which is developed by Güler [2, 3]. This method was generalized to singular systems with higher-order Lagrangian [4, 5]. A treatment of singular Lagrangian system as field system was studied in [6, 7]. In this work we shall generalize the latter formulation to the higher-order Lagrangian system. In section two we make a brief discussion of the Hamilton-Jacobi formulation or canonical method to investigate the higher-order singular system. The generalization of the treatment of singular Lagrangian system as field system to the higher-order Lagrangian will be done in section three. An example will be solved in section four.

2. Hamilton-Jacobi Formulation of the Higher-Order Singular Lagrangian

The higher-order Lagrangian is described by the function $L(q_i, \overset{(1)}{q_i}, \overset{(2)}{q_i}, \overset{(3)}{q_i}, ..., t)$, where $\overset{(s)}{q_i} = \frac{d^s q_i}{dt^s}$, $s = 1, 2, ...$ and $i = 1, 2, 3, ..., n$. We are concerned with Lagrangian of K-order, which takes the form $L(q_i, \overset{(1)}{q_i}, \overset{(2)}{q_i}, ..., \overset{(K)}{q_i}, t)$.
The system is regular if the rank of the Hessian matrix [4]

$$A_{ij} = \frac{\partial^2 L}{\partial \overset{(K)}{q_i} \partial \overset{(K)}{q_j}} \tag{1}$$

is n, and singular if the rank is $n-r$, $r<n$. The generalized momenta are defined as

$$p_{(K-1)i} = \frac{\partial L}{\partial \overset{(K)}{q_i}}, \tag{2}$$

$$p_{(s-1)i} = \frac{\partial L}{\partial \overset{(K)}{q_i}} - \dot{p}_{(s)i}; \quad s = 1, ..., K-1, \tag{3}$$

where $p_{(K-1)i}$ and $p_{(s-1)i}$ are the momenta conjugated to the coordinates $q_{i(K-1)}$ and $q_{i(s-1)}$, respectively. Since the rank of the Hessian matrix is $n-r$, one may solve the derivatives $\overset{(K)}{q_i}$ as functions of the coordinates $q_{(s)i}$, the momenta $p_{(K-1)b}$, and the unsolved derivatives $\overset{(K)}{q_i}$, as follows:

$$\overset{(K)}{q_i} \equiv f_{(K)a}(q_{(s)i}; p_{(K-1)b}; \overset{(K)}{q_\mu}), \tag{4}$$

where $a, b = 1, ..., n-r$, and $\mu = 0, n-r+1, ..., n$.
Also the momenta variables will not be independent of one another, and one may write

$$p_{(u)\alpha} = -H_{(u)\alpha}(q_{(s)j}; p_{(s)a}); \qquad u, s = 0, ..., K-1, s \leq u, \tag{5}$$

where we are assuming that the expression for the momentum $p_{(u)\mu}$ depends on all momenta $p_{(s)a}$.

The Hamiltonian H_0 is defined as

$$H_0 = \sum_{u=0}^{K-2} p_{(u)a} \overset{(u+1)}{q_a} + p_{(K-1)a} f_{(K)a} + \sum_{u=0}^{K-1} \overset{(u+1)}{q_\mu} p_{(u)\mu}|_{p_{(s)\nu}=-H_{(s)\nu}}$$

$$-L\left(\overset{(s)}{q_i}, \overset{(K)}{q_\mu}, \overset{(K)}{q_a} = f_{(K)a}\right), \tag{6}$$

where $\mu, \nu = 0, n-r+1, ..., n; a = 1, ..., n-r$. Here H_0 does not depend explicitly upon the derivatives $\overset{(K)}{q_\mu}$, that is:

$$\frac{\partial H_0}{\partial \overset{(K)}{q_\mu}} = 0. \tag{7}$$

Now let us consider the following notations. The time parameter will be denoted $t_{(s)0} \equiv \overset{(s)}{q_0}$ (for any value of s). The coordinates $\overset{(s)}{q_\mu}$ will be denoted $t_{(s)\mu}$. The momenta $p_{(s)\mu}$ will be denoted as $P_{(s)\mu}$. And the momentum $p_{(s)0} \equiv P_{(s)0}$ will be defined as

$$P_{(s)0} \equiv \frac{\partial S}{\partial t}, \tag{8}$$

where S is the action, and $H_{(s)0} \equiv H_0$.

Now, to obtain an extremum of the action integral, we must find a function $S(t_{(c)\mu}; q_{(c)a}, t)$ $(c = 0, ..., K-1)$ that satisfies the set of Hamilton-Jacobi partial differential equations

$$H'_{(s)\mu} \equiv P_{(s)\mu} + H_{(s)\mu}\left(t_{(u)\mu}; q_{(u)a}; p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}\right) = 0. \tag{9}$$

where $s, u = 0, ..., K-1$ and $\mu = 0, n-r+1, ..., n$.
The equations of motion are written as total differential equations as:

$$dq_{(u)i} = \sum_{s=0}^{K-1} \frac{\partial H'_{(s)\mu}}{\partial p_{(u)i}} dt_{(s)\mu}, \qquad i = 1, ..., n, \tag{10}$$

$$dp_{(u)c} = -\sum_{s=0}^{K-1} \frac{\partial H'_{(s)\mu}}{\partial q_{(u)c}} dt_{(s)\mu}, \tag{11}$$

where $u = 0, 1, ..., K-1$, $c = 0, 1, ..., n$ and $\mu = 0, n-r+1, ..., n$. Making $Z \equiv S(t_{(s)\mu}; q_{(s)a})$ and using the momenta definitions together with eq. (10), we have

$$dZ = \sum_{d=0}^{K-1} \left(-H_{(d)\nu} + \sum_{s=0}^{K-1} p_{(s)a} \frac{\partial H'_{(d)\nu}}{\partial p_{(s)a}} \right) dt_{(d)\nu}. \tag{12}$$

This equation together with eqs. (10) and (11) are the total differential equations for the characteristics curves of the Hamilton-Jacobi partial differential equation given by (9); and if they form a completely integrable set, their solutions determine $S(t_{(s)\mu}; q_{(s)a})$ uniquely from the initial conditions.

3. A Treatment of the Higher-Order Singular Lagrangian as Field System

In previous work, the first-order singular Lagrangian system was treated as field system or continuous system [6]. In this section we shall generalize this proposal to the higher-order singular Lagrangian systems. The higher-order singular Lagrangian $L = L(q_i, \overset{(1)}{q_i}, \overset{(2)}{q_i}, ..., \overset{(K)}{q_i}, t)$ can be treated as field system, where the fields q_a are expressed in term of the independent coordinates as

$$q_a \equiv q_a(t, x_\mu), \qquad x_\mu \equiv q_\mu, \tag{13}$$

where $\mu = 0, n-r+1, ..., n, a = 1, ..., n-r$. The Euler-Lagrange equation of the higher-order singular Lagrangian takes the form

$$\frac{\partial L'}{\partial q_a} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial L'}{\partial(\partial_\mu q_a)}\right) + \frac{\partial^2}{\partial x_{\mu_1}\partial x_{\mu_2}}\left(\frac{\partial L'}{\partial(\partial_{\mu_1}\partial_{\mu_2} q_a)}\right)$$

$$-... + \frac{(-1)^K \partial^K}{\partial x_{\mu_1}\partial x_{\mu_2}...\partial x_{\mu_K}}\left(\frac{\partial L'}{\partial(\partial_{\mu_1}...\partial_{\mu_2}\partial_{\mu_K} q_a)}\right) = 0, \tag{14}$$

where

$$\partial_\mu q_a \equiv \frac{\partial q_a}{\partial x_\mu}, \tag{15}$$

$$\partial_{\mu_1}\partial_{\mu_2} q_a \equiv \frac{\partial^2 q_a}{\partial x_{\mu_1}\partial x_{\mu_2}}, \tag{16}$$

$$\partial_{\mu_1}\partial_{\mu_2}...\partial_{\mu_K} q_a = \frac{\partial^K q_a}{\partial x_{\mu_1}\partial x_{\mu_2}...\partial x_{\mu_K}}, \tag{17}$$

and the modified Lagrangian L' is defined as

$$L'\left(x_\mu, q_a, \partial_\mu q_a, \partial_{\mu_1}\partial_{\mu_2} q_a, \partial_{\mu_1}\partial_{\mu_2}...\partial_{\mu_K} q_a, \overset{(1)}{x}_\mu, \overset{(2)}{x}_\mu, ..., \overset{(K)}{x}_\mu\right)$$

$$\begin{aligned} &\equiv L(x_\mu, q_a, \overset{(1)}{q}_a = (\partial_\mu q_a)\overset{(1)}{x}_\mu, \overset{(2)}{q}_a = \partial_{\mu_2}(\partial_{\mu_1} q_a \overset{(1)}{x}_{\mu 1})\overset{(1)}{x}_{\mu 2}, ..., \overset{(K)}{q}_a \\ &\quad = \partial_{\mu_K}(...(\partial_{\mu_2}(\partial_{\mu_1} q_a \overset{(1)}{x}_{\mu 1})\overset{(1)}{x}_{\mu 2}), ..., \overset{(1)}{x}_{\mu K}), \overset{(1)}{x}_\mu, \overset{(2)}{x}_\mu, ..., \overset{(K)}{x}_\mu) \end{aligned} \tag{18}$$

with $\overset{(s)}{x}_\mu = \frac{d^s x_\mu}{dt^s}$ and $\overset{(1)}{x}_0 = 1$.

The constraints equations can be written as

$$dG_0 = -\frac{\partial L'}{\partial t} dt, \tag{19}$$

$$dG_{(u)\mu} = -\frac{\partial L'}{\partial q_\mu} dt, \tag{20}$$

where

$$G_0 = H_0(\overset{(u)}{q_i}, p_{(u)a}, t), \tag{21}$$

and

$$G_{(u)\mu} = H_{(u)\mu}(\overset{(u)}{q_i}, p_{(u)a}, t) \tag{22}$$

are the canonical Hamiltonian and the constraints equations, respectively. Both H_0 and $H_{(u)\mu}$ are obtained from the canonical Hamiltonian formulation of singular system which has been discussed in section 2. The solution of Euler-Lagrange equation (14), together with the constraint equations (19) and (20), gives the solution of the system.

An example

Consider the third-order singular Lagrangian [8]

$$L = \overset{(3)}{q_1}\overset{(3)}{q_2} + \overset{(2)}{q_1}(\overset{(2)}{q_2} - \overset{(2)}{q_3}) + \overset{(1)}{q_1}(\overset{(1)}{q_2} - \overset{(1)}{q_3}) - q_1 q_3. \tag{23}$$

Since the rank of the Hessian matrix of this Lagrangian is two, the system can be treated as field system in the form

$$q_1 = q_1(q_3, t), \tag{24}$$

$$q_2 = q_2(q_3, t). \tag{25}$$

The first, second and third derivatives with respect to t of (24) and (25), respectively, are:

$$\overset{(1)}{q_1} = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3}\overset{(1)}{q_3}, \tag{26}$$

$$\overset{(1)}{q_2} = \frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial q_3}\overset{(1)}{q_3}, \tag{27}$$

$$\overset{(2)}{q_1} = \frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial t \partial q_3}\overset{(1)}{q_3} + \frac{\partial^2 q_1}{\partial q_3^2}(\overset{(1)}{q_3})^2 + \frac{\partial q_1}{\partial q_3}\overset{(2)}{q_3}, \tag{28}$$

$$\overset{(2)}{q_2} = \frac{\partial^2 q_2}{\partial^2 t} + 2\frac{\partial^2 q_2}{\partial t \partial q_3}\overset{(1)}{q_3} + \frac{\partial^2 q_2}{\partial q_3^2}(\overset{(1)}{q_3})^2 + \frac{\partial q_2}{\partial q_3}\overset{(2)}{q_3}, \tag{29}$$

and

$$\overset{(3)}{q}_1 = \frac{\partial^3 q_1}{\partial t^3} + 3\frac{\partial^3 q_1}{\partial t^2 \partial q_3}\overset{(1)}{q}_3 + 3\frac{\partial^3 q_1}{\partial t \partial q_3^2}(\overset{(1)}{q}_3)^2 + 3\frac{\partial^2 q_1}{\partial t \partial q_3}\overset{(2)}{q}_3 + \frac{\partial^3 q_1}{\partial q_3^3}(\overset{(1)}{q}_3)^3$$

$$+3\frac{\partial^2 q_1}{\partial q_3^2}\overset{(1)}{q}_3\overset{(2)}{q}_3 + \frac{\partial q_1}{\partial q_3}\overset{(3)}{q}_3, \tag{30}$$

$$\overset{(3)}{q}_2 = \frac{\partial^3 q_2}{\partial t^2} + 3\frac{\partial^3 q_2}{\partial t^2 \partial q_3}\overset{(1)}{q}_3 + 3\frac{\partial^3 q_2}{\partial t \partial q_3^2}(\overset{(1)}{q}_3)^2 + 3\frac{\partial^2 q_2}{\partial t \partial q_3}\overset{(2)}{q}_3 + \frac{\partial^3 q_2}{\partial q_3^3}(\overset{(1)}{q}_3)^3$$

$$+3\frac{\partial^2 q_2}{\partial q_3^2}\overset{(1)}{q}_3\overset{(2)}{q}_3 + \frac{\partial q_2}{\partial q_3}\overset{(3)}{q}_3. \tag{31}$$

Substituting (26)–(31) into (23) we get the "modified Lagrangian" function

$$L' = (\frac{\partial^3 q_1}{\partial t^3} + 3\frac{\partial^3 q_1}{\partial t^2 \partial q_3}\overset{(1)}{q}_3 + 3\frac{\partial^3 q_1}{\partial t \partial q_3^2}(\overset{(1)}{q}_3)^2 + 3\frac{\partial^2 q_1}{\partial t \partial q_3}\overset{(2)}{q}_3 + \frac{\partial^3 q_1}{\partial q_3^3}(\overset{(1)}{q}_3)^3$$

$$+3\frac{\partial^2 q_1}{\partial q_3^2}\overset{(1)}{q}_3\overset{(2)}{q}_3 + \frac{\partial q_1}{\partial q_3}\overset{(3)}{q}_3)(\frac{\partial^3 q_2}{\partial t^2} + 3\frac{\partial^3 q_2}{\partial t^2 \partial q_3}\overset{(1)}{q}_3 + 3\frac{\partial^3 q_2}{\partial t \partial q_3^2}(\overset{(1)}{q}_3)^2 + 3\frac{\partial^2 q_2}{\partial t \partial q_3}\overset{(2)}{q}_3$$

$$+\frac{\partial^3 q_2}{\partial q_3^3}\left(\overset{(1)}{q}_3\right)^3 + 3\frac{\partial^2 q_2}{\partial q_3^2}\overset{(1)}{q}_3\overset{(2)}{q}_3 + \frac{\partial q_2}{\partial q_3}\overset{(3)}{q}_3) + (\frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial t \partial q_3}\overset{(1)}{q}_3 + \frac{\partial^2 q_1}{\partial q_3^2}(\overset{(1)}{q}_3)^2$$

$$+\frac{\partial q_1}{\partial q_3}\overset{(2)}{q}_3)\left[\left(\frac{\partial^2 q_2}{\partial^2 t} + 2\frac{\partial^2 q_2}{\partial t \partial q_3}\overset{(1)}{q}_3 + \frac{\partial^2 q_2}{\partial q_3^2}(\overset{(1)}{q}_3)^2 + \frac{\partial q_2}{\partial q_3}\overset{(2)}{q}_3\right) - \overset{(3)}{q}_2\right]$$

$$+\left(\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3}\overset{(1)}{q}_3\right)\left[\frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial q_3}\overset{(1)}{q}_3 - \overset{(1)}{q}_3\right] - q_1 q_3. \tag{32}$$

The Euler-Lagrange equations (14) in q_1 and q_2 then read as

$$\frac{\partial L'}{\partial q_1} - \frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial\left(\frac{\partial q_1}{\partial t}\right)}\right) - \frac{\partial}{\partial q_3}\left(\frac{\partial L'}{\partial\left(\frac{\partial q_1}{\partial q_3}\right)}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial L'}{\partial\left(\frac{\partial^2 q_1}{\partial t^2}\right)}\right)$$

$$+2\frac{\partial^2}{\partial t \partial q_3}\left(\frac{\partial L'}{\partial\left(\frac{\partial^2 q_1}{\partial t \partial q_3}\right)}\right) + \frac{\partial^2}{\partial q_3^2}\left(\frac{\partial L'}{\partial\left(\frac{\partial^2 q_1}{\partial q_3^2}\right)}\right) - \frac{\partial^3}{\partial t^3}\left(\frac{\partial L'}{\partial\left(\frac{\partial^3 q_1}{\partial t^3}\right)}\right)$$

$$-3\frac{\partial^3}{\partial t \partial q_3^2}\left(\frac{\partial L'}{\partial\left(\frac{\partial^3 q_1}{\partial t \partial q_3^2}\right)}\right) - 3\frac{\partial^3}{\partial t^2 \partial q_3}\left(\frac{\partial L'}{\partial\left(\frac{\partial^3 q_1}{\partial t^2 \partial q_3}\right)}\right) - \frac{\partial^3}{\partial q_3^3}\left(\frac{\partial L'}{\partial\left(\frac{\partial^3 q_1}{\partial q_3^3}\right)}\right) = 0, \tag{33}$$

and

$$\frac{\partial L'}{\partial q_2} - \frac{\partial}{\partial t}(\frac{\partial L'}{\partial(\frac{\partial q_2}{\partial t})}) - \frac{\partial}{\partial q_3}(\frac{\partial L'}{\partial(\frac{\partial q_2}{\partial q_3})}) + \frac{\partial^2}{\partial t^2}(\frac{\partial L'}{\partial(\frac{\partial^2 q_2}{\partial t^2})}) + 2\frac{\partial^2}{\partial t \partial q_3}(\frac{\partial L'}{\partial(\frac{\partial^2 q_2}{\partial t \partial q_3})})$$

$$+\frac{\partial^2}{\partial q_3^2}\Big(\frac{\partial L'}{\partial(\frac{\partial^2 q_2}{\partial q_3^2})}\Big)-\frac{\partial^3}{\partial t^3}\Big(\frac{\partial L'}{\partial(\frac{\partial^3 q_2}{\partial t^3})}\Big)-3\frac{\partial^3}{\partial t\partial q_3^2}\Big(\frac{\partial L'}{\partial(\frac{\partial^3 q_2}{\partial t\partial q_3^2})}\Big)$$

$$-3\frac{\partial^3}{\partial t^2\partial q_3}\Big(\frac{\partial L'}{\partial(\frac{\partial^3 q_2}{\partial t^2\partial q_3})}\Big)-\frac{\partial^3}{\partial q_3^3}\Big(\frac{\partial L'}{\partial(\frac{\partial^3 q_2}{\partial q_3^3})}\Big)=0. \tag{34}$$

Using equation (32), equations (33) and (34) become

$$-q_3-\frac{\partial \overset{(1)}{q}_2}{\partial t}+\overset{(2)}{q}_3-\frac{\partial \overset{(2)}{q}_3}{\partial q_3}\overset{(3)}{q}_3-\frac{\partial \overset{(2)}{q}_2}{\partial q_3}\overset{(2)}{q}_3-\frac{\partial \overset{(1)}{q}_2}{\partial q_3}\overset{(1)}{q}_3+\frac{\partial^2 \overset{(2)}{q}_2}{\partial t^2}-\overset{(4)}{q}_3$$

$$+6\frac{\partial^2 \overset{(3)}{q}_2}{\partial t\partial q_3}\overset{(2)}{q}_3+4\frac{\partial^2 \overset{(2)}{q}_2}{\partial t\partial q_3}\overset{(1)}{q}_3+3\frac{\partial^2 \overset{(3)}{q}_2}{\partial q_3^2}\overset{(1)}{q}_3\overset{(2)}{q}_3+\frac{\partial^2 \overset{(2)}{q}_2}{\partial q_3^2}q_3^2\left(\overset{(1)}{q}_3\right)^2-\frac{\partial^3 \overset{(3)}{q}_2}{\partial t^3}$$

and

$$-9\frac{\partial^3 \overset{(3)}{q}_2}{\partial t\partial q_3^2}\left(\overset{(1)}{q}_3\right)^2-9\frac{\partial^3 \overset{(3)}{q}_2}{\partial t^2\partial q_3}\overset{(1)}{q}_3-\frac{\partial^3 \overset{(3)}{q}_2}{\partial {q_3}^3}\left(\overset{(1)}{q}_3\right)^2=0, \tag{35}$$

$$-\frac{\partial \overset{(1)}{q}_1}{\partial t}-\frac{\partial \overset{(1)}{q}_1}{\partial q_3}\overset{(1)}{q}_3-\frac{\partial \overset{(2)}{q}_1}{\partial q_3}\overset{(2)}{q}_3-\frac{\partial \overset{(3)}{q}_1}{\partial q_3}\overset{(3)}{q}_3+\frac{\partial^2 \overset{(2)}{q}_1}{\partial t^2}+6\frac{\partial^2 \overset{(3)}{q}_1}{\partial t\partial q_3}\overset{(2)}{q}_3$$

$$+4\frac{\partial^2 \overset{(2)}{q}_1}{\partial t\partial q_3}\overset{(1)}{q}_3+3\frac{\partial^2 \overset{(3)}{q}_1}{\partial q_3^2}\overset{(1)}{q}_3\overset{(2)}{q}_3+\frac{\partial^2 \overset{(2)}{q}_1}{\partial q_3^2}\left(\overset{(1)}{q}_3\right)^2-\frac{\partial^3 \overset{(3)}{q}_1}{\partial t^3}-9\frac{\partial^3 \overset{(3)}{q}_1}{\partial t\partial q_3^2}\left(\overset{(1)}{q}_3\right)^2$$

$$-9\frac{\partial^3 \overset{(3)}{q}_1}{\partial t^2\partial q_3}\overset{(1)}{q}_3-\frac{\partial^3 \overset{(3)}{q}_1}{\partial q_3^3}\left(\overset{(1)}{q}_3\right)^2=0. \tag{36}$$

From the canonical formulation (see Appendix A) the constraint equation (22) formulates as

$$G_3=\left(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_3}\overset{(1)}{q}_3\right)-\left(\frac{\partial \overset{(2)}{q}_1}{\partial t}+\frac{\partial \overset{(2)}{q}_1}{\partial q_3}\overset{(1)}{q}_3\right). \tag{37}$$

The total variation is

$$dG_3=\Big[\Big(\frac{\partial \overset{(1)}{q}_1}{\partial t}+\frac{\partial \overset{(1)}{q}_1}{\partial q_3}\overset{(1)}{q}_3\Big)-\Big(\frac{\partial^2 \overset{(2)}{q}_1}{\partial t^2}+2\frac{\partial^2 \overset{(2)}{q}_1}{\partial t\partial q_3}\overset{(1)}{q}_3+\frac{\partial^2 \overset{(2)}{q}_1}{\partial q_3^2}(\overset{(1)}{q}_3)^2$$

$$+\frac{\partial \overset{(2)}{q}_1}{\partial q_3}\overset{(2)}{q}_3\Big)\Big]dt. \tag{38}$$

Thus the constraint equation (20) reads as

$$\Big(\frac{\partial^2 \overset{(2)}{q}_1}{\partial t^2}+2\frac{\partial^2 \overset{(2)}{q}_1}{\partial t\partial q_3}\overset{(1)}{q}_3+\frac{\partial^2 \overset{(2)}{q}_1}{\partial q_3^2}(\overset{(1)}{q}_3)^2+\frac{\partial \overset{(2)}{q}_1}{\partial q_3}\overset{(2)}{q}_3\Big)-\Big(\frac{\partial \overset{(1)}{q}_1}{\partial t}+\frac{\partial \overset{(1)}{q}_1}{\partial q_3}\overset{(1)}{q}_3\Big)+q_1=0. \tag{39}$$

The total derivative of (39) can be written as

$$\Big(\frac{\partial^3 \overset{(3)}{q}_1}{\partial t^3}+3\frac{\partial^3 \overset{(3)}{q}_1}{\partial t^2\partial q_3}\overset{(1)}{q}_3+3\frac{\partial^3 \overset{(3)}{q}_1}{\partial t\partial q_3^2}(\overset{(1)}{q}_3)^2+3\frac{\partial^2 \overset{(3)}{q}_1}{\partial t\partial q_3}\overset{(2)}{q}_3+\frac{\partial^3 \overset{(3)}{q}_1}{\partial q_3^3}(\overset{(1)}{q}_3)^3$$

$$+3\frac{\partial^2\overset{(3)}{q_1}}{\partial q_3^2}\overset{(1)}{q_3}\overset{(2)}{q_3}+\frac{\partial\overset{(2)}{q_1}}{\partial q_3}\overset{(3)}{q_3}-(\frac{\partial^2\overset{(2)}{q_1}}{\partial t^2}+2\frac{\partial^2\overset{(2)}{q_1}}{\partial t\partial q_3}\overset{(1)}{q_3}+\frac{\partial^2\overset{(2)}{q_1}}{\partial q_3^2}(\overset{(1)}{q_3})^2$$

$$+\frac{\partial\overset{(1)}{q_1}}{\partial q_3}\overset{(2)}{q_3})+(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_3}\overset{(1)}{q_3})=0. \tag{40}$$

Subtracting (40) from (36) we get

$$2\frac{\partial^2\overset{(2)}{q_1}}{\partial t\partial q_3}\overset{(1)}{q_3}-2\frac{\partial\overset{(2)}{q_1}}{\partial q_3}\overset{(2)}{q_3}-6\frac{\partial^3\overset{(3)}{q_1}}{\partial t\partial q_3^2}(\overset{(1)}{q_3})^2-6\frac{\partial^3\overset{(3)}{q_1}}{\partial t^2\partial q_3}\overset{(1)}{q_3}+9\frac{\partial^2\overset{(3)}{q_1}}{\partial t\partial q_3}\overset{(2)}{q_3}$$

$$+6\frac{\partial^2\overset{(3)}{q_1}}{\partial q_3^2}\overset{(1)}{q_3}\overset{(2)}{q_3}=0. \tag{41}$$

Using equation (41), Euler-Lagrange equation (36) becomes

$$-\frac{d\overset{(1)}{q_1}}{dt}+\frac{d^2\overset{(2)}{q_1}}{dt^2}+\frac{d^3\overset{(3)}{q_1}}{dt^3}=0, \tag{42}$$

which has a well known solution.

From equation (64) (in appendix A), the variation of the constraint equation

$$G_0=H_0(q_a,x_\mu,p_a=\frac{\partial L}{\partial q_a})$$

takes the form

$$dG_0=(\overset{(4)}{q_3}-\overset{(2)}{q_3}+q_3)(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_3}\overset{(1)}{q_3})+[(\frac{\partial^2\overset{(2)}{q_1}}{\partial t^2}+2\frac{\partial^2\overset{(2)}{q_1}}{\partial t\partial q_3}\overset{(1)}{q_3}$$

$$+\frac{\partial^2\overset{(2)}{q_1}}{\partial q_3^2}(\overset{(1)}{q_3})^2+\frac{\partial\overset{(2)}{q_1}}{\partial q_3}\overset{(2)}{q_3})-(\frac{\partial\overset{(1)}{q_1}}{\partial t}+\frac{\partial\overset{(1)}{q_1}}{\partial q_3}\overset{(1)}{q_3})+q_1]\overset{(1)}{q_3}$$

$$+[(\frac{\partial^3\overset{(3)}{q_2}}{\partial t^3}+3\frac{\partial^3\overset{(3)}{q_2}}{\partial t^2\partial q_3}\overset{(1)}{q_3}+3\frac{\partial^3\overset{(3)}{q_2}}{\partial t\partial q_3^2}(\overset{(1)}{q_3})^2+3\frac{\partial^2\overset{(3)}{q_2}}{\partial t\partial q_3}\overset{(2)}{q_3}$$

$$+\frac{\partial^3\overset{(3)}{q_2}}{\partial q_3^2}(\overset{(1)}{q_3})^3+3\frac{\partial^2\overset{(3)}{q_2}}{\partial q_3^2}\overset{(1)}{q_3}\overset{(2)}{q_3}+\frac{\partial\overset{(3)}{q_2}}{\partial q_3}\overset{(3)}{q_3})-(\frac{\partial^2\overset{(2)}{q_2}}{\partial t^2}+2\frac{\partial^2\overset{(2)}{q_2}}{\partial t\partial q_3}\overset{(1)}{q_3}$$

$$+\frac{\partial^2\overset{(2)}{q_2}}{\partial q_3^2}(\overset{(1)}{q_3})^2+\frac{\partial\overset{(2)}{q_2}}{\partial q_3}\overset{(2)}{q_3})+(\frac{\partial\overset{(1)}{q_2}}{\partial t}+\frac{\partial\overset{(1)}{q_2}}{\partial q_3}\overset{(1)}{q_3})](\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_3}\overset{(1)}{q_3})+[(\frac{\partial^3\overset{(3)}{q_1}}{\partial t^3}$$

$$+3\frac{\partial^3\overset{(3)}{q_1}}{\partial t^2\partial q_3}\overset{(1)}{q_3}+3\frac{\partial^3\overset{(3)}{q_1}}{\partial t\partial \overset{(2)}{q_3}}(\overset{(1)}{q_3})^2+3\frac{\partial^2\overset{(3)}{q_1}}{\partial t\partial q_3}\overset{(2)}{q_3}+\frac{\partial^3\overset{(3)}{q_1}}{\partial q_3^3}(\overset{(1)}{q_3})^3+3\frac{\partial^2\overset{(3)}{q_1}}{\partial q_3^2}\overset{(1)}{q_3}\overset{(2)}{q_3}$$

$$+\frac{\partial \overset{(3)}{q_1}}{\partial q_3}\overset{(3)}{q_3}) - (\frac{\partial^2 \overset{(2)}{q_1}}{\partial t^2} + 2\frac{\partial^2 \overset{(2)}{q_1}}{\partial t \partial q_3}\overset{(1)}{q_3} + \frac{\partial^2 \overset{(2)}{q_1}}{\partial q_3^2}(\overset{(1)}{q_3})^2 + \frac{\partial \overset{(2)}{q_1}}{\partial q_3}\overset{(2)}{q_3})$$

$$+(\frac{\partial \overset{(1)}{q_1}}{\partial t} + \frac{\partial \overset{(1)}{q_1}}{\partial q_3}\overset{(1)}{q_3})](\frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial q_3}\overset{(1)}{q_3}) = 0. \tag{43}$$

Using equation (39) and(40), equation (43) leads us to

$$(\frac{\partial^3 \overset{(3)}{q_2}}{\partial t^2} + 3\frac{\partial^3 \overset{(3)}{q_2}}{\partial t^2 \partial q_3}\overset{(1)}{q_3} + 3\frac{\partial^3 \overset{(3)}{q_2}}{\partial t \partial q_3^2}(\overset{(1)}{q_3})^2 + 3\frac{\partial^2 \overset{(3)}{q_2}}{\partial t \partial q_3}\overset{(2)}{q_3} + \frac{\partial^3 \overset{(3)}{q_2}}{\partial q_3^3}(\overset{(1)}{q_3})^3$$

$$+3\frac{\partial^2 \overset{(3)}{q_2}}{\partial q_3^2}\overset{(1)}{q_3}\overset{(2)}{q_3} + \frac{\partial \overset{(3)}{q_2}}{\partial q_3}\overset{(3)}{q_3}) - (\frac{\partial^2 \overset{(2)}{q_2}}{\partial^2 t} + 2\frac{\partial^2 \overset{(2)}{q_2}}{\partial t \partial q_3}\overset{(1)}{q_3} + \frac{\partial^2 \overset{(2)}{q_2}}{\partial q_3^2}(\overset{(1)}{q_3})^2$$

$$+\frac{\partial \overset{(2)}{q_2}}{\partial q_3}\overset{(2)}{q_3}) + (\frac{\partial \overset{(1)}{q_2}}{\partial t} + \frac{\partial \overset{(1)}{q_2}}{\partial q_3}\overset{(1)}{q_3}) + \overset{(4)}{q_3} - \overset{(2)}{q_3} + q_3 = 0. \tag{44}$$

Adding (35) and (44), we get

$$-6\frac{\partial^3 \overset{(3)}{q_2}}{\partial t^2 \partial q_3}\overset{(1)}{q_3} - 6\frac{\partial^3 \overset{(3)}{q_2}}{\partial t \partial q_3^2}(\overset{(1)}{q_3})^2 + 9\frac{\partial^2 \overset{(3)}{q_{(2)}}}{\partial t \partial q_3}\overset{(2)}{q_3} + 6\frac{\partial^2 \overset{(3)}{q_2}}{\partial q_3^2}\overset{(1)}{q_3}\overset{(2)}{q_3} + 2\frac{\partial^2 \overset{(2)}{q_2}}{\partial t \partial q_3}\overset{(1)}{q_3}$$

$$-2\frac{\partial \overset{(2)}{q_2}}{\partial q_3}\overset{(2)}{q_3} = 0. \tag{45}$$

Using equation (45), Euler-lagrange equation (35) becomes

$$\overset{(6)}{q_2} - \overset{(4)}{q_2} + \overset{(4)}{q_3} + \overset{(2)}{q_2} - \overset{(2)}{q_3} + q_3 = 0, \tag{46}$$

or

$$\overset{(4)}{F} - \overset{(2)}{F} + F = 0. \tag{47}$$

The two equations (42) and (47) give us the solution of the system.

A Hamilton-Jacobi Treatment of the Lagrangian in eq.(23)

The canonical momenta (2)–(3) are

$$p_{(2)1} = \frac{\partial L}{\partial \overset{(3)}{q_1}} = \overset{(3)}{q_2}, \tag{48}$$

$$p_{(2)2} = \frac{\partial L}{\partial \overset{(3)}{q_2}} = \overset{(3)}{q_1}, \tag{49}$$

$$p_{(2)3} = \frac{\partial L}{\partial \overset{(3)}{q_3}} = 0, \tag{50}$$

$$p_{(1)1} = \frac{\partial L}{\partial \overset{(2)}{q_1}} - \dot{p}_{(2)1} = \overset{(2)}{q_2} - \overset{(2)}{q_3} - \overset{(4)}{q_2}, \tag{51}$$

$$p_{(1)2} = \frac{\partial L}{\partial \overset{(2)}{q_2}} - \dot{p}_{(2)2} = \overset{(2)}{q_1} - \overset{(4)}{q_1}, \tag{52}$$

$$p_{(1)3} = \frac{\partial L}{\partial \overset{(2)}{q_3}} - \dot{p}_{(2)3} = -\overset{(2)}{q_1}, \tag{53}$$

$$p_{(0)1} = \frac{\partial L}{\partial \overset{(1)}{q_1}} - \dot{p}_{(1)1} = \overset{(1)}{q_2} - \overset{(1)}{q_3} - \overset{(3)}{q_2} + \overset{(3)}{q_3} + \overset{(5)}{q_2}, \tag{54}$$

$$p_{(0)2} = \frac{\partial L}{\partial \overset{(1)}{q_2}} - \dot{p}_{(1)2} = \overset{(1)}{q_1} - \overset{(3)}{q_1} + \overset{(5)}{q_1}, \tag{55}$$

$$p_{(0)3} = \frac{\partial L}{\partial \overset{(1)}{q_3}} - \dot{p}_{(1)3} = -\overset{(1)}{q_1} + \overset{(3)}{q_1}. \tag{56}$$

Equations (49) and (50) can be solved for $\overset{(3)}{q_1}$ and $\overset{(3)}{q_2}$ as

$$\overset{(3)}{q_1} = p_{(2)2} = f_{(3)1}, \tag{57}$$

$$\overset{(3)}{q_2} = p_{(2)1} = f_{(3)2}. \tag{58}$$

Since the momenta are not independent, $p_{(u)\mu}$ can be written as

$$p_{(0)3} = -\overset{(1)}{q_1} + p_{(2)2} = -H_{(0)3}, \tag{59}$$

$$p_{(1)3} = -\overset{(2)}{q_1} = -H_{(1)3}, \tag{60}$$

$$p_{(2)3} = 0 = -H_{(2)3}. \tag{61}$$

The Hamiltonian (6), takes the form

$$\begin{aligned} H_0 = \; & p_{(0)1}\overset{(1)}{q_1} + p_{(0)2}\overset{(1)}{q_2} + p_{(1)1}\overset{(2)}{q_1} + p_{(1)2}\overset{(2)}{q_2} + p_{(2)1}f_{(3)1} + p_{(2)2}f_{(3)2} + \overset{(1)}{q_3}p_{(0)3} \\ & + \overset{(2)}{q_3}p_{(2)3} - L(q_i, \overset{(1)}{q_i}, \overset{(2)}{q_i}, \overset{(3)}{q_3}, \overset{(3)}{q_1} = f_{(3)1}, \overset{(3)}{q_2} = f_{(3)2}), \end{aligned} \tag{62}$$

or

$$H_0 = p_{(0)1}\overset{(1)}{q_1} + p_{(0)2}\overset{(1)}{q_2} + p_{(1)1}\overset{(2)}{q_1} + p_{(1)2}\overset{(2)}{q_2} + p_{(2)1}p_{(2)2} + \overset{(1)}{q_3}p_{(2)2}$$

$$-\overset{(2)}{q_1}\overset{(2)}{q_2} - \overset{(1)}{q_1}\overset{(1)}{q_2} + q_1 q_3. \tag{63}$$

The set of Hamilton-Jacobi equations (9) read as

$$H'_{(0)3} = p_{(0)3} + H_{(0)3} = p_{(0)3} + \overset{(1)}{q_1} - p_{(2)2} = 0, \tag{64}$$

$$H'_{(1)3} = p_{(1)3} + H_{(1)3} = p_{(1)3} + \overset{(2)}{q_1} = 0, \tag{65}$$

$$H'_{(2)3} = p_{(2)3} + H_{(2)3} = p_{(2)3} = 0, \tag{66}$$

$$H'_0 = p_0 + H_0 = 0. \tag{67}$$

The equations of motion (10)–(11) can be written as

$$dq_1 = \frac{\partial H'_0}{\partial p_{(0)1}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(0)1}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(0)1}}d\overset{(1)}{q_3} + \frac{\partial H'_{(2)3}}{\partial p_{(0)1}}d\overset{(2)}{q_3} = \overset{(1)}{q_1}dt, \tag{68}$$

$$dq_2 = \frac{\partial H'_0}{\partial p_{(0)2}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(0)2}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(0)2}}d\overset{(1)}{q_3} + \frac{\partial H'_{(2)3}}{\partial p_{(0)2}}d\overset{(2)}{q_3} = \overset{(1)}{q_2}dt, \tag{69}$$

$$d\overset{(1)}{q_1} = \frac{\partial H'_0}{\partial p_{(1)1}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(1)1}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(1)1}}d\overset{(1)}{q_3} + \frac{\partial H'_{(2)3}}{\partial p_{(1)1}}d\overset{(2)}{q_3} = \overset{(2)}{q_1}dt, \tag{70}$$

$$d\overset{(1)}{q_2} = \frac{\partial H'_0}{\partial p_{(1)2}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(1)2}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(1)2}}d\overset{(1)}{q_3} + \frac{\partial H'_{(2)3}}{\partial p_{(1)2}}d\overset{(2)}{q_3} = \overset{(2)}{q_2}dt, \tag{71}$$

$$d\overset{(2)}{q_1} = \frac{\partial H'_0}{\partial p_{(2)1}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(2)1}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(2)1}}d\overset{(1)}{q_3} + \frac{\partial H'_{(2)3}}{\partial p_{(2)1}}d\overset{(2)}{q_3} = p_{(2)2}dt, \tag{72}$$

$$d\overset{(2)}{q_2} = \frac{\partial H'_0}{\partial p_{(2)2}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(2)2}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(2)2}}d\overset{(1)}{q_3} + \frac{\partial H'_{(2)3}}{\partial p_{(2)2}}d\overset{(2)}{q_3} = (p_{(2)1}$$

$$+\overset{(1)}{q_2})dt - dq_3, \tag{73}$$

$$dp_{(0)1} = -\frac{\partial H'_0}{\partial q_1}dt - \frac{\partial H'_{(0)3}}{\partial q_1}dq_3 - \frac{\partial H'_{(1)3}}{\partial q_1}d\overset{(1)}{q}_3 - \frac{\partial H'_{(2)3}}{\partial q_1}d\overset{(2)}{q}_3 = -q_3 dt, \tag{74}$$

$$dp_{(0)2} = -\frac{\partial H'_0}{\partial q_2}dt - \frac{\partial H'_{(0)3}}{\partial q_2}dq_3 - \frac{\partial H'_{(1)3}}{\partial q_2}d\overset{(1)}{q}_3 - \frac{\partial H'_{(2)3}}{\partial q_2}d\overset{(2)}{q}_3 = 0, \tag{75}$$

$$dp_{(0)3} = -\frac{\partial H'_0}{\partial q_3}dt - \frac{\partial H'_{(0)3}}{\partial q_3}dq_3 - \frac{\partial H'_{(1)3}}{\partial q_3}d\overset{(1)}{q}_3 - \frac{\partial H'_{(2)3}}{\partial q_3}d\overset{(2)}{q}_3 = -q_1 dt, \tag{76}$$

$$dp_{(1)1} = -\frac{\partial H'_0}{\partial \overset{(1)}{q}_1}dt - \frac{\partial H'_{(0)3}}{\partial \overset{(1)}{q}_1}dq_3 - \frac{\partial H'_{(1)3}}{\partial \overset{(1)}{q}_1}d\overset{(1)}{q}_3 - \frac{\partial H'_{(2)3}}{\partial \overset{(1)}{q}_1}d\overset{(2)}{q}_3$$

$$= -(p_{(0)1} - \overset{(1)}{q}_2)dt - dq_3, \tag{77}$$

$$dp_{(1)2} = -\frac{\partial H'_0}{\partial \overset{(1)}{q}_2}dt - \frac{\partial H'_{(0)3}}{\partial \overset{(1)}{q}_2}dq_3 - \frac{\partial H'_{(1)3}}{\partial \overset{(1)}{q}_2}d\overset{(1)}{q}_3 - \frac{\partial H'_{(2)3}}{\partial \overset{(1)}{q}_2}d\overset{(2)}{q}_3$$

$$= -(p_{(0)2} - \overset{(1)}{q}_1)dt, \tag{78}$$

$$dp_{(1)3} = -\frac{\partial H'_0}{\partial \overset{(1)}{q}_3}dt - \frac{\partial H'_{(0)3}}{\partial \overset{(1)}{q}_3}dq_3 - \frac{\partial H'_{(1)3}}{\partial \overset{(1)}{q}_3}d\overset{(1)}{q}_3 - \frac{\partial H'_{(2)3}}{\partial \overset{(1)}{q}_3}d\overset{(2)}{q}_3 = -p_{(2)2}dt, \tag{79}$$

$$dp_{(2)1} = -\frac{\partial H'_0}{\partial \overset{(2)}{q}_1}dt - \frac{\partial H'_{(0)3}}{\partial \overset{(2)}{q}_1}dq_3 - \frac{\partial H'_{(1)3}}{\partial \overset{(2)}{q}_1}d\overset{(1)}{q}_3 - \frac{\partial H'_{(2)3}}{\partial \overset{(2)}{q}_1}d\overset{(2)}{q}_3$$

$$= -(p_{(1)1} - \overset{(2)}{q}_2)dt - d\overset{(1)}{q}_3, \tag{80}$$

$$dp_{(2)2} = -\frac{\partial H'_0}{\partial \overset{(2)}{q}_2}dt - \frac{\partial H'_{(0)3}}{\partial \overset{(2)}{q}_2}dq_3 - \frac{\partial H'_{(1)3}}{\partial \overset{(2)}{q}_2}d\overset{(1)}{q}_3 - \frac{\partial H'_{(2)3}}{\partial \overset{(2)}{q}_2}d\overset{(2)}{q}_3$$

$$= -(p_{(1)2} - \overset{(2)}{q}_1)dt, \tag{81}$$

$$dp_{(2)3} = -\frac{\partial H'_0}{\partial \overset{(2)}{q}_3}dt - \frac{\partial H'_{(0)3}}{\partial \overset{(2)}{q}_3}dq_3 - \frac{\partial H'_{(1)3}}{\partial \overset{(2)}{q}_3}d\overset{(1)}{q}_3 - \frac{\partial H'_{(2)3}}{\partial \overset{(2)}{q}_3}d\overset{(2)}{q}_3 = 0. \tag{82}$$

The system of total differential equations (68)–(82)is integrable if the variation of $H'_{(s)3}$ is identically zero.

The variation of $H'_{(0)3}$ is

$$dH'_{(0)3} = dp_{(0)3} + d\overset{(1)}{q}_1 - dp_{(2)2} = (-q_1 + p_{(1)2})dt. \tag{83}$$

Since $dH'_{(0)3}$ is not identically zero we have a new constraint:

$$H''_{(0)3} = -q_1 + p_{(1)2}. \tag{84}$$

The variation of $H''_{(0)3}$ is

$$dH''_{(0)3} = -dq_1 + dp_{(1)2} = -p_{(0)2}dt. \tag{85}$$

Again, as $dH''_{(0)3}$ is not identically zero, we have

$$H'''_{(0)3} = -p_{(0)2}. \tag{86}$$

Using (75), the variation of $H'''_{(0)3}$ is identically zero,

$$dH'''_{(0)3} = -dp_{(0)2} = 0. \tag{87}$$

From (87) we have

$$p_{(0)2} = C. \tag{88}$$

Using equation (55) we get

$$\overset{(1)}{q_1} - \overset{(3)}{q_1} + \overset{(5)}{q_1} = C \tag{89}$$

or

$$\overset{(2)}{q_1} - \overset{(4)}{q_1} + \overset{(6)}{q_1} = 0. \tag{90}$$

The variation of $H'_{(1)3}$ in (65) is

$$dH'_{(1)3} = dp_{(1)3} + d\overset{(2)}{q_1} = 0. \tag{91}$$

Finally, the variation of $H'_{(2)3}$ is

$$dH'_{(2)3} = dp_{(2)3} = 0. \tag{92}$$

From (87), (91) and (92), we conclude that the system is integrable. The equivalent partial differential equation of (73) is

$$\frac{\partial \overset{(2)}{q_2}}{\partial q_3} = -1, \tag{93}$$

and its solution is

$$\overset{(2)}{q_2} = -q_3 + F(t) \tag{94}$$

or

$$F(t) = \overset{(2)}{q_2} + q_3. \tag{95}$$

The second and forth derivatives of (95) can be written as

$$\overset{(2)}{F} = \overset{(4)}{q_2} + \overset{(2)}{q_3}, \tag{96}$$

$$\overset{(4)}{F} = \overset{(6)}{q_2} + \overset{(4)}{q_3}. \tag{97}$$

Equations (54) and (74) together take the form,

$$\overset{(6)}{q_2} - \overset{(4)}{q_2} + \overset{(4)}{q_3} + \overset{(2)}{q_2} - \overset{(2)}{q_3} + q_3 = 0 \tag{98}$$

Using equations (95), (96) and (97), equation (98) becomes

$$\overset{(4)}{F} - \overset{(2)}{F} + F = 0. \tag{99}$$

Equations (90) and (99) are equivalent to (42) and (47), respectively.

4. Conclusion

The treatment of the first-order singular Lagrangian system as field system has been discussed in previous work [6]. The development of this treatment to a second-order singular Lagrangian was done in [7]. In this work, the same method is generalized to study the higher-order singular Lagrangian system, which contains the time derivative of the acceleration. The Euler-Lagrange equations for field system are used to obtain the equations of motion for the higher-order singular Lagrangian system. Simultaneous solution of the Euler-Lagrange equations with the constraints equations gives us the solution of the dynamical system. As in the first and the second order singular Lagrangian systems, constraints equations are obtained from the Hamilton-Jacobi approach for singular system. This means that both Hamiltonian and Lagrangian formulations of singular systems are mixed.

References

[1] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva University,new York, (1964), and Cand.*J.Math*, **2**, (1950), 129.

[2] Y. Güler, *Nouva Cimento B*, **107**, (1992), 1143.

[3] Y. Güler, *Nuova Cimento B*, **107**, (1992), 1143.

[4] B. M. pimental, and R. G. Teixrira, *Nouva Cimento B*, **113**, (1998), 805.

[5] B. M. Pimental,and R. G. Teixrira,*Nouva Cimento B*, **111**, (1996), 841.

[6] N. I. Farahat and Y. Güler, *Phys.Rev.A*, **51**, (1995), 68.

[7] N. I. Farahat and Kh. A. Ramadan, *Hadronic Jornal*, **25**, (2002), 291.

[8] Y. Saito, R. Sugano, T. Ohta, and T. Kimura, *J.Math.Phys.***30**, (1989), 1122.