Tkachenko Modes of the Square Vortex Lattice in a two-component Bose-Einstein Condensate

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Abstract

We study Tkachenko modes of the square vortex lattice of a two-component Bose-Einstein condensate (BEC) in the mean-field quantum Hall regime, considering the coupling of these modes with density excitations. We derive the hydrodynamic equations and obtain the dispersion relations of the excitation modes. We find that there are two types of excitations, gapped inertial modes and gapless Tkachenko modes. These modes have two branches which we call acoustic and optical modes in analogy with phonons. The former has quadratic while the latter has linear wave-number dependence in both inertial and Tkachenko modes. Acoustic Tkachenko mode is found to be anisotropic while the other three modes are isotropic. The anisotropy of the acoustic Tkachenko mode reflects the four-fold symmetry of the square lattice.

1. Introduction

Unlike ordinary fluids, superfluids have rotationless velocity field which leads formation of quantized vortices to carry the additional angular momentum. As the rotation frequency is increased, more vortices are formed and they arrange into an Abrikosov lattice. These triangular vortex lattices were proposed by Abrikosov [1] for type II superconductors under magnetic field, and they have been observed in type-II superconductors [2], superfluid helium [3], Bose-Einstein Condensed gases [4, 5] and recently in ultracold fermion superfluids [6].

The lowest energetic excitation of a vortex lattice is the Tkachenko mode [7] which is the oscillation of vortices around their equilibrium points. These oscillations are elliptical and major axis lies perpendicular to the propagation of the wave. They can be considered as transverse sound modes in a vortex lattice and they were introduced by V. K. Tkachenko for an infinite array of vortices in an incompressible fluid, superfluid helium. Tkachenko showed that triangular lattice is the energetically favorable one among other bravais lattices and then proved that this configuration is stable in the long wavelength region (long with respect to intervortex spacing). The observation of vortex lattices in He II was done by Yarmchuk *et al.* [3], in 1979 and after a year Tkachenko modes were observed [8]. These modes are important for the dynamics of the superfluid [9, 10, 11], vortex melting [12], and periodic pulse of neutron stars [13].

From the experimental realization of atomic Bose-Einstein Condensate (BEC), there has been a growing interest on rotating BECs both experimentally and theoretically. Studying vortex lattices in a BEC has a lot of advantages over other superfluids and superconductors. Analytically condensates can be investigated using only mean-field theory since the interactions are weak, and experimentally they can be easily imaged since the vortex core size is larger, moreover a wide range of rotation frequency is attainable. Large vortex lattices have been created in rotating harmonically trapped BECs [4, 5]. In a recent experiment at JILA, vortex lattice dynamics has been studied, and Tkachenko modes over a large range of rotation frequencies are observed [14]. In the experiment, a resonant laser beam was focused on the center of the condensate to excite the Tkachenko modes. As the rotation frequency is increased, a clear reduction in the Tkachenko

mode frequencies is observed. JILA group has also achieved rotation frequencies up to 99% of the trapping frequency for a one-component BEC [15].

Tkachenko modes of trapped BECs have also been studied theoretically by a number of groups [16–23]. Baym described four different physical regimes according to rotational frequency [17]. For slow rotation $\Omega < sk_0$ where s is the sound velocity and k_0 is the lowest wave-number in the trap, Thomas-Fermi approximation is applicable and compressibility of the cloud is negligible. As in helium II, Tkachenko mode is found to be linear in k. For $sk_0 \ll \Omega \ll ms^2$ where m is the atomic mass of the boson, compression becomes important and quadratic wave-number relation is found. When the rotation frequency exceeds ms^2 and approaches to the trap frequency ω of the condensate, centrifugal force is nearly equal to confining force and the cloud can be treated as a two-dimensional system. In this regime mean-field description still holds and the system resembles to the two-dimensional quantum Hall liquids where the particles are confined to the lowest Landau levels (LLL). A faster rotation results in the break down of the mean-field description and the large ratio of the number of vortices to particles causes the vortex lattice to melt [24].

BEC experiments are also flexible to create mixtures of superfluids such as spinor condensates. The theoretical study of two-component vortex lattices in the LLL regime has been done by Mueller and Ho [25], and they showed that five types of lattice geometries are possible as the interaction parameters are changed. Kasamatsu, Tsubota, and Ueda numerically calculated the phase diagram in a larger domain of rotation frequency [26]. For the experimental side, JILA group has been able to create a two-component BEC and study its behavior under rotation [27]. They found that equilibrium structure is an interlaced square lattice of two-components, whereas experimental parameters require the rectangular lattice to be more favorable according to [25]. This is probably because the condensate was not in the LLL regime. JILA group also tried to measure the frequencies of the Tkachenko modes but they are found to be heavily damped. With this motivation, we calculate the dispersion relation for the Tkachenko modes of the square lattice in the mean-field quantum Hall regime.

We consider an infinite two-component vortex lattice assuming that both components have the same number of atoms and the same interaction strength ($g_1 = g_2 = g$) within each component. Although our hydrodynamic approach can be applied to slower rotation, the calculations of the elastic constants of the lattice are done in the LLL regime. It is found by Mueller and Ho [25] that, square lattice is energetically favorable for $0.373 < \alpha < 0.926$ where $\alpha = g/g_{12}$ and g_{12} is the interaction strength between the components.

The spectrum of the excitations of a two-component vortex lattice is expected to be richer than a single-component vortex lattice. There is only one branch of Tkachenko modes in one-component, whereas the two-component lattice should have two branches, in analogy with phonon modes of a diatomic solid compared with a monoatomic solid. With the same analogy, we call these branches acoustic and optical branches. These branches have two modes, one is elliptically polarized Tkachenko mode and the other is gapped longitudinal inertial mode. Acoustic Tkachenko mode is excited when two vortices inside the unit cell of the lattice oscillate in phase. In other words, acoustic modes are oscillations of the "center of mass" of the unit cell, while the vortices positions with respect to the center of mass remain stationary. For an optical Tkachenko mode, vortices of different components oscillate in opposite phase, leaving the "center of mass" of each unit cell stationary. In our approach there is symmetry under the exchange of components since the interactions within the components are same. This makes the above definitions of optical and acoustic unambiguous. In the next section we derive the hydrodynamic equations for the two-component BEC and then we use the elastic constants for the square lattice calculated in Ref. [28] to find dispersion relations. We conclude by the comparison of our theory with the experiment done by JILA [27].

2. Hydrodynamic Theory

After Tkachenko showed that the vortex lattice in a rotating superfluid supports collective elastic waves, a new hydrodynamic approach should have to be developed since Bekarevich-Khalatikov hydrodynamics could not explain these modes. Sonin [9] and Williams and Fetter [10] developed a continuum hydrodynamic theory taking into account the shearing of vortex lattice and vortex line bending. Later, Baym and Chandler used an equivalent coarse graining procedure to derive these modes [11]. Baym also applied his method to vortex lattices of BECs considering the compressibility of the system [17].

We generalize the continuum hydrodynamic approach to a vortex lattice in a two-component BEC where we use $i = 1, 2$ as component index and set $\hbar = 1$. The degree of freedoms of the system are superfluid velocity fields $\vec{v}_i(\vec{r}, t)$, vortex displacement fields $\vec{\epsilon}_i(\vec{r}, t)$, and densities of each component $n_i(\vec{r}, t)$. We study in the co-rotating plane and the system is effectively two dimensional in the plane perpendicular to the rotation axis. For a rotating system, the relation $\vec{v} = \frac{1}{m}\vec{\nabla}\phi$ turns into

$$\vec{v}_i + 2\vec{\Omega} \times \vec{\epsilon}_i = \frac{1}{m}\vec{\nabla}\phi_i, \tag{1}$$

where ϕ is the phase of the order parameter. This equation links the long wavelength superfluid velocity to the deviations of vortices and reduce the number of variables from ten to eight. The relation can be seen directly by taking curl of this equation which shows that the long wavelength average of the velocity field is not irrotational, but linked to the compressions of the vortex lattice

$$\vec{\nabla} \times \vec{v}_i = -2\vec{\Omega}\vec{\nabla} \cdot \vec{\epsilon}_i, \qquad i = 1, 2. \tag{2}$$

By taking the time derivative of Eq. (1) we get the superfluid acceleration equation that holds for each component

$$m\left(\frac{\partial \vec{v}_i}{\partial t} + 2\vec{\Omega} \times \frac{\partial \vec{\epsilon}_i}{\partial t}\right) = -\vec{\nabla}\mu_i. \tag{3}$$

Here μ_i is the chemical potential of component i, and $\mu_1 = g(n_1 + \alpha n_2)$, $\mu_2 = g(n_2 + \alpha n_1)$.

Particle number conservation and momentum conservation gives,

$$\frac{\partial n_i}{\partial t} + \vec{\nabla} \cdot (n_i \vec{v}_i) = 0, \tag{4}$$

$$m\left(n_i \frac{\partial \vec{v}_i}{\partial t} + 2n_i \vec{\Omega} \times \vec{v}_i\right) + \vec{\nabla}P_i = -\vec{\sigma}_i, \tag{5}$$

respectively. Here P_i is the pressure which is related to the chemical potential as $\vec{\nabla}P_i = n_i\vec{\nabla}\mu_i$, and $\vec{\sigma}$ is the elastic stress vector. For a weakly interacting two-component condensate we can use

$$\begin{align} \vec{\nabla}P_1 &= gn\vec{\nabla}n_1 + \alpha gn\vec{\nabla}n_2, \\ \vec{\nabla}P_2 &= gn\vec{\nabla}n_2 + \alpha gn\vec{\nabla}n_1. \end{align} \tag{6}$$

Taking curl of Eq.(5) and substituting Eq.(2), we have

$$2m\vec{\Omega} \times \left(\frac{\partial \vec{\epsilon}_i}{\partial t} - \vec{v}_i\right) = \frac{\vec{\sigma}_i}{n}. \tag{7}$$

The curl and divergence of these equations lead to

$$\vec{\nabla} \cdot \left(\frac{\partial \vec{\epsilon}_i}{\partial t} - \vec{v}_i\right) = \left(\frac{\vec{\nabla} \times \vec{\sigma}_i}{2\Omega nm}\right)_z, \tag{8}$$

and

$$\left(\vec{\nabla} \times \frac{\partial \vec{\epsilon}}{\partial t}\right)_z + 2\Omega\vec{\nabla} \cdot \vec{\epsilon}_i = -\frac{\vec{\nabla} \cdot \vec{\sigma}_i}{2\Omega nm}. \tag{9}$$

Similarly, the divergence of the superfluid acceleration equation gives for each component

$$\begin{align} \left(-\frac{\partial^2}{\partial t^2} + \frac{gn}{m}\nabla^2\right)n_1 + \alpha\frac{gn}{m}\nabla^2 n_2 &= 2\Omega n\left(\vec{\nabla} \times \frac{\partial \vec{\epsilon}_1}{\partial t}\right)_z, \\ \left(-\frac{\partial^2}{\partial t^2} + \frac{gn}{m}\nabla^2\right)n_2 + \alpha\frac{gn}{m}\nabla^2 n_1 &= 2\Omega n\left(\vec{\nabla} \times \frac{\partial \vec{\epsilon}_2}{\partial t}\right)_z. \end{align} \tag{10}$$

Since we have a symmetry under the exchange of component 1 with component 2, we can define the symmetric and antisymmetric variables as

$$
\begin{aligned}
n_+ &= n_1 + n_2, & \vec{\epsilon}_+ &= \vec{\epsilon}_1 + \vec{\epsilon}_2, & \vec{\sigma}_+ &= \vec{\sigma}_1 + \vec{\sigma}_2 \\
n_- &= n_1 - n_2, & \vec{\epsilon}_- &= \vec{\epsilon}_1 - \vec{\epsilon}_2, & \vec{\sigma}_- &= \vec{\sigma}_1 - \vec{\sigma}_2.
\end{aligned}
\tag{11}
$$

Using these variables, we obtain two sets of three equations. By taking the time derivative of Eq. (8), we can find the relation of the dynamics of the vortex lattice and its coupling with the density modes

$$
\vec{\nabla} \cdot \frac{\partial^2 \vec{\epsilon}_\pm}{\partial t^2} + \frac{1}{n} \frac{\partial^2 n_\pm}{\partial t^2} = \frac{1}{2nm\Omega} \frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{\sigma}_\pm \right)_z.
\tag{12}
$$

The coupled equation for the inertial modes from Eq. (10) is

$$
-\frac{\partial^2 n_\pm}{\partial t^2} + (1 \pm \alpha) \frac{gn}{m} \nabla^2 n_\pm = 2n\Omega \left(\vec{\nabla} \times \frac{\partial \vec{\epsilon}_\pm}{\partial t} \right)_z.
\tag{13}
$$

The polarization of the Tkachenko modes can be obtained from Eq. (9)

$$
\left(\vec{\nabla} \times \frac{\partial \vec{\epsilon}_\pm}{\partial t} \right)_z + 2\Omega \vec{\nabla} \cdot \vec{\epsilon}_\pm = -\frac{1}{2mn\Omega} \vec{\nabla} \cdot \vec{\sigma}_\pm.
\tag{14}
$$

Eqs.(14-12), form a linear set of six equations. We can find the stress vectors by taking the functional derivative of the elastic energy with respect to vortex displacement fields,

$$
\vec{\sigma}_i = \frac{\delta E_{\text{elastic}}}{\delta \vec{\epsilon}_i}.
\tag{15}
$$

In the mean-field quantum Hall regime energy of the lattice can be calculated exactly [25]. It is also possible to find the energy of the slightly deformed lattices and the energy difference should be quadratic in the long-wavelength approximation [28].

We should specify our lattice geometry to find the stress vector. We write the elastic energy of the square lattice due to optical and acoustic deformations as

$$
E_{\text{elastic}} = E_{\text{elastic}}^{\text{ac}} + E_{\text{elastic}}^{\text{op}}
\tag{16}
$$

with

$$
\begin{aligned}
E_{\text{elastic}}^{\text{ac}} &= \frac{gn^2}{2} \int d^2r \left[C_u \left(\frac{\partial \epsilon_+^x}{\partial y} + \frac{\partial \epsilon_+^y}{\partial x} \right)^2 - 4C_v \frac{\partial \epsilon_+^x}{\partial x} \frac{\partial \epsilon_+^y}{\partial y} + \right] \\
E_{\text{elastic}}^{\text{op}} &= \frac{gn^2 m\Omega}{2\pi} \int d^2r C_{ab} \left(\vec{\epsilon}_- \right)^2.
\end{aligned}
\tag{17}
$$

The elastic constants can be found numerically, the variations of C_u, C_v and C_{ab} with respect to interaction parameter α are shown in Figure 1.

Since the stresses $\sigma_\pm$, depend only on the lattice displacements $\epsilon_\pm$, with the same sign, "+" equations with symmetric variables are decoupled from the "−" equations with antisymmetric variables. Thus the "+" set describes the acoustic Tkachenko modes and their coupling with the in-phase sound mode, while the "−" set describes the optical Tkachenko modes and their coupling to the out-of-phase sound mode.

Fourier transforming the polarization equation 14, we get

$$
\epsilon_+^y = \frac{N_1}{D_1} \epsilon_+^x,
\tag{18}
$$

with

$$
\begin{aligned}
N_1 &= \left[-i\omega k_y - 2\Omega k_x - 2gn \frac{C_u - C_v}{m\Omega} k_x k_y^2 \right], \\
D_1 &= \left[-i\omega k_x + 2\Omega k_y + 2gn \frac{C_u + C_v}{m\Omega} k_y k_x^2 \right].
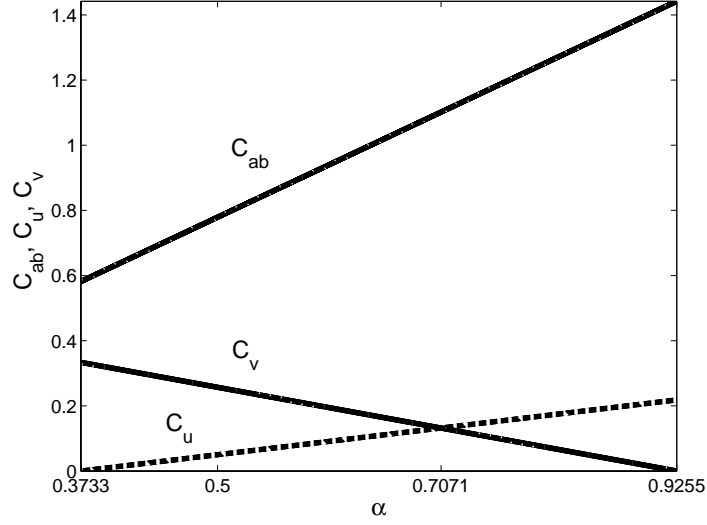\end{aligned}
\tag{19}
$$

Figure 1. Elastic constants (C_{ab}, C_u, C_v) of square lattice. As the components attract each other more, C_{ab} increases linearly. C_u and C_v are equal at $\alpha = 0.7071$ and they vanish at opposite limits [28].

Using these results in the Fourier transforms of Eqs. (12,13) and scaling k as $k' = \sqrt{\frac{gn}{m}}k$ in the long wavelength limit (up to second order in k'), the acoustic inertial mode frequency is given by,

$$\omega_I^{\mathrm{ac}} = 2\Omega + \frac{1 + \alpha + 2C_u}{4\Omega}k'^2,\tag{20}$$

and the acoustic Tkachenko mode frequency is

$$\omega_T^{\mathrm{ac}} = \sqrt{\frac{1+\alpha}{2}}\sqrt{\left[Cu\cos^2(2\theta) + C_v\sin^2(2\theta)\right]}\frac{k'^2}{\Omega}.\tag{21}$$

where θ is the angle from the $\hat{x}$ direction when the basis vectors of the vortex lattice are taken along $\hat{x}$ and $\hat{y}$.

The gapped optical inertial mode and the optical Tkachenko mode are calculated to the lowest order in k' and $\frac{gn}{\Omega}$ as

$$\omega_I^{\mathrm{op}} = 2\Omega\sqrt{1 + \frac{gn}{\pi\Omega}C_{ab}} + (1-\alpha)k',\tag{22}$$

and,

$$\omega_T^{\mathrm{op}} = \sqrt{\frac{1-\alpha}{\pi + \frac{gn}{\Omega}C_{ab}}}\sqrt{\frac{gn}{2\Omega}C_{ab}}\,k',\tag{23}$$

respectively. The dispersion relations are plotted in Figure 2 and Figure 3. Acoustic modes have quadratic wave-number dependence whereas optical modes have linear. Inertial modes are gapped by twice the rotation frequency, and anisotropy of the acoustic Tkachenko mode shows the symmetry of the square lattice. At the value of α where $C_u = C_v$ this mode is also isotropic like the other three modes.

3. Conclusion

We generalized the hydrodynamics of the vortex lattice of the one-component rotating BEC to the two-component case. Considering a square lattice geometry which is experimentally observed and using the elastic constants found for the square lattice in the rapidly rotating regime; we derived the dispersion relations for the excitations of the vortex lattice and density of a two-component BEC. We found that there are two branches for these modes, acoustic and optical. Acoustic modes are the center of mass oscillations
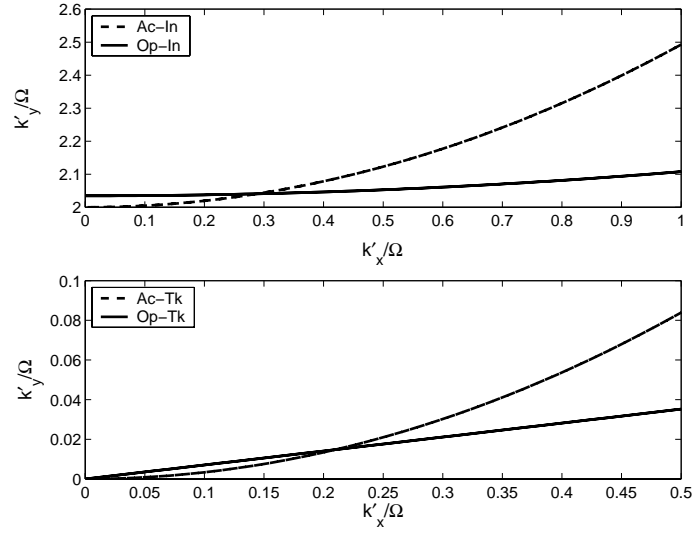
Figure 2. Dispersion relations for all 4 modes of the square lattice at $\alpha = 0.7071$. All of the modes are isotropic at this value of the interaction parameter α.
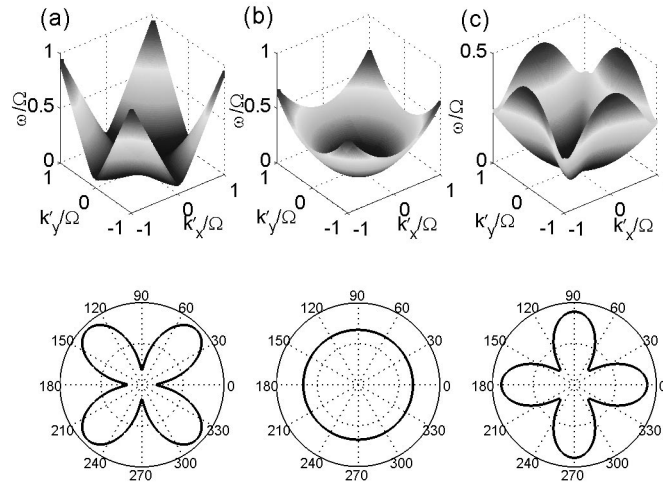

Figure 3. 3D dispersion relation for anisotropic acoustic Tkachenko mode.(a) $\alpha = 0.4$ (b) $\alpha = 0.7071$ Since at this point of α elastic constants C_u and C_v are equal eq. (21) becomes isotropic. The mode's isotropy is clear from the lower polar plot. (c) $\alpha = 0.9$ The below panel shows the polar plots of the above spectrums.

where the vortices of different components oscillate in phase. For optical modes, vortices oscillate out of phase and center of mass of the unit cell remains stationary. Optical Tkachenko mode and inertial modes are found to be isotropic. The anisotropy of acoustic Tkachenko modes reflect the four fold symmetry of the square lattice except at $\alpha = 0.7071$ where the mode is isotropic (See Figure 3).

The experiment of JILA demonstrates that square lattice is stable in the spinor condensate of Rb. However, they could not measure Tkachenko frequencies since they are found to be heavily damped. When comparing our results with the experiment, one should note that our calculations are valid in the LLL regime and we did not consider boundary effects. Despite the fact that there is a discrepancy between our considerations and the experiments, our results of anisotropic Tkachenko modes can explain the heavy damping for these oscillations. The laser beam used in the experiment excites the Tkachenko mode isotropically which result in different frequencies in different directions and leads to dephasing effect. Another fact is that we found two Tkachenko modes, optical and acoustic which should also exist in the experiment's conditions. Since these modes are coupled, they can be excited together. To decrease the damping due to these effects an elliptically polarized laser beam can be focused and then measurement of the Tkachenko modes can be possible.

Acknowledgments

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