Ramifications of Lineland

Daniel GRUMILLER[1], **Rene MEYER**[1,2]

[1]*Institute for Theoretical Physics, University of Leipzig Augustusplatz 10-11,*
D-04109 Leipzig-GERMANY
e-mail: grumiller@itp.uni-leipzig.de
[2]*Max Planck Institute for Mathematics in the Sciences Inselstrasse 22,*
D-04103 Leipzig-GERMANY
e-mail: Rene.Meyer@itp.uni-leipzig.de

Abstract

A non-technical overview on gravity in two dimensions is provided. Applications discussed in this work comprise 2D type 0A/0B string theory, Black Hole evaporation/thermodynamics, toy models for quantum gravity, for numerical General Relativity in the context of critical collapse and for solid state analogues of Black Holes. Mathematical relations to integrable models, non-linear gauge theories, Poisson-sigma models, KdV surfaces and non-commutative geometry are presented.

Key Words: Black Holes in String Theory, 2D Gravity, Integrable Models.

1. Introduction

The study of gravity in 2D — boring to some, fascinating to others [1] — has the undeniable disadvantage of eliminating a lot of structure that is present in higher dimensions; for instance, the Riemann tensor is determined already by the Ricci scalar, i.e., there is no Weyl curvature and no trace-free Ricci part. On the other hand, it has the undeniable advantage of eliminating a lot of structure that is present in higher dimensions; for instance, non-perturbative results may be obtained with relative ease due to technical simplifications, thus allowing one to understand some important conceptual issues arising in classical and quantum gravity which are universal and hence of relevance also for higher dimensions.

The scope of this non-technical overview is broad rather than focussed, since there exist already various excellent reviews and textbooks presenting the technical pre-requisites in detail,[1] and because the broadness envisaged here may lead to a cross-fertilization between otherwise only loosely connected communities. Some recent results are presented in more detail. It goes without saying that the topics selected concur with the authors' preferences; by no means it should be concluded that an issue or a reference omitted here is devoid of interest.

The common link between all applications mentioned here is 2D dilaton gravity,[2]

$$S_{\text{2DG}} = \frac{1}{2} \int \mathrm{d}^2 x \, \sqrt{-g} \, \left[X R + U(X) \, (\nabla X)^2 - 2V(X) \, \right] , \tag{1}$$

---

[1]For instance, the status of the field in the late 1980ies is summarized in [2].
[2]The 2D Einstein-Hilbert action will not be discussed except in section 6.1.

the action of which depends functionally on the metric $g_{\mu\nu}$ and on the scalar field X. Note that very often, in particular in the context of string theory, the field redefinition $X = e^{-2\phi}$ is employed; the field ϕ is the dilaton of string theory, hence the name "dilaton gravity". However, it is emphasized that the natural interpretation of X need not be the one of a dilaton field — it may also play the role of surface area, dual field strength, coordinate of a suitable target space or black hole (BH) entropy, depending on the application. The curvature scalar R and covariant derivative ∇ are associated with the Levi-Civita connection and Minkowskian signature is implied unless stated otherwise. The potentials U, V define the model; several examples will be provided below. A summary is contained in table 1.

This proceedings contribution is organized as follows: section 2 is devoted to a reformulation of (1) as a non-linear gauge theory, which considerably simplifies the construction of all classical solutions; section 3 discusses applications in 2D string theory; section 4 summarizes applications in BH physics; section 5 demonstrates how to reconstruct geometry from matter in a quantum approach; section 6 contains not only mathematical issues but also some open problems.

2. Gravity as non-linear gauge theory

It has been known for a long time how to obtain all classical solutions of (1) not only locally, but globally. Two ingredients turned out to be extremely useful: a reformulation of (1) as a first order action and the imposition of a convenient (axial or Eddington-Finkelstein type) gauge, rather than using conformal gauge.[3] Subsequently we will briefly recall these methods. For a more comprehensive review cf. [4].

Table 1. Selected list of models

Model (cf. (1) or (3))	$U(X)$	$V(X)$	$w(X)$ (cf. (4))
1. Schwarzschild [5]	$-\frac{1}{2X}$	$-\lambda^2$	$-2\lambda^2\sqrt{X}$
2. Jackiw-Teitelboim [6, 7]	0	$-\Lambda X$	$-\frac{1}{2}\Lambda X^2$
3. Witten BH/CGHS [8, 9]	$-\frac{1}{X}$	$-2b^2X$	$-2b^2X$
4. CT Witten BH [8, 9]	0	$-2b^2$	$-2b^2X$
5. SRG ($D > 3$)	$-\frac{D-3}{(D-2)X}$	$-\lambda^2 X^{(D-4)/(D-2)}$	$-\lambda^2\frac{D-2}{D-3}X^{(D-3)/(D-2)}$
6. $(A)dS_2$ ground state [10]	$-\frac{a}{X}$	$-\frac{B}{2}X$	$a \neq 2: -\frac{B}{2(2-a)}X^{2-a}$
7. Rindler ground state [11]	$-\frac{a}{X}$	$-\frac{B}{2}X^a$	$-\frac{B}{2}X$
8. BH attractor [12]	0	$-\frac{B}{2}X^{-1}$	$-\frac{B}{2}\ln X$
9. All above: ab-family [13]	$-\frac{a}{X}$	$-\frac{B}{2}X^{a+b}$	$b \neq -1: -\frac{B}{2(b+1)}X^{b+1}$
10. Liouville gravity [14]	a	$be^{\alpha X}$	$a \neq -\alpha: \frac{b}{a+\alpha}e^{(a+\alpha)X}$
11. Scattering trivial [15]	generic	0	const.
12. Reissner-Nordström [16]	$-\frac{1}{2X}$	$-\lambda^2 + \frac{Q^2}{X}$	$-2\lambda^2\sqrt{X} - 2Q^2/\sqrt{X}$
13. Schwarzschild-$(A)dS$ [17]	$-\frac{1}{2X}$	$-\lambda^2 - \ell X$	$-2\lambda^2\sqrt{X} - \frac{2}{3}\ell X^{3/2}$
14. Katanaev-Volovich [18]	α	$\beta X^2 - \Lambda$	$\int^X e^{\alpha y}(\beta y^2 - \Lambda)\,\mathrm{d}y$
15. Achucarro-Ortiz [19]	0	$\frac{Q^2}{X} - \frac{J}{4X^3} - \Lambda X$	$Q^2\ln X + \frac{J}{8X^2} - \frac{1}{2}\Lambda X^2$
16. KK reduced CS [20, 21]	0	$\frac{1}{2}X(c - X^2)$	$-\frac{1}{8}(c - X^2)^2$
17. Symmetric kink [22]	generic	$-X\Pi_{i=1}^n(X^2 - X_i^2)$	cf. [22]
18. 2D type 0A/0B [23, 24]	$-\frac{1}{X}$	$-2b^2X + \frac{b^2q^2}{8\pi}$	$-2b^2X + \frac{b^2q^2}{8\pi}\ln X$
19. exact string BH[25, 26]	(31)	(31)	(33)

---

[3]In string theory almost exclusively conformal gauge is used. A notable exception is [3].

2.1. First order formulation

The Jackiw-Teitelboim model (cf. the second model in table (1)) allows a gauge theoretic formulation based upon $(A)dS_2$,

$$[P_a, P_b] = \Lambda \varepsilon_{ab} J, \qquad [P_a, J] = \varepsilon_a{}^b P_b, \tag{2}$$

with Lorentz generator J, translation generators P_a and $\Lambda \neq 0$. A corresponding first order action, $S = \int X_A F^A$, has been introduced in [27]. The field strength $F = \mathrm{d}A + [A, A]/2$ contains the $SO(1,2)$ connection $A = e^a P_a + \omega J$, and the Lagrange multipliers X_A transform under the coadjoint representation. This example is exceptional insofar as it allows a formulation in terms of a *linear* (Yang-Mills type) gauge theory. Similarly, the fourth model in table 1 allows a gauge theoretic formulation [28] based upon the centrally extended Poincarè algebra [29]. The generalization to non-linear gauge theories [30] allowed a comprehensive treatment of all models (1) with $U = 0$, which has been further generalized to $U \neq 0$ in [31]. The corresponding first order gravity action

$$S_{\text{FOG}} = -\int \left[X_a T^a + XR + \epsilon \left(X^+ X^- U(X) + V(X) \right) \right] \tag{3}$$

is equivalent to (1) (with the same potentials U, V) upon elimination of the auxiliary fields X_a and the torsion-dependent part of the spin-connection. Here is our notation: $e^a = e^a_\mu dx^\mu$ is the dyad 1-form. Latin indices refer to an anholonomic frame, Greek indices to a holonomic one. The 1-form ω represents the spin-connection $\omega^a{}_b = \varepsilon^a{}_b \omega = \varepsilon^a{}_b \omega_\mu \, \mathrm{d}x^\mu$ with the totally antisymmetric Levi-Civita symbol ε_{ab} ($\varepsilon_{01} = +1$). With the flat metric η_{ab} in light-cone coordinates ($\eta_{+-} = 1 = \eta_{-+}$, $\eta_{++} = 0 = \eta_{--}$) it reads $\varepsilon^\pm{}_\pm = \pm 1$. The torsion 2-form present in the first term of (3) is given by $T^\pm = (\mathrm{d} \pm \omega) \wedge e^\pm$. The curvature 2-form $R^a{}_b$ can be represented by the 2-form R defined by $R^a{}_b = \varepsilon^a{}_b R$ with $R = \mathrm{d}\omega$. It appears in the second term in (3). Since no confusion between 0-forms and 2-forms should arise the Ricci scalar is also denoted by R. The volume 2-form is denoted by $\epsilon = e^+ \wedge e^-$. Signs and factors of the Hodge-$*$ operation are defined by $*\epsilon = 1$. It should be noted that (3) is a specific Poisson-sigma model [31] with a 3D target space, with target space coordinates $X, X^\pm$, see section 6.3 below. A second order action similar to (1) has been introduced in [32].

2.2. Generic classical solutions

It is useful to introduce the following combinations of the potentials U and V:

$$I(X) := \exp \int^X U(y) \, \mathrm{d}y, \quad w(X) := \int^X I(y) V(y) \, \mathrm{d}y \tag{4}$$

The integration constants may be absorbed, respectively, by rescalings and shifts of the "mass", see equation (10) below. Under dilaton dependent conformal transformations $X^a \to X^a/\Omega$, $e^a \to e^a \Omega$, $\omega \to \omega + X_a e^a \, \mathrm{d} \ln \Omega / \mathrm{d}X$ the action (3) is mapped to a new one of the same type with transformed potentials $\tilde{U}, \tilde{V}$. Hence, it is not invariant. It turns out that only the combination $w(X)$ as defined in (4) remains invariant, so conformally invariant quantities may depend on w only. Note that I is positive apart from eventual boundaries (typically, I may vanish in the asymptotic region and/or at singularities). One may transform to a conformal frame with $\tilde{I} = 1$, solve all equations of motion and then perform the inverse transformation. Thus, it is sufficient to solve the classical equations of motion for $\tilde{U} = 0$,

$$\mathrm{d}X + \tilde{X}^- \tilde{e}^+ - \tilde{X}^+ \tilde{e}^- = 0, \tag{5}$$

$$(\mathrm{d} \pm \tilde{\omega}) \tilde{X}^\pm \mp \tilde{e}^\pm \tilde{V}(X) = 0, \tag{6}$$

$$(\mathrm{d} \pm \tilde{\omega}) \wedge \tilde{e}^\pm = 0, \tag{7}$$

which is what we are going to do now. Note that the equation containing $\mathrm{d}\tilde{\omega}$ is redundant, whence it is not displayed.

Let us start with an assumption: $\tilde{X}^+ \neq 0$ for a given patch. To get some physical intuition as to what this condition could mean: the quantities X^a, which are the Lagrange multipliers for torsion, can be expressed as directional derivatives of the dilaton field by virtue of (5) (e.g. in the second order formulation a term of the form $X^a X_a$ corresponds to $(\nabla X)^2$). For those who are familiar with the Newman-Penrose formalism: for spherically reduced gravity the quantities X^a correspond to the expansion spin coefficients ρ and ρ' (both are real). If $\tilde{X}^+$ vanishes a (Killing) horizon is encountered and one can repeat the calculation below with indices $+$ and $-$ swapped everywhere. If both vanish in an open region by virtue of (5) a constant dilaton vacuum emerges, which will be addressed separately below. If both vanish on isolated points the Killing horizon bifurcates there and a more elaborate discussion is needed [33]. The patch implied by $\tilde{X}^+ \neq 0$ is a "basic Eddington-Finkelstein patch", i.e., a patch with a conformal diagram which, roughly speaking, extends over half of the bifurcate Killing horizon and exhibits a coordinate singularity on the other half. In such a patch one may redefine $\tilde{e}^+ = \tilde{X}^+ Z$ with a new 1-form Z. Then (5) implies $\tilde{e}^- = dX/\tilde{X}^+ + \tilde{X}^- Z$ and the volume form reads $\tilde{\epsilon} = \tilde{e}^+ \wedge \tilde{e}^- = Z \wedge dX$. The $+$ component of (6) yields for the connection $\tilde{\omega} = -d\tilde{X}^+/\tilde{X}^+ + Z\tilde{V}(X)$. One of the torsion conditions (7) then leads to $dZ = 0$, i.e., Z is closed. Locally (in fact, in the whole patch) it is also exact: $Z = du$. It is emphasized that, besides the integration of (9) below, this is the only integration needed! After these elementary steps one obtains already the conformally transformed line element in Eddington-Finkelstein (EF) gauge

$$d\tilde{s}^2 = 2\tilde{e}^+ \tilde{e}^- = 2\,du\,dX + 2\tilde{X}^+ \tilde{X}^-\,du^2\,, \tag{8}$$

which nicely demonstrates the power of the first order formalism. In the final step the combination $\tilde{X}^+ \tilde{X}^-$ has to be expressed as a function of X. This is possible by noting that the linear combination $\tilde{X}^+ \times [(6)$ with $-$ index$] + \tilde{X}^- \times [(6)$ with $+$ index$]$ together with (5) establishes a conservation equation,

$$d(\tilde{X}^+ \tilde{X}^-) + \tilde{V}(X)\,dX = d(\tilde{X}^+ \tilde{X}^- + w(X)) = 0\,. \tag{9}$$

Thus, there is always a conserved quantity $(dM = 0)$, which in the original conformal frame reads

$$M = -X^+ X^- I(X) - w(X)\,, \tag{10}$$

where the definitions (4) have been inserted. It should be noted that the two free integration constants inherent to the definitions (4) may be absorbed by rescalings and shifts of M, respectively. The classical solutions are labelled by M, which may be interpreted as mass (see section 4.2). Finally, one has to transform back to the original conformal frame (with conformal factor $\Omega = I(X)$). The line element (8) by virtue of (10) may be written as

$$ds^2 = 2I(X)\,du\,dX - 2I(X)(w(X) + M)\,du^2\,. \tag{11}$$

Evidently there is always a Killing vector $K \cdot \partial = \partial/\partial u$ with associated Killing norm $K^2 = -2I(w + M)$. Since $I \neq 0$ Killing horizons are encountered at $X = X_h$ where X_h is a solution of

$$w(X_h) + M = 0\,. \tag{12}$$

It is recalled that (11) is valid in a basic EF patch, e.g., an outgoing one. By redoing the derivation above, but starting from the assumption $X^- \neq 0$ one may obtain an ingoing EF patch, and by gluing together these patches appropriately one may construct the Carter-Penrose diagram, cf. [34, 33, 4].

As pointed out in the introduction the full geometric information resides in the Ricci scalar. The one related to the generic solution (11) reads

$$R = \frac{2}{I(X)} \frac{d}{dX}\Big(U(X)(M + w(X)) + I(X)V(X)\Big)\,. \tag{13}$$

There are two important special cases: for $U = 0$ the Ricci scalar simplifies to $R = 2V'(X)$, while for $w(X) \propto 1/I(X)$ it scales proportional to the mass, $R = 2MU'(X)/I(X)$. The latter case comprises so-called Minkowskian ground state models (for examples cf. the first, third, fifth and last line in table 1). Note that for many models in table 1 the potential $U(X)$ has a singularity at $X = 0$ and consequently a curvature singularity arises.

2.3. Constant dilaton vacua

For sake of completeness it should be mentioned that in addition to the family of generic solutions (11), labelled by the mass M, isolated solutions may exist, so-called constant dilaton vacua (cf. e.g. [22]), which have to obey[4] $X = X_{\mathrm{CDV}} = $ const. with $V(X_{CDV}) = 0$. The corresponding geometry has constant curvature, i.e., only Minkowski, Rindler or $(A)dS_2$ are possible space-times for constant dilaton vacua.[5] The Ricci scalar is determined by

$$R_{\mathrm{CDV}} = 2V'(X_{\mathrm{CDV}}) = \text{const.} \tag{14}$$

Examples are provided by the last eighth entries in table 1. For instance, 2D type 0A strings with an equal number q of electric and magnetic D0 branes (cf. the penultimate entry in table 1) allow for an AdS_2 vacuum with $X_{\mathrm{CDV}} = q^2/(16\pi)$ and $R_{\mathrm{CDV}} = -4b^2$ [37].

2.4. Topological generalizations

In 2D there are neither gravitons nor photons, i.e. no propagating physical modes exist [38]. This feature makes the inclusion of Yang-Mills fields in 2D dilaton gravity or an extension to supergravity straightforward. Indeed, both generalizations can be treated again in the first order formulation as a Poisson-sigma model, cf. e.g. [39]. In addition to M (see (10)) more locally conserved quantities (Casimir functions) may emerge and the integrability concept is extended.

As a simple example we include an abelian Maxwell field, i.e., instead of (3) we take

$$S_{\mathrm{MDG}} = -\int \left[X_a T^a + XR + BF + \epsilon \left(X^+ X^- U(X,B) + V(X,B) \right) \right], \tag{15}$$

where B is an additional scalar field and $F = \mathrm{d}A$ is the field strength 2-form. Variation with respect to A immediately establishes a constant of motion, $B = Q$, where Q is some real constant, the $U(1)$ charge. Variation with respect to B may produce a relation that allows to express B as a function of the dilaton and the dual field strength $*F$. For example, suppose that $V(X,B) = V(X) + \frac{1}{2}B^2$. Then, variation with respect to B gives $B = -*F$. Inserting this back into the action yields a standard Maxwell term. The solution of the remaining equations of motion reduces to the case without Maxwell field. One just has to replace B by its on-shell value Q in the potentials U, V.

Concerning supergravity we just mention a couple of references for further orientation [40, 41, 36].

2.5. Non-topological generalizations

To get a non-topological theory one can add scalar or fermionic matter. The action for a real, self-interacting and non-minimally coupled scalar field $\mathcal{T}$,

$$S_{\mathcal{T}} = \frac{1}{2} \int \left[F(X) \, \mathrm{d}\mathcal{T} \wedge *\mathrm{d}\mathcal{T} + \epsilon f(X,\mathcal{T}) \right], \tag{16}$$

---

[4]Incidentally, for the generic case (11) the value of the dilaton on an extremal Killing horizon is also subject to these two constraints.

[5]In quintessence cosmology in 4D such solutions serve as late time dS_4 attractor [35]. In 2D dilaton supergravity solutions preserving both supersymmetries are necessarily constant dilaton vacua [36].

in our convention requires $F < 0$ for the kinetic term to have the correct sign; e.g. $F = -\kappa$ or $F = -\kappa X$.

While scalar matter couples to the metric and the dilaton, fermions[6] couple directly to the Zweibein $(A \overleftrightarrow{\mathrm{d}} B = A \, \mathrm{d}B - (\mathrm{d}A)B)$,

$$S_\chi = \int \left[\frac{i}{2} F(X) \, (*e^a) \wedge (\overline{\chi} \gamma_a \overleftrightarrow{\mathrm{d}} \chi) + \epsilon H(X) g(\overline{\chi}\chi) \right], \tag{17}$$

but not — and this is a peculiar feature of 2D — to the spin connection. The self-interaction is at most quartic (a constant term may be absorbed in $V(X)$),

$$g(\overline{\chi}\chi) = m\overline{\chi}\chi + \lambda(\overline{\chi}\chi)^2. \tag{18}$$

The quartic term (henceforth: Thirring term [43]) can also be recast into a classically equivalent form by introducing an auxiliary vector potential,

$$\lambda \int \epsilon(\overline{\chi}\chi)^2 = \frac{\lambda}{2} \int [A \wedge *A + 2A \wedge (*e_a)\overline{\chi}\gamma^a\chi], \tag{19}$$

which lacks a kinetic term and thus does not propagate by itself.

We speak of minimal coupling if the coupling functions $F(X), f(X, \mathcal{T}), H(X)$ do not depend on the dilaton X, and of nonminimal coupling otherwise.

As an illustration we present the spherically reduced Einstein-massless-Klein-Gordon model (EMKG). It emerges from dimensional reduction of 4D Einstein-Hilbert (EH) gravity (cf. the first model in table 1) with a minimally coupled scalar field, with the choices $f(X, \tau) = 0$ and

$$w(X) = -2\lambda^2\sqrt{X}, \quad F(X) = -\kappa X, \quad I(X) = \frac{1}{\sqrt{X}}, \tag{20}$$

where λ is an irrelevant scale parameter and κ encodes the (also irrelevant) Newton coupling. Minimally coupled Dirac fermions in four dimensions yield upon dimensional reduction two 2-spinors coupled to each other through intertwinor terms, which is not covered by (17) (see [44] for details on spherical reduction of fields of arbitrary spin and the spherical reduced standard model).

With matter the equation of motion (6) and the conservation law (9) obtain contributions $W^\pm = \delta(S_{\mathcal{T}} + S_\chi)/\delta e^\mp$ and $X^-W^+ + X^+W^-$, respectively, destroying integrability because Z is not closed anymore: $\mathrm{d}Z = W^+ \wedge Z/X^+$. In special cases exact solutions can be obtained:

1. For (anti-)chiral fermions and (anti-)selfdual scalars with $W^+ = 0$ ($W^- = 0$) the geometric solution (8) is still valid [4] and the second equation of motion (6) implies $W^- = W_u^- \, \mathrm{d}u$. Such solutions have been studied e.g. in [45, 46]. They arise also in the Aichelburg-Sexl limit [47] of boosted BHs [48].

2. A one parameter family of static solutions of the EMKG has been discovered in [49]. Studies of static solutions in generic dilaton gravity may be found in [50, 51]. A static solution for the line-element with time-dependent scalar field (linear in time) has been discussed for the first time in [52]. It has been studied recently in more detail in [53].

3. A (continuously) self-similar solution of the EMKG has been discoverd in [54].

4. Specific models allow for exact solutions even in the presence of more general matter sources; for instance, the conformally transformed CGHS model (fourth in table 1), Rindler ground state models (seventh in table 1) and scattering trivial models (eleventh in table 1).

---

[6] We use the same definition for the Dirac matrices as in [42].

3. Strings in 2D

Strings propagating in a 2D target space are comparatively simple to describe because the only propagating degree of freedom is the tachyon (and if the latter is switched off the theory becomes topological). Hence several powerful methods exist to describe the theory efficiently, e.g. as matrix models. In particular, strings in non-trivial backgrounds may be studied in great detail. Here are some references for further orientation: For the matrix model description of 2D type 0A/0B string theory cf. [55, 23] (for an extensive review on Liouville theory and its relation to matrix models and strings in 2D cf. [14]; some earlier reviews are refs. [56]; the matrix model for the 2D Euclidean string BH has been constructed in [57]; a study of Liouville theory from the 2D dilaton gravity point of view may be found in [58]). The low energy effective action for 2D type 0A/0B string theory in the presence of RR fluxes has been studied from various aspects e.g. in [59, 23, 37, 24].

3.1. Target space formulation of 2D type 0A/0B string theory

For sake of definiteness focus will be on 2D type 0A with an equal number q of electric and magnetic D0 branes, but other cases may be studied as well. For vanishing tachyon the corresponding target space action is given by (setting $\kappa^2 = 1$)

$$S_{0A} = \frac{1}{2} \int d^2x \sqrt{-g} \left[e^{-2\phi} \left(R - 4 \left(\nabla\phi \right)^2 + 4b^2 \right) - \frac{b^2 q^2}{4\pi} \right] , \qquad (21)$$

Obviously, this is a special case of the generic model (1), with U, V given by the penultimate model in table 1, to which all subsequent considerations — in particular thermodynamical issues — apply. Note that the dilaton fields X and ϕ are related by $X = \exp(-2\phi)$. The constant $b^2 = 2/\alpha'$ defines the physical scale. In the absence of D0 branes, $q = 0$, the model simplifies to the Witten BH, cf. the third line in table 1.

The action defining the tachyon sector up to second order in $\mathcal{T}$ is given by (cf. (16))

$$S_{\mathcal{T}} = \frac{1}{2} \int d^2x \sqrt{-g} \left[F(X) g^{\mu\nu} (\partial_\mu \mathcal{T})(\partial_\nu \mathcal{T}) + f(X, \mathcal{T}) \right] , \qquad (22)$$

with

$$F(X) = X , \quad f(\mathcal{T}, X) = b^2 \mathcal{T}^2 \left(X - \frac{q^2}{2\pi} \right) . \qquad (23)$$

The total action is $S_{0A} + S_{\mathcal{T}}$.

3.2. Exact string Black Hole

The exact string black hole (ESBH) was discovered by Dijkgraaf, Verlinde and Verlinde more than a decade ago [25]. The construction of a target space action for it which does not display non-localities or higher order derivatives had been an open problem which could be solved only recently [26]. There are several advantages of having such an action available: the main point of the ESBH is its non-perturbative aspect, i.e., it is believed to be valid to all orders in the string-coupling α'. Thus, a corresponding action captures non-perturbative features of string theory and allows, among other things, a thorough discussion of ADM mass, Hawking temperature and Bekenstein–Hawking entropy of the ESBH which otherwise requires some ad-hoc assumption. Therefore, we will devote some space to its description. At the perturbative level actions approximating the ESBH are known: to lowest order in α' one has (21) with $q = 0$. Pushing perturbative considerations further Tseytlin was able to show that up to 3 loops the ESBH is consistent with sigma model conformal invariance [60]. In the strong coupling regime the ESBH asymptotes to the Jackiw–Teitelboim model [6]. The exact conformal field theory methods used in [25], based upon the $SL(2, \mathbb{R})/U(1)$

gauged Wess–Zumino–Witten model, imply the dependence of the ESBH solutions on the level k. A different (somewhat more direct) derivation leading to the same results for dilaton and metric was presented in [61]. For a comprehensive history and more references [62] may be consulted.

In the notation of [63] for Euclidean signature the line element of the ESBH is given by

$$ds^2 = f^2(x)\,d\tau^2 + dx^2\,, \tag{24}$$

with

$$f(x) = \frac{\tanh(bx)}{\sqrt{1 - p\tanh^2(bx)}}\,. \tag{25}$$

Physical scales are adjusted by the parameter $b \in \mathbb{R}^+$ which has dimension of inverse length. The corresponding expression for the dilaton,

$$\phi = \phi_0 - \ln\cosh(bx) - \frac{1}{4}\ln\left(1 - p\tanh^2(bx)\right), \tag{26}$$

contains an integration constant ϕ_0. Additionally, there are the following relations between constants, string-coupling α', level k and dimension D of string target space:

$$\alpha'b^2 = \frac{1}{k-2}\,, \qquad p := \frac{2}{k} = \frac{2\alpha'b^2}{1 + 2\alpha'b^2}\,, \qquad D - 26 + 6\alpha'b^2 = 0\,. \tag{27}$$

For $D = 2$ one obtains $p = \frac{8}{9}$, but like in the original work [25] we will treat general values of $p \in (0;1)$ and consider the limits $p \to 0$ and $p \to 1$ separately: for $p = 0$ one recovers the Witten BH geometry; for $p = 1$ the Jackiw–Teitelboim model is obtained. Both limits exhibit singular features: for all $p \in (0;1)$ the solution is regular globally, asymptotically flat and exactly one Killing horizon exists. However, for $p = 0$ a curvature singularity (screened by a horizon) appears and for $p = 1$ space-time fails to be asymptotically flat. In the present work exclusively the Minkowskian version of (24)

$$ds^2 = f^2(x)\,d\tau^2 - dx^2\,, \tag{28}$$

will be needed. The maximally extended space-time of this geometry has been studied in [64]. Winding/momentum mode duality implies the existence of a dual solution, the Exact String Naked Singularity (ESNS), which can be acquired most easily by replacing $bx \to bx + i\pi/2$, entailing in all formulas above the substitutions $\sinh \to i\cosh$, $\cosh \to i\sinh$.

After it had been realized that the nogo result of [65] may be circumvented without introducing superfluous physical degrees of freedom by adding an abelian BF-term, a straightforward reverse-engineering procedure allowed to construct uniquely a target space action of the form (1), supplemented by aforementioned BF-term,

$$S_{\mathrm{ESBH}} = -\int \left[X_a T^a + X_{\mathrm{ESBH}}R + \epsilon\left(X^+ X^- U_{\mathrm{ESBH}} + V_{\mathrm{ESBH}}\right)\right] - \int BF\,, \tag{29}$$

where B is a scalar field and $F = dA$ an abelian field strength 2-form. *Per constructionem* S_{ESBH} reproduces as classical solutions precisely (25)–(28) not only locally but globally. A similar action has been constructed for the ESNS. The relation $(X - \gamma)^2 = \operatorname{arcsinh}^2\gamma$ in conjunction with the definition $\gamma := \exp\left(-2\phi\right)/B$ may be used to express the auxiliary dilaton field X entering the action (1) in terms of the "true" dilaton field ϕ and the auxiliary field B. The two branches of the square root function correspond to the ESBH (main branch) and the ESNS (second branch), respectively:

$$X_{\mathrm{ESBH}} = \gamma + \operatorname{arcsinh}\gamma\,, \qquad X_{\mathrm{ESNS}} = \gamma - \operatorname{arcsinh}\gamma\,. \tag{30}$$
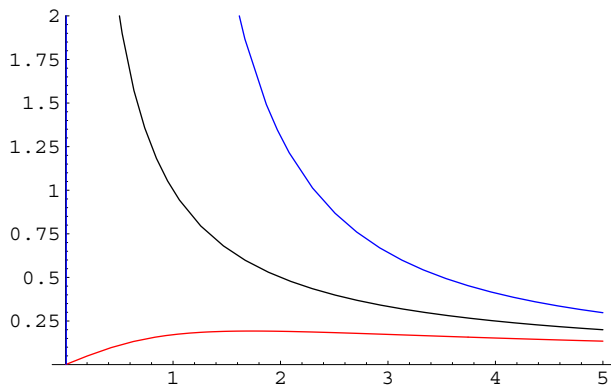
Figure 1. The potentials $U(\gamma)$ for the ESNS, the Witten BH and the ESBH.

The potentials read [26]

$$V_{\mathrm{ESBH}} = -2b^2\gamma\,, \quad U_{\mathrm{ESBH}} = -\frac{1}{\gamma N_+(\gamma)}\,, \qquad V_{\mathrm{ESNS}} = -2b^2\gamma\,, \quad U_{\mathrm{ESNS}} = -\frac{1}{\gamma N_-(\gamma)}\,, \tag{31}$$

with

$$N_\pm(\gamma) = 1 + \frac{2}{\gamma}\left(\frac{1}{\gamma} \pm \sqrt{1 + \frac{1}{\gamma^2}}\right). \tag{32}$$

Note that $N_+ N_- = 1$. The conformally invariant combination (4),

$$w_{\mathrm{ESBH}} = -b\left(1 + \sqrt{\gamma^2 + 1}\right), \qquad w_{\mathrm{ESNS}} = -b\left(1 - \sqrt{\gamma^2 + 1}\right), \tag{33}$$

of the potentials shows that the ESBH/ESNS is a Minkowskian ground state model, $w \propto 1/I$. In figure 1 the potential U is plotted as function of the auxiliary dilaton γ. The lowest branch is associated with the ESBH, the one on top with the ESNS and the one in the middle with the Witten BH (i.e., the third entry in table 1). The regularity of the ESBH is evident, as well as the convergence of all three branches for $\gamma \to \infty$, encoding (T-)self-duality of the Witten BH. For small values of the dilaton the discrepancy between the ESBH, the ESNS and the Witten BH is very pronounced. Note that U remains bounded globally only for the ESBH, concurring with the absence of a curvature singularity.

The two constants of motion — mass and charge — may be parameterized by k and ϕ_0, respectively. Thus, the level k is not fixed a priori but rather emerges as a constant of motion, namely essentially the ADM mass. A rough interpretation of this — from the stringy point of view rather unexpected — result has been provided in [26] and coincides with a similar one in [63]. There is actually a physical reason why k defines the mass: in the presence of matter the conservation equation $dM = 0$ (with M from (10)) acquires a matter contribution, $dM = W^{(m)}$, where $W^{(m)} = d\mathcal{C}^{(m)}$ is an exact 1-form defined by the energy-momentum tensor (cf. section 5 of [4] or [66]). In a nutshell, the addition of matter deforms the total mass which now consists of a geometric and a matter part, M and $\mathcal{C}^{(m)}$, respectively. Coming back to the ESBH, the interpretation of k as mass according to the preceding discussion implies that the addition of matter should "deform" k. But this is precisely what happens: adding matter will in general change the central charge and hence the level k. Thus, from an intrinsically 2D dilaton gravity point of view the interpretation of k as mass is not only possible but favored.

It could be interesting to generalize the target space action of 2D type 0A/0B, (21), as to include the non-perturbative corrections implicit in the ESBH by adding (22) (not necessarily with the choice (23)) to the ESBH action (29). However, it is not quite clear how to incorporate the term from the D0 branes — perturbatively one should just add $b^2 q^2/8\pi$ to V in (31), but non-perturbatively this need not be correct. More results and speculations concerning applications of the ESBH action can be found in [26].

4.   Black Holes

BHs are fascinating objects, both from a theoretical and an experimental point of view [67]. Many of the features which are generic for BHs are already exhibited by the simplest members of this species, the Schwarzschild and Reissner-Nordström BHs (sometimes the Schwarzschild BH even is dubbed as "Hydrogen atom of General Relativity"). Since both of them, after integrating out the angular part, belong to the class of 2D dilaton gravity models (the first and twelfth model in table 1), the study of (3) at the classical, semi-classical and quantum level is of considerable importance for the physics of BHs.

4.1.   Classical analysis

In section 2.2 it has been recalled briefly how to obtain all classical solutions in basic EF patches, (11). By looking at the geodesics of test particles and completeness properties it is straightforward to construct all Carter-Penrose diagrams for a generic model (3) (or, equivalently, (1)). For a detailed description of this algorithm cf. [34, 33, 4] and references therein.

4.2.   Thermodynamics

Mass   The question of how to define "the" mass in theories of gravity is notoriously cumbersome. A nice clarification for D = 4 is contained in [68]. The main conceptual point is that any mass definition is meaningless without specifying 1. the ground state space-time with respect to which mass is being measured and 2. the physical scale in which mass units are being measured. Especially the first point is emphasized here. In addition to being relevant on its own, a proper mass definition is a pivotal ingredient for any thermodynamical study of BHs. Obviously, any mass-to-temperature relation is meaningless without defining the former (and the latter). For a large class of 2D dilaton gravities these issues have been resolved in [69]. One of the key ingredients is the existence [70, 71] of a conserved quantity (10) which has a deeper explanation in the context of first order gravity [72] and Poisson-sigma models [31]. It establishes the necessary prerequisite for all mass definitions, but by itself it does not yet constitute one. Ground state and scale still have to be defined. Actually, one can take M from (10) provided the two ambiguities from integration constants in (4) are fixed appropriately. This is described in detail in appendix A of [51]. In those cases where this notion makes sense M then coincides with the ADM mass.

Hawking temperature   There are many ways to calculate the Hawking temperature, some of them involving the coupling to matter fields, some of them being purely geometrical. Because of its simplicity we will restrict ourselves to a calculation of the geometric Hawking temperature as derived from surface gravity (cf. e.g. [73]). If defined in this way it turns out to be independent of the conformal frame. However, it should be noted that identifying Hawking temperature with surface gravity is somewhat naive for space-times which are not asymptotically flat. But the difference is just a redshift factor and for quantities like entropy or specific heat actually (34) is the relevant quantity as it coincides with the period of Euclidean time (cf. e.g. [74]). Surface gravity can be calculated by taking the normal derivative $\mathrm{d}/\mathrm{d}X$ of the Killing norm (cf. (11)) evaluated on one of the Killing horizons $X = X_h$, where X_h is a solution of (12), thus yielding

$$T_H = \frac{1}{2\pi}\Big|w'(X)\Big|_{X=X_h}.\tag{34}$$

The numerical prefactor in (34) can be changed e.g. by a redefinition of the Boltzmann constant. It has been chosen in accordance with refs. [75, 4].

Entropy   In 2D dilaton gravity there are various ways to calculate the Bekenstein-Hawking entropy. Using two different methods (simple thermodynamical considerations, i.e., $\mathrm{d}M = T\,\mathrm{d}S$, and Wald's Noether charge

technique [76]) Gegenberg, Kunstatter and Louis-Martinez were able to calculate the entropy for rather generic 2D dilaton gravity [77]: entropy equals the dilaton field evaluated at the Killing horizon,

$$S = 2\pi X_h \,. \tag{35}$$

There exist various ways to count the microstates by appealing to the Cardy formula [78] and to recover the result (35). However, the true nature of these microstates remains unknown in this approach, which is a challenging open problem. Many different proposals have been made [79].

Specific heat By virtue of $C_s = T \, \mathrm{d}S / \, \mathrm{d}T$ the specific heat reads

$$C_s = 2\pi \left. \frac{w'}{w''} \right|_{X=X_h} = \gamma_S \, T_H \,, \tag{36}$$

with $\gamma_S = 4\pi^2 \, \mathrm{sign}\,(w'(X_h))/w''(X_h)$. Because it is determined solely by the conformally invariant combination of the potentials, w as defined in (4), specific heat is independent of the conformal frame, too. On a curious sidenote it is mentioned that (36) behaves like an electron gas at low temperature with Sommerfeld constant γ_S (which in the present case may have any sign). If C_s is positive and $C_s T^2 \gg 1$ one may calculate logarithmic corrections to the canonical entropy from thermal fluctuations and finds [80]

$$S_{\mathrm{can}} = 2\pi X_h + \frac{3}{2} \ln \left| w'(X_h) \right| - \frac{1}{2} \ln \left| w''(X_h) \right| + \dots \,. \tag{37}$$

Hawking-Page like phase transition In their by now classic paper on thermodynamics of BHs in AdS, Hawking and Page found a critical temperature signalling a phase transition between a BH phase and a pure AdS phase [17]. This has engendered much further research, mostly in the framework of the AdS/CFT correspondence (for a review cf. [81]). This transition is displayed most clearly by a change of the specific heat from positive to negative sign: for Schwarzschild-AdS (cf. the thirteenth entry in table 1) the critical value of X_h is given by $X_h^c = \ell^2/3$. For $X_h > X_h^c$ the specific heat is positive, for $X_h < X_h^c$ it is negative.[7] By analogy, a similar phase transition may be expected for other models with corresponding behavior of C_s. Interesting speculations on a phase transition at the Hagedorn temperature $T_h = k/(2\pi)$ induced by a tachyonic instability have been presented recently in the context of 2D type 0A strings (cf. the penultimate model in table 1) by Olsson [83]. From equation (22) of that work one can check easily that indeed the specific heat (at fixed q), $C_s = (q^2/8)(T/T_h)/(1 - T/T_h)$, changes sign at $T = T_h$.

4.3. Semi-classical analysis

After the influential CGHS paper [9] there has been a lot of semi-classical activity in 2D, most of which is summarized in [84, 75, 4]. In many applications one considers (1) coupled to a scalar field (16) with $F = \mathrm{const.}$ (minimal coupling). Technically, the crucial ingredient for 1-loop effects is the Weyl anomaly (cf. e.g. [85]) $< T_\mu^\mu > = R/(24\pi)$, which — together with the semi-classical conservation equation $\nabla_\mu < T^{\mu\nu} > = 0$ — allows to derive the flux component of the energy momentum tensor after fixing some relevant integration constant related to the choice of vacuum (e.g. Unruh, Hartle-Hawking or Boulware). This method goes back to Christensen and Fulling [86]. For non-minimal coupling, e.g. $F \propto X$, there are some important modifications — for instance, the conservation equation no longer is valid but acquires a right hand side proportional to $F'(X)$. The first calculation of the conformal anomaly in that case has been performed by Mukhanov, Wipf and Zelnikov [87]. It has been confirmed and extended e.g. in [88].

---

[7]Actually, in the original work [17] Hawking and Page did not invoke the specific heat directly. The consideration of the specific heat as an indicator for a phase transition is in accordance with the discussion in [82].

4.4. Long time behavior

The semi-classical analysis, while leading to interesting results, has the disadvantage of becoming unreliable as the mass of the evaporating BH drops to zero. The long time behavior of an evaporating BH presents a challenge to theoretical physics and touches relevant conceptual issues of quantum gravity, such as the information paradox. There are basically two strategies: top-down, i.e., to construct first a full quantum theory of gravity and to discuss BH evaporation as a particular application thereof, and bottom-up, i.e., to sidestep the difficulties inherent to the former approach by invoking "reasonable" ad-hoc assumptions. The latter route has been pursued in [12]. A crucial technical ingredient has been Izawa's result [89] on consistent deformations of 2D BF theory, while the most relevant physical assumption has been boundedness of the asymptotic matter flux during the whole evaporation process. Together with technical assumptions which can be relaxed, the dynamics of the evaporating BH has been described by means of consistent deformations of the underlying gauge symmetries with only one important deformation parameter. In this manner an attractor solution, the endpoint of the evaporation process, has been found (cf. the eighth model in table 1).

Ideologically, this resembles the exact renormalization group approach, cf. e.g. [90, 91] and references therein, which is based upon Weinberg's idea of "asymptotic safety".[8] There are, however, several conceptual and technical differences, especially regarding the truncation of "theory space": in 4D a truncation to EH plus cosmological constant, undoubtedly a very convenient simplification, may appear to be somewhat ad-hoc, whereas in 2D a truncation to (3) comprises not only infinitely many different theories, but essentially[9] *all* theories with the same field content as (3) and the same kind of local symmetries (Lorentz transformations and diffeomorphisms).

The global structure of an evaporating BH can also be studied, and despite of the differences between various approaches there seems to be partial agreement on it, cf. e.g. [93, 94, 12, 95, 96, 91]. The crucial insight might be that a BH in the mathematical sense (i.e., an event horizon) actually never forms, but only some trapped region, cf. figure 5 in [96].

4.5. Killing horizons kill horizon degrees

As pointed out by Carlip [97], the fact that very different approaches to explain the entropy of BHs nevertheless agree on the result urgently asks for some deeper explanation. Carlip's suggestion was to consider an underlying symmetry, somehow attached to the BH horizon, as the key ingredient, and he noted that requiring the presence of a horizon imposes constraints on the physical phase space. Actually, the change of the phase-space structure due to a constraint which imposes the existence of a horizon in space-time is an issue which is of considerable interest by itself.

In a recent work [98] we could show that the classical physical phase space is smaller as compared to the generic case if horizon constraints are imposed. Conversely, the number of gauge symmetries is larger for the horizon scenario. In agreement with a conjecture by 't Hooft [99], we found that physical degrees of freedom are converted into gauge degrees of freedom at a horizon. We will now sketch the derivation of this result briefly for the action (3) which differs from the one used in [98] by a (Gibbons-Hawking) boundary term. For sake of concreteness we will suppose the boundary is located at $x^1 = $ const. Consistency of the variational principle then requires

$$X^+ \delta e_0^- + X^- \delta e_0^+ + X \delta \omega_0 = 0 \tag{38}$$

at the boundary. Note that one has to fix the parallel component of the spin-connection at the boundary rather than the dilaton field, which is the main difference to [98]. The generic case imposes $\delta e_0^\pm = 0 = \delta \omega_0$, while a horizon allows the alternative prescription $\delta e_0^- = X^- = 0 = \delta \omega_0$. One can now proceed in the same

---

[8]In the present context also [92] should be mentioned.

[9]Actually, one should replace in (3) the term $X^+ X^- U(X) + V(X)$ by $\mathcal{V}(X^+ X^-, X)$. However, only (3) allows for standard supergravity extensions [41].

way as in [98], i.e., derive the constraints (the only boundary terms in the secondary constraints are now X and $X^{\pm}$, while the primary ones have none) and calculate the constraint algebra. All primary constraints and the Lorentz constraint turn out to be first class, even at the boundary, whereas the Poisson bracket between the two diffeomorphism constraints (G_2, G_3 in the notation of [98]) acquires a boundary term of the form

$$X(U'X^+X^- + V') + U(X)X^+X^- - V(X).\tag{39}$$

Notably, it vanishes only for $V \propto X$ and $U \propto 1/X$, e.g. for the second, third and sixth model in table 1, i.e., $(A)dS_2$ ground state models. The boundary constraints for the generic case convert all primary constraints into second class constraints. The construction of the reduced phase space works in the same way as in section 6 of [98], thus establishing again one physical degree of freedom "living on the boundary". Actually, this had been known already before [100]. The horizon constraints, however, lead to more residual gauge symmetries and to a stronger fixing of free functions — in fact, no free function remains and the reduced phase space is empty. Thus, the physical degree of freedom living on a generic boundary is killed by a Killing horizon.

It would be interesting to generalize this physics-to-gauge conversion at a horizon to the case with matter. Obviously, it will no longer be a Killing horizon, but one can still employ the (trapping) horizon condition $X^- = 0$.

4.6. Critical collapse

Critical phenomena in gravitational collapse have been discovered in the pioneering numerical investigations of Choptuik [101]. He studied a free massless scalar field coupled to spherically symmetric EH gravity in 4D (the EMKG) with sophisticated numerical techniques that allowed him to analyze the transition in the space of initial data between dispersion to infinity and the formation of a BH. Thereby the famous scaling law

$$M_{BH} \propto (p - p_*)^\gamma,\tag{40}$$

has been established, where $p \in [0,1]$ is a free parameter characterizing a one-parameter family of initial data with the property that for $p < p_*$ a BH never forms while for $p > p_*$ a BH always forms with mass M_{BH} determined by (40) for p sufficiently close to p_*. The critical parameter $p_* \in (0,1)$ may be found by elaborate numerical analysis and depends on the specific family under consideration; but the critical exponent $\gamma \approx 0.37$ is universal, albeit model dependent. Other systems may display a different critical behavior, cf. the review [102]. The critical solution $p = p_*$, called the "Choptuon", in general exhibits remarkable features, e.g. discrete or continuous self-similarity and a naked singularity.

Since the original system studied by Choptuik, (20), is a special case of (1) (with U, V as given by the first line in table 1) coupled to (16), it is natural to inquire about generalizations of critical phenomena to arbitrary 2D dilaton gravity with scalar matter. Indeed, in [103] a critical exponent $\gamma = 1/2$ has been derived analytically for the RST model [104], a semi-classical generalization of the CGHS model (cf. the third line in table 1). Later, in [105] critical collapse within the CGHS model has been considered and $\gamma \approx 1/2$ has been found numerically. More recently the generalization of the original Choptuik system to D dimensions has been considered [106, 107, 108]. For $3.5 \leq D \leq 14$ the approximation $\gamma(D) \approx 0.47(1 - \exp(-0.41D))$ shows that γ increases monotonically[10] with D. Since formally the CGHS corresponds to the limit D $\to \infty$ one may expect that $\gamma(D)$ asymptotes to the value $\gamma \approx 1/2$.

In the remainder of this subsection we will establish evolution equations for generic 2D dilaton gravity with scalar matter, to be implemented numerically analogously to [109, 110]. In these works for various

---

[10]In [107] a maximum in γ near D=11 has been found. The most recent study suggests it is an artifact of numerics [108]. Another open question concerns the limit D $\to$ 3: does γ remain finite?

reasons Sachs-Bondi gauge has been used. Thus we employ

$$e_0^+ = 0 \,, \quad e_0^- = 1 \,, \quad x^0 = X \,, \tag{41}$$

while the remaining Zweibein components are parameterized as

$$e_1^- = \alpha(u, X) \,, \quad e_1^+ = I(X)e^{2\beta(u,X)} \,. \tag{42}$$

In the gauge (41) with the parameterization (42) the line element reads

$$\mathrm{d}s^2 = 2I(X)e^{2\beta(u,X)} \, \mathrm{d}u \, (\mathrm{d}X + \alpha(u, X) \, \mathrm{d}u) \,. \tag{43}$$

A trapping horizon emerges either if $\alpha = 0$ or $\beta \to \infty$. The equations of motion may be reduced to the following set:

$$\text{Slicing condition :} \qquad \partial_X \alpha(u, X) = -e^{2\beta(u,X)} w'(X) \tag{44}$$

$$\text{Hamiltonian constraint :} \qquad \partial_X \beta(u, X) = -F(X)(\partial_X \mathcal{T}(u, X))^2 \tag{45}$$

$$\text{Klein} - \text{Gordon equation :} \qquad \Box \, \mathcal{T}(u, X) = 0 \tag{46}$$

with

$$\Box = 2\partial_X \partial_u - 2\partial_X(\alpha(u, X)\partial_X) - \frac{F'(X)}{F(X)} \left(2\alpha(u, X)\partial_X - \partial_u\right) \,. \tag{47}$$

These equations should be compared with (2.12a), (2.12b) in [109] or with (2.4) (and for the Klein-Gordon equation also (2.3)) in [110], where they have been derived for spherically symmetric EH gravity in 4D. In the present case they are valid for generic 2D dilaton gravity coupled non-minimally to a free massless scalar field. Thus, the set of equations (44)-(47) is a suitable starting point for numerical simulations in generic 2D dilaton gravity. The Misner-Sharp mass function

$$m(u, X) = -X^+ X^- I(X) - w(X) = -\alpha(u, X)e^{-2\beta(u,X)} - w(X) \tag{48}$$

allows to rewrite the condition for a trapped surface as $\alpha e^{-2\beta} = 0$ (cf. (12) with (10)). Thus, as noted before, either α has to vanish or $\beta \to \infty$; it is the latter type of horizon that is of relevance for numerical simulations of critical collapse. One may use the Misner-Sharp function instead of α and thus obtains instead of (44)

$$\partial_X m(u, X) = (m(u, X) + w(X))2F(X)(\partial_X \mathcal{T}(u, X))^2 \,. \tag{49}$$

To monitor the emergence of a trapped surface numerically one has to check whether

$$m(u_0, X_h) + w(X_h) \approx 0 \tag{50}$$

is fulfilled to a certain accuracy at a given retarded time u_0; the quantity X_h corresponds to the value of the dilaton field at the horizon. By analogy to (2.16) of [110] one may now introduce a compactified "radial" coordinate, e.g. $X/(1 + X)$, although there may be more convenient choices.

As a consistency check the original Choptuik system in the current notation will be reproduced. We recall that (20) describes the EMKG. Using $\mathrm{d}r = I(X) \, \mathrm{d}X$ the evolution equations for geometry read:

$$\partial_r \beta = \frac{\kappa}{2} r(\partial_r \mathcal{T})^2 \tag{51}$$

$$\partial_r \alpha = \lambda^2 e^{2\beta} \tag{52}$$

They look almost the same as (2.4) in [110]. The coupling constant κ just has to be fixed appropriately in (51) (i.e. $\kappa = 4\pi$). Also, the scaling constant λ must be fixed. Note that the line element reads

$$\mathrm{d}s^2 = 2\frac{2}{r}e^{2\beta(u,X(r))} \, \mathrm{d}u \left(\mathrm{d}r\frac{r}{2} + \alpha(u, X(r)) \, \mathrm{d}u\right) = 2e^{2\beta} \, \mathrm{d}u \left(\mathrm{d}r + \frac{2\alpha}{r} \, \mathrm{d}u\right) \tag{53}$$

This shows that β here really coincides with β in [110] and α here coincides, up to a numerical factor, with V there (and there are some signs due to different conventions).

4.7. Quasinormal modes

The term "quasinormal modes" refers to some set of modes with a complex frequency, associated with small perturbations of a BH. For $U = 0$ and monomial V in [111] quasinormal modes arising from a scalar field, (16) with $f = 0$ and $F \propto X^p$, have been studied in the limit of high damping by virtue of the "monodromy approach", and the relation

$$e^{\omega/T_H} = -(1 + 2\cos(\pi(1 - p))) \tag{54}$$

for the frequency ω has been found (T_H is Hawking temperature as defined in (34)). Minimally coupled scalar fields ($p = 0$) lead to the trivial result $\omega/T_H = 2\pi i n$. High damping implies that the integer n has to be large. For the important case of $p = 1$ (relevant for the first and fifth entry in table 1) one obtains from (54)

$$\frac{\omega}{T_H} = 2\pi i\left(n + \frac{1}{2}\right) + \ln 3. \tag{55}$$

The result (55) coincides with the one obtained for the Schwarzschild BH with 4D methods, both numerically [112] and analytically [113]. Moreover, consistency with D > 4 is found as well [114]. This shows that the 2D description of BHs is reliable also with respect to highly damped quasinormal modes.

4.8. Solid state analogues

BH analogues in condensed matter systems go back to the seminal paper by Unruh [115]. Due to the amazing progress in experimental condensed matter physics, in particular Bose-Einstein condensates, in the past decade the subject of BH analogues has flourished, cf. e.g. [116] and references therein.

In some cases the problem effectively reduces to 2D. It is thus perhaps not surprising that an analogue system for the Jackiw-Teitelboim model has been found [117] for a cigar shaped Bose-Einstein condensate. More recently this has led to some analogue 2D activity [118]. Note, however, that some issues, like the one of backreaction, might not be modelled very well by an effective action method [119]. Indeed, 2D dilaton gravity with matter could be of interest in this context, because these systems might allow not just kinematical but dynamical equivalence, i.e., not only the fluctuations (e.g. phonons) behave as the corresponding gravitational ones (e.g. Hawking quanta), but also the background dynamics does (e.g. the flow of the fluid or the metric, respectively). Such a system would be a necessary pre-requisite to study issues of mass and entropy in an analogue context. At least for static solutions this is possible [120], but of course the non-static case would be much more interesting. Alas, it is not only more interesting but also considerably more difficult, and a priori there is no reason why one should succeed in finding a fully fledged analogue model of 2D dilaton gravity with matter. Still, one can hope and try.

5. Geometry from matter

In first order gravity (3) coupled to scalar (16) or fermionic (17) matter the geometry can be quantized exactly: after analyzing the constraints, fixing EF gauge

$$(\omega_0, e_0^-, e_0^+) = (0, 1, 0) \tag{56}$$

and constructing a BRST invariant Hamiltonian, the path integral can be evaluated exactly and a (nonlocal) effective action is obtained [121]. Subsequently the matter fields can be quantized by means of ordinary perturbation theory. To each order all backreactions are included automatically by this procedure.

Although geometry has been integrated out exactly, it can be recovered off-shell in the form of interaction vertices of the matter fields, some of which resemble virtual black holes (VBHs) [122, 123, 15]. This
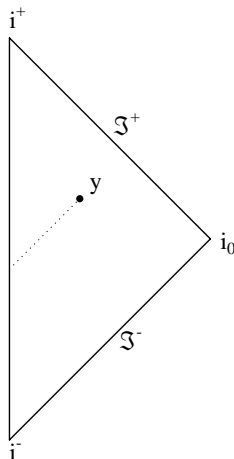
Figure 2. VBH

metamorphosis of geometry however does not take place in the matterless case [124], where the quantum effective action coincides with the classical action in EF gauge. We hasten to add that one should not take this off-shell geometry at face value — this would be like over-interpreting the role of virtual particles in a loop diagram. But the simplicity of such geometries and the fact that all possible configurations are summed over are both nice qualitative features of this picture.

A Carter-Penrose diagram of a typical VBH configuration is depicted in figure 2. The curvature scalar of such effective geometries is discontinuous and even has a δ-peak. A typical effective line element (for the EMKG) reads

$$(\mathrm{d}s)^2 = 2\,\mathrm{d}r\,\mathrm{d}u + \left(1 - \theta(r_y - r)\delta(u - u_y)\left(\frac{2m}{r} + ar - d\right)\right)(\mathrm{d}u)^2\,, \tag{57}$$

It obviously has a Schwarzschild part with r_y-dependent "mass" m and a Rindler part with r_y-dependent "acceleration" a, both localized on a lightlike cut. This geometry is nonlocal in the sense that it depends not just on the coordinates r, u but additionally on a second point r_y, u_y. While the off-shell geometry (57) is highly gauge dependent, the ensuing S-matrix — the only *physical* observable in this context [125] — appears to be gauge independent, although a formal proof of this statement, e.g. analogously to [126], is lacking.

5.1. Scalar matter

After integrating out geometry and the ghost sector (for $f(X, \mathcal{T}) = 0$), the effective Lagrangian (w is defined in (4))

$$\mathcal{L}_{\mathcal{T}}^{\mathrm{eff}} = F(\hat{X})(\partial_0 \mathcal{T})(\partial_1 \mathcal{T}) - w'(\hat{X}) + \text{sources} \tag{58}$$

contains the quantum version of the dilaton field $\hat{X} = \hat{X}(\nabla_0^{-2}(\partial_0 \mathcal{T})^2)$, depending non-locally on $\mathcal{T}$. The quantity $\hat{X}$ solves the equation of motion of the classical dilaton field, with matter terms and external sources for the geometric variables in EF gauge. The simplicity of (58) is in part due to the gauge choice (56) and in part due to the linearity of the gauge fixed Lagrangian in the remaining gauge field components, thus producing delta-functionals upon path integration.

In principle, the interaction vertices can be extracted by expanding the nonlocal effective action in a power series of the scalar field $\mathcal{T}$. However, this becomes cumbersome already at the $\mathcal{T}^4$ level. Fortunately, the localization technique introduced in [121] simplifies the calculations considerably. It relies on
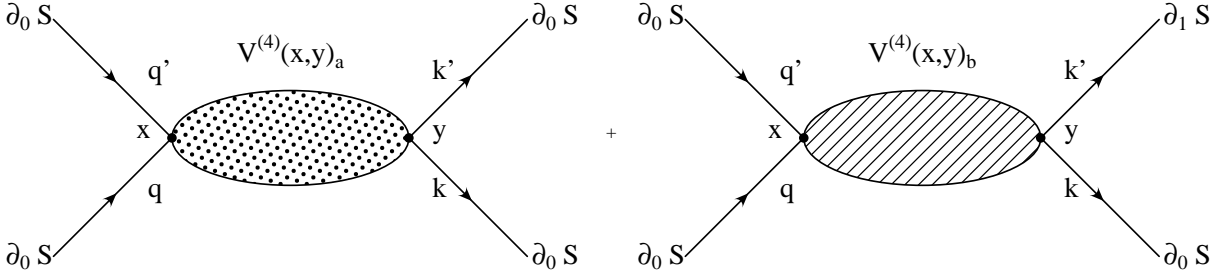
Figure 3. Non-local 4-point vertices

two observations: First, instead of dealing with complicated nonlocal kernels one may solve corresponding differential equations after imposing asymptotic conditions on the solutions. Second, instead of taking the n-th functional derivative of the action with respect to bilinear combinations of $\mathcal{T}$, the matter fields may be localized at n different space-time points, which mimics the effect of functional differentiation. For tree-level calculations it is then sufficient to solve the classical equations of motion in the presence of these sources, which is achieved most easily via appropriate matching conditions.

It turns out (as anticipated from (57)) that the conserved quantity (10) is discontinuous for a VBH. This phenomenon is generic [15].[11] The corresponding Feynman diagrams are contained in figure 3.[12] For free, massless, non-minimally coupled scalars ($F \neq$ const.) both the symmetric and the non-symmetric 4-point vertex

$$V^{(4)} = \int \mathrm{d}^2 x\, \mathrm{d}^2 y (\partial_0 \mathcal{T})_x^2 \left[V_a(x,y)(\partial_0 \mathcal{T})_y^2 + V_b(x,y)(\partial_0 \mathcal{T})_y (\partial_1 \mathcal{T})_y \right] \tag{59}$$

are given in [15], and have the following properties:

1. They are local in one coordinate (e.g. containing $\delta(x^1 - y^1)$) and nonlocal in the other.

2. They vanish in the local limit ($x^0 \to y^0$). Additionally, V_b vanishes for minimal coupling $F = $ const.

3. The symmetric vertex depends only on the conformal invariant combination $w(X)$ and the asymptotic value M_∞ of (10). The non-symmetric one is independent of U, V and M_∞. Thus if M_∞ is fixed in all conformal frames, both vertices are conformally invariant.

4. They respect the $\mathbb{Z}_2$ symmetry $F(X) \mapsto -F(X)$.

It should be noted that the class of models with $UV + V' = 0$ and $F = $ const (containing the CGHS model, the seventh and eleventh entry in table 1) shows "scattering triviality", i.e., the classical vertices vanish, and scattering can only arise from higher order quantum backreactions. For these models the VBH has no classically observable consequences, but at 1-loop level physical observables like the specific heat are modified appreciably [127].

The 2D Klein-Gordon equation relevant for the construction of asymptotic states is also conformally invariant. For minimal coupling it simplifies considerably, and a complete set of asymptotic states can be obtained explicitly. Since both, asymptotic states and vertices, only depend on $w(X)$ and M_∞, at tree level conformal invariance holds nonperturbatively (to all orders in $\mathcal{T}$), but it is broken at 1-loop level due to the conformal anomaly. Because asymptotically geometry does not fluctuate, a standard Fock space may be built with creation/annihilation operators $a^\pm(k)$ obeying the standard commutation relations. The S-matrix for two ingoing (q, q') into two outgoing (k, k') asymptotic modes is determined by (cf. (59))

$$T(q, q'; k, k') = \frac{1}{2} \langle 0 \left| a^-(k) a^-(k') V^{(4)} a^+(q) a^+(q') \right| 0 \rangle . \tag{60}$$

---

[11] With the exception of scattering trivial models, cf. the eleventh entry in table 1.

[12] The scalar field $\mathcal{T}$ is denoted by S in these graphs.

The simple choice $M_\infty = 0$ yields a "standard QFT vacuum" $|0\rangle$, provided the model under consideration has a Minkowskian ground state (e.g. the first, third, fifth and last model in table 1).

For the physically interesting case of the EMKG model such an S-matrix was obtained in [128, 123]. Both the symmetric and the non-symmetric vertex contribute, each giving a divergent contribution to the S-matrix, but the sum of both turned out to be finite! The whole calculation is highly nontrivial, involving cancellations of polylogarithmic terms, but at the end giving the surprisingly simple result

$$T(q, q'; k, k') = -\frac{i\kappa\delta\left(k + k' - q - q'\right)}{2(4\pi)^4|kk'qq'|^{3/2}} E^3 \tilde{T}, \tag{61}$$

with ingoing (q, q') and outgoing (k, k') spatial momenta, total energy $E = q + q'$,

$$\tilde{T} := \frac{1}{E^3}\left[\Pi \ln \frac{\Pi^2}{E^6} + \frac{1}{\Pi} \sum_{p\in\{k,k',q,q'\}} p^2 \ln \frac{p^2}{E^2} \cdot \left(3kk'qq' - \frac{1}{2}\sum_{r\neq p}\sum_{s\neq r,p} \left(r^2 s^2\right)\right)\right], \tag{62}$$

and the momentum transfer function $\Pi = (k + k')(k - q)(k' - q)$. The factor $\tilde{T}$ is invariant under rescaling of the momenta $p \mapsto ap$, and the whole amplitude transforms monomial like $T \mapsto a^{-4}T$. It should be noted that due to the non-locality of the vertices there is just one δ-function of momentum conservation (but no separate energy conservation) present in (61). This is advantageous because it eliminates the problem of "squared δ-functions" that is otherwise present in 2D theories of massless scalar fields (cf. e.g. [129]). In this sense gravity acts as a regulator of the theory.

The corresponding differential cross section also reveals interesting features [123]:

1. For vanishing Π forward scattering poles are present.

2. There is an approximate self-similarity close to the forward scattering peaks. Far away from them it is broken, however.

3. It is CPT invariant.

4. An ingoing s-wave can decay into three outgoing ones. Although this may be expected on general grounds, within the present formalism it is possible to provide explicit results for the decay rate.

Although it seems straightforward to generalize (60) to arbitrary n-point vertices, no such calculation has been attempted so far. This is related to the fact that the derivation of (61) has been somewhat tedious and lengthy. Thus, it could be worthwhile to find a more efficient way to obtain this interesting S-matrix element.

5.2. Fermionic matter

Recently we considered 2D dilaton gravity (3) coupled to fermions (17) along the lines of the previous subsection. The results will be published elsewhere, but we give a short summary with emphasis on differences to the scalar case.

The constraint analysis for the general case (17) has been worked out first in [42]. Three first class constraints generating the two diffeomorphisms and the local Lorentz symmetry and four well-known second class constraints relating the four real components of the Dirac spinor to their canonical momenta are present in the system. As anticipated the Hamiltonian is fully constrained. After introducing the Dirac bracket the constructions of the BRST charge and the gauge fixed Hamiltonian are straightforward. Path integration over geometry is even simpler than in the scalar case, because the second class constraints are implemented in the path integral through delta functionals, allowing to integrate out the fermion momenta. The effective

Lagrangian

$$
\begin{aligned}
\mathcal{L}_\chi^{\text{eff}} \;=\;\; & \frac{i}{\sqrt{2}}F(\hat{X})(\chi_1^{*}\overleftrightarrow{\partial_1}\chi_1) \\
& +I(\hat{X})\left(\frac{i}{\sqrt{2}}F(\hat{X})(\chi_0^{*}\overleftrightarrow{\partial_0}\chi_0) + H(\hat{X})g(\overline{\chi}\chi) - V(\hat{X})\right) + \text{sources}
\end{aligned}
\tag{63}
$$

again depends on the quantum version $\hat{X} = \hat{X}(\nabla_0^{-2}(\chi_1^{*}\overleftrightarrow{\partial_0}\chi_1))$ of the dilaton field and exhibits non-locality in the matter field.

Some properties remain the same as compared to previous studies with scalar matter. For instance, the VBH phenomenon is still present, now even for the eleventh model in table 1. In fact, the conserved quantity (10) now becomes continuous only for the trivial case $F(X) = 0$. But there are also some notable differences. For example, the non-selfinteracting system already has three 4-point vertices, two of them being the symmetric and asymmetric vertices of the scalar case and a new third one, arising from the first term in the second line of (63). All vertices show the first two properties listed above, and the symmetric and non-symmetric ones also the third one.

The new vertex however does not vanish for minimal coupling, and thus in contrast to the scalar case there are two vertices present even for this simple case. It is not conformally invariant, but rather transforms additively because it contains a term proportional to $U(X)$. However, since also the external legs have a conformal weight, conformal invariance of the tree-level S-matrix still is expected to hold, despite of the non-invariance of some of the vertices and some of the asymptotic modes.

At 1-loop level and for minimal coupling conformal symmetry is broken and, exactly as in the case of scalar matter, the conformal anomaly can be integrated to the non-local Polyakov action [130]. This has been applied e.g. in[131]. A possible Thirring term can be reformulated using (19) and integrated by use of the chiral anomaly, giving a Wess-Zumino [132] contribution to the effective action. In this case, a path integral over the auxiliary vector potential remains, with a highly non-local self-interaction. Whether this treatment is favourable over treating the Thirring term directly as an interaction vertex has to be decided by application.

Another peculiar feature of 2D field theories is bosonization, e.g. the quantum equivalence of the Thirring model and the Sine-Gordon model, both in flat 1+1 dimensions [133]. This issue has been addressed recently on a curved background by Frolov, Kristjánsson and Thorlacius [134] to investigate the effect of pair-production on BH space times in regions of small curvature (as compared to the microscopic length scale of quantum theory). In the framework of first order gravity it may be possible to investigate the question of bosonization even outside this simple framework, since one is able to integrate out geometry non-perturbatively.

6.   Mathematical issues

In the absence of matter many of the interesting features discussed in the previous three sections are absent: there is no tachyon dynamics, no Hawking radiation, no interesting semi-classical behavior, no critical collapse, no quasinormal modes, no relevant solid state analogue, no scattering processes and no reconstruction of geometry from matter. Nevertheless, some basic features remain, like the global structure of the classical solutions or the physics-to-gauge conversion mentioned in section 4.5. Mathematically, however, the absence of matter bears some attractiveness and reveals beautiful structures responsible for the classical integrability of (3). They may allow some relevant generalizations of (3), e.g. in the context of non-commutative gravity.

6.1.   Remarks on the Einstein-Hilbert action in 2D

In 2D the Einstein tensor vanishes identically for any 2D metric and thus conveys no useful information. Similarly, the 2D EH action, supplemented appropriately by boundary and corner terms, just counts the number of holes of a compact Riemannian manifold, cf. e.g. [135]. Thus, as compared to (1) or (3) the study of "pure" 2D gravity, i.e., without coupling to a dilaton field, is of rather limited interest. If one adds a cosmological constant term one may study quantum gravity in 2D by means of dynamical triangulations, cf. e.g. [136] and references therein. The EH part of the action plays no essential role, however.

It is possible to consider EH gravity in $2 + \varepsilon$ dimensions, an idea which seems to go back to [137]. After taking the limit $\varepsilon \to 0$ in a specific way [138] one obtains again a dilaton gravity model (1) with $V = 0$ and $U = \text{const.}$ (cf. the eleventh model in table 1). That such a limit can be very subtle has been shown recently by Jackiw [139] in the context of Weyl invariant scalar field dynamics: if one simply drops the EH term in equation (3.5) of that work the Liouville model is obtained (cf. the tenth model in table 1), but Weyl invariance is lost.

6.2.   Relations to 3D: Chern-Simons and BTZ

The gravitational Chern-Simons term [140] and the 3D BTZ BH [141] have inspired a lot of further research. Here we will focus on relations to (1) and (3): dimensional reduction of the BTZ to 2D has been performed in [19], cf. the fifteenth model in table 1. A reduction of the gravitational Chern-Simons term from 3D to 2D has been performed in [20], cf. the sixteenth model in table 1. Recently [142], such reductions have been exploited to calculate the entropy of a BTZ BH in the presence of gravitational Chern-Simons terms, something which is difficult to achieve in 3D because there is no manifestly covariant formulation of the Chern-Simons term, whereas the reduced theory is manifestly covariant. It is not unlikely that also other open problems of 3D gravity may be tackled with 2D methods.

6.3.   Integrable systems, Poisson-sigma models and KdV surfaces

Some of the pioneering work has been mentioned already in section 2.1 and in table 1. In two seminal papers by Kummer and Schwarz [143] the usefulness of light-cone gauge for the Lorentz frame and EF gauge for the curved metric has been demonstrated for the fourteenth model in table 1, which is a rather generic one as it has non-vanishing U and non-monomial V. A Hamiltonian analysis [72] revealed an interesting (W-)algebraic structure of the secondary constraints together with the fields $X, X^{\pm}$ as generators. The center of this algebra consists of the conserved quantity (10) and its first derivative, $\partial_1 M$ (which, of course, vanishes on the surface of constraints). Consequently, it has been shown by Schaller and Strobl [31] that (3) is a special case of a Poisson-sigma model,[13]

$$S_{\text{PSM}} = -\int_{\mathcal{M}} \left[X^I \, dA_I - \frac{1}{2} P^{IJ} A_J \wedge A_I \right] , \tag{64}$$

with a 3D target space, the coordinates of which are $X^I = \{X, X^+, X^-\}$. The gauge fields comprise the Cartan variables, $A_I = \{\omega, e^-, e^+\}$. Because the dimension of the Poisson manifold is odd the Poisson tensor $(I, J \in \{X, \pm\})$

$$P^{X\pm} = \pm X^{\pm} , \qquad P^{+-} = X^+ X^- U(X) + V(X) , \qquad P^{IJ} = -P^{JI} , \tag{65}$$

cannot have full rank. Therefore, always a Casimir function, (10), exists, which may be interpreted as "mass". Note that (65) indeed fulfills the required Jacobi-identities, $P^{IL} \partial_L P^{JK} + perm\,(IJK) = 0$. For a generic (graded) Poisson-sigma model (64) the commutator of two symmetry transformations

$$\delta X^I = P^{IJ} \varepsilon_J , \qquad\qquad \delta A_I = -d\varepsilon_I - \left(\partial_I P^{JK} \right) \varepsilon_K A_J , \tag{66}$$

---

[13]Dirac-sigma models [144] are a recent generalization thereof.

is a (non-linear) symmetry modulo the equations of motion. Only for P^{IJ} linear in X^I a Lie algebra is obtained; cf. the second model in table 1. For (65) the symmetries (66) on-shell correspond to local Lorentz transformations and diffeomorphisms. Generalizations discussed in section 2.4 are particularly transparent in this approach; essentially, one has to add more target space coordinates to the Poisson manifold, some of which will be fermionic in supergravity extensions, cf. e.g. [39].

Actually, there exist various approaches to integrability of gravity models in 2D, cf. e.g. [145], and we can hardly do them justice here. We will just point out a relation to Korteweg-de Vries (KdV) surfaces as discussed recently in [146]. These are 2D surfaces embedded in 3D Minkowski space arising from the KdV equation $\partial_t w = \partial_x^3 w + 6 w \partial_x w$, with line element (cf. (11) in [146]; u there coincides with w here) $ds^2 = 2\, dX\, du - (4\lambda - w(X, u))\, du^2$, where $X \propto x$, $u \propto t$ and λ is some constant. For static KdV solutions, $\partial_u w = 0$, this line element is also a solution of (3) as can bee seen from (11), with λ playing the role of the mass M. In the non-static case it describes a solution of (3) coupled to some energy-momentum tensor. It could be of interest to pursue this relation in more depth.

6.4. Torsion and non-metricity

For $U = 0$ the equation of motion $R = 2V'(X)$, if invertible, allows to rewrite the action (1) as $S_{\mathrm{R}} = \int d^2 x \sqrt{-g} f(R)$, cf. e.g. [147] and references therein. As compared to such theories, the literature on models with torsion $\tau^a = *T^a$,

$$S_{\mathrm{RT}} = \int d^2 x \sqrt{-g} f(R, \tau^a \tau_a) \,, \tag{67}$$

is relatively scarce and consists mainly of elaborations based upon the fourteenth model in table 1, where $f = A\tau^a \tau_a + BR^2 + CR + \Lambda$, also known as "Poincarè gauge theory", cf. [148] and references therein. This model in particular (and a large class of models of type (67)) allows an equivalent reformulation as (3). Thus, they need not be discussed separately.

A generalization which includes also effects from non-metricity has been studied in [149]. Elimination of non-metricity leads again to models of type (1), (3), but one has to be careful with such reformulations as test-particles moving along geodesics or, alternatively, along auto-parallels, may "feel" the difference. Thus, it could be of interest to generalize (3) (which already contains torsion if $U \neq 0$) as to include non-metricity, thus dropping the requirement that the connection $\omega^a{}_b$ is proportional to $\varepsilon^a{}_b$. However, a formulation as Poisson-sigma model (64) (with 6D target space) seems to be impossible as there are only trivial solutions to the Jacobi identities.

6.5. Non-commutative gravity

In the 1970ies/1980ies theories have been supersymmetrized, in the 1990ies/2000s theories have been "non-commutativized", for reviews cf. e.g. [150]. The latter procedure still has not stopped as the original idea, namely to obtain a fully satisfactory non-commutative version of gravity, has not been achieved so far. In order to get around the main conceptual obstacles it is tempting to consider the simplified framework of 2D.

There it is possible to construct non-commutative dilaton gravity models with a usual (non-twisted) realization of gauge symmetries.[14] A non-commutative version of the Jackiw-Teitelboim model (cf. the second entry in table 1),

$$S_{\mathrm{NCJT}} = -\frac{1}{2} \int d^2 x \, \varepsilon^{\mu\nu} \left[X_a \star T^a_{\mu\nu} + X_{ab} \star \left(R^{ab}_{\mu\nu} - \Lambda e^a_\mu \star e^b_\nu \right) \right] \,, \tag{68}$$

---

[14] Another approach has been pursued in [151].

has been constructed in [152] and then quantized in [153]. A non-commutative version of the fourth model in table 1 was suggested in [154]. For a definition of the Moyal-$\star$ and further notation cf. these two references. A crucial change as compared to (3), besides the $\star$, is the appearance of a second dilaton field ψ in $2X_{ab} = X\varepsilon_{ab} - i\psi\eta_{ab}$. However, interesting as these results may be, there seems to be no way to generalize them to generic 2D dilaton gravity without twisting the gauge symmetries [155]. Moreover, the fact that the metric can be changed by "Lorentz transformations" seems questionable from a physical point of view, cf. [156] for a similar problem.

An important step towards constructing a satisfactory non-commutative gravity was recently made by Wess and collaborators [157], who understood how one can construct diffeomorphism invariants, including the EH action, on non-commutative spaces (see also [158] for a real formulation). There is, however, a price to pay. The diffeomorphism group becomes twisted, i.e., there is a non-trivial coproduct [159]. Recently it could be shown [160] that twisted gauge symmetries close for arbitrary gauge groups and thus a construction of twisted-invariant actions is straightforward. The main element in that construction (cf. also [159, 161, 157, 158] and [162]) is the twist operator

$$\mathcal{F} = \exp\mathcal{P}, \qquad \mathcal{P} = \frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu \,, \tag{69}$$

which acts on the tensor products of functions $\phi_1 \otimes \phi_2$. With the multiplication map $\mu(\phi_1 \otimes \phi_2) = \phi_1 \cdot \phi_2$ and (69) the Moyal-Weyl representation of the star product,

$$\phi_1 \star \phi_2 = \mu \circ \mathcal{F}(\phi_1 \otimes \phi_2) = \mu_\star(\phi_1 \otimes \phi_2)\,, \tag{70}$$

can be constructed. Consider now generators u of some symmetry transformations which form a Lie algebra. If one knows the action of these transformations on primary fields, $\delta_u\phi = u\phi$, the action on tensor products is defined by the coproduct Δ. In the undeformed case the coproduct is primitive, $\Delta_0(u) = u \otimes 1 + 1 \otimes u$ and $\delta_u(\phi_1 \otimes \phi_2) = \Delta_0(u)(\phi_1 \otimes \phi_2) = u\phi_1 \otimes \phi_2 + \phi_1 \otimes u\phi_2$ satisfies the usual Leibniz rule. The action of symmetry generators on elementary fields is left undeformed, but the coproduct is twisted,

$$\Delta(u) = \exp(-\mathcal{P})\Delta_0(u)\exp(\mathcal{P})\,. \tag{71}$$

Obviously, twisting preserves the commutation relations. Therefore, the commutators of gauge transformations for an arbitrary gauge group close.

It seems plausible that a corresponding generalization to twisted non-linear gauge symmetries will be a crucial technical pre-requisite to a successful construction of generic non-commutative 2D dilaton gravity[164].[15] It would allow, among other things, a thorough discussion of non-commutative BHs, along the lines of sections 2-5.

Acknowledgments

---

[15]The relation of (64) to a specific Lie algebroid [163] could be helpful in this context.

References

[1] Such a life, with all vision limited to a Point, and all motion to a Straight Line, seemed to me inexpressibly dreary; and I was surprised to note the vivacity and cheerfulness of the King. [Edwin A. Abbot, "Flatland — A Romance of Many Dimensions." Dover Publications 1992, New York. (first published under the pseudonym A. Square in 1884, Seeley & Co., London)].

[2] J. Brown, *Lower Dimensional Gravity*. World Scientific, 1988.

[3] A. M. Polyakov, *Mod. Phys. Lett.*, **A2**, (1987), 893.

[4] D. Grumiller, W. Kummer, and D. V. Vassilevich, *Phys. Rept.*, **369**, (2002), 327, `hep-th/0204253`.

[5] P. Thomi, B. Isaak, and P. Hájíček, *Phys. Rev.*, **D30**, (1984), 1168.
P. Hájíček, *Phys. Rev.*, **D30**, (1984), 1178.

[6] C. Teitelboim, *Phys. Lett.*, **B126**, (1983), 41.

[7] R. Jackiw, *Nucl. Phys.*, **B252**, (1985), 343.

[8] E. Witten, *Phys. Rev.*, **D44**, (1991), 314.
G. Mandal, A. M. Sengupta, and S. R. Wadia, *Mod. Phys. Lett.*, **A6**, (1991), 1685.
S. Elitzur, A. Forge, and E. Rabinovici, *Nucl. Phys.*, **B359**, (1991), 581.

[9] C. G. Callan, Jr., S. B. Giddings, J. A. Harvey, and A. Strominger, *Phys. Rev.*, **D45**, (1992), 1005, `hep-th/9111056`.

[10] J. P. S. Lemos and P. M. Sa, *Phys. Rev.*, **D49**, (1994), 2897, `arXiv:gr-qc/9311008`.

[11] A. Fabbri and J. G. Russo, *Phys. Rev.*, **D53**, (1996), 6995, `hep-th/9510109`.

[12] D. Grumiller, *JCAP*, **05**, (2004), 005, `gr-qc/0307005`.

[13] M. O. Katanaev, W. Kummer, and H. Liebl, *Nucl. Phys.*, **B486**, (1997), 353, `gr-qc/9602040`.

[14] Y. Nakayama, *Int. J. Mod. Phys.*, **A19**, (2004), 2771, `hep-th/0402009`.

[15] D. Grumiller, W. Kummer, and D. V. Vassilevich, *European Phys. J.*, **C30**, (2003), 135, `hep-th/0208052`.

[16] H. Reissner, *Ann. Phys.*, **50**, (1916), 106.
G. Nordström, *Proc. Kon. Ned. Akad. Wet.*, **20**, (1916), 1238.

[17] S. W. Hawking and D. N. Page, *Commun. Math. Phys.*, **87**, (1983), 577.

[18] M. O. Katanaev and I. V. Volovich, *Phys. Lett.*, **B175**, (1986), 413; *Ann. Phys.*, **197**, (1990), 1.

[19] A. Achucarro and M. E. Ortiz, *Phys. Rev.*, **D48**, (1993), 3600, `hep-th/9304068`.

[20] G. Guralnik, A. Iorio, R. Jackiw, and S. Y. Pi, *Ann. Phys.*, **308**, (2003), 222, `hep-th/0305117`.

[21] D. Grumiller and W. Kummer, *Ann. Phys.*, **308**, (2003), 211, `hep-th/0306036`.
L. Bergamin, D. Grumiller, A. Iorio, and C. Nuñez, *JHEP*, **11**, (2004), 021, `hep-th/0409273`.

[22] L. Bergamin, `hep-th/0509183`.

[23] M. R. Douglas *et al.*, `hep-th/0307195`.

[24] S. Gukov, T. Takayanagi, and N. Toumbas, *JHEP*, **03**, (2004), 017, `hep-th/0312208`.

[25] R. Dijkgraaf, H. Verlinde, and E. Verlinde, *Nucl. Phys.*, **B371**, (1992), 269.

[26] D. Grumiller, *JHEP*, **05**, (2005), 028, `hep-th/0501208`.

[27] K. Isler and C. A. Trugenberger, *Phys. Rev. Lett.*, **63**, (1989), 834.
A. H. Chamseddine and D. Wyler, *Phys. Lett.*, **B228**, (1989), 75.

[28] H. Verlinde, in *Trieste Spring School on Strings and Quantum Gravity*, p. 178. April, 1991. the same lectures have been given at MGVI in Japan, June, 1991.

[29] D. Cangemi and R. Jackiw, *Phys. Rev. Lett.*, **69**, (1992), 233, `hep-th/9203056`.
A. Achucarro, *Phys. Rev. Lett.*, **70**, (1993), 1037, `hep-th/9207108`.

[30] N. Ikeda and K. I. Izawa, *Prog. Theor. Phys.*, **90**, (1993), 237, `hep-th/9304012`.

[31] P. Schaller and T. Strobl, *Mod. Phys. Lett.*, **A9**, (1994), 3129, `hep-th/9405110`.

[32] J. G. Russo and A. A. Tseytlin, *Nucl. Phys.*, **B382**, (1992), 259, `arXiv:hep-th/9201021`.
S. D. Odintsov and I. L. Shapiro, *Phys. Lett.*, **B263**, (1991), 183.
T. Banks and M. O'Loughlin, *Nucl. Phys.*, **B362**, (1991), 649.
R. B. Mann, A. Shiekh, and L. Tarasov, *Nucl. Phys.*, **B341**, (1990), 134.

[33] T. Klösch and T. Strobl, *Class. Quant. Grav.*, **13**, (1996), 2395, `arXiv:gr-qc/9511081`.

[34] T. Klösch and T. Strobl, *Class. Quant. Grav.*, **13**, (1996), 965, `arXiv:gr-qc/9508020`.

[35] J.-G. Hao and X.-Z. Li, *Phys. Rev.*, **D68**, (2003), 083514, `hep-th/0306033`.

[36] L. Bergamin, D. Grumiller, and W. Kummer, *J. Phys.*, **A37** (2004), 3881, `hep-th/0310006`.

[37] D. M. Thompson, *Phys. Rev.*, **D70**, (2004), 106001, `hep-th/0312156`.

[38] D. Birmingham, M. Blau, M. Rakowski, and G. Thompson, *Phys. Rept.*, **209**, (1991), 129.

[39] T. Strobl, `hep-th/0011240`. Habilitation thesis.

[40] Y.-C. Park and A. Strominger, *Phys. Rev.*, **D47**, (1993), 1569, `arXiv:hep-th/9210017`.
J. M. Izquierdo, *Phys. Rev.*, **D59**, (1999), 084017, `arXiv:hep-th/9807007`.
T. Strobl, *Phys. Lett.*, **B460**, (1999), 87, `arXiv:hep-th/9906230`.
M. Ertl, W. Kummer, and T. Strobl, *JHEP*, **01**, (2001), 042, `arXiv:hep-th/0012219`.
M. Ertl, PhD thesis, Technische Universität Wien, 2001. `arXiv:hep-th/0102140`.
L. Bergamin and W. Kummer, *Phys. Rev.*, **D68**, (2003), 104005, `hep-th/0306217`; *Eur. Phys. J.*, **C39**, (2005), 41, `hep-th/0402138`; *Eur. Phys. J.*, **C39**, (2005), S53, `hep-th/0411204`.
L. Bergamin, D. Grumiller, and W. Kummer, *JHEP*, **05**, (2004), 060, `hep-th/0404004`.

[41] L. Bergamin and W. Kummer, *JHEP*, **05**, (2003), 074, `hep-th/0209209`.

[42] R. Meyer, `hep-th/0512267`.

[43] W. E. Thirring, *Annals Phys.*, **3**, (1958), 91.

[44] H. Balasin, C. G. Boehmer, and D. Grumiller, *Gen. Rel. Grav.*, **37**, (2005), 1435, `gr-qc/0412098`.

[45] W. Kummer, *HADRON STRUCTURE '92*, D. Bruncko and J. Urban, eds. September, 1992. Stara Lesna, Czechoslovakia.

[46] H. Pelzer and T. Strobl, *Class. Quant. Grav.*, **15**, (1998), 3803, `arXiv:gr-qc/9805059`.

[47] P. C. Aichelburg and R. U. Sexl, *Gen. Rel. Grav.*, **2**, (1971), 303.

[48] H. Balasin and D. Grumiller, *Class. Quant. Grav.*, **21**, (2004), 2859, `gr-qc/0312086`.

[49] I. Z. Fisher, *Zh. Eksp. Teor. Fiz.*, **18**, (1948), 636, `gr-qc/9911008`.

[50] A. T. Filippov and D. Maison, *Class. Quant. Grav.*, **20**, (2003), 1779, `gr-qc/0210081`.

[51] D. Grumiller and D. Mayerhofer, *Class. Quant. Grav.*, **21**, (2004), 5893, `gr-qc/0404013`.

[52] M. Wyman, *Phys. Rev.*, **D24**, (1981), 839.

[53] A. H. Bilge and D. Daghan, `gr-qc/0508020`.

[54] M. D. Roberts, *Gen. Rel. Grav.*, **21**, (1989), 907.

[55] T. Takayanagi and N. Toumbas, *JHEP*, **07**, (2003), 064, `hep-th/0307083`.

[56] P. Ginsparg and G. W. Moore, `hep-th/9304011`.
     P. Di Francesco, P. H. Ginsparg, and J. Zinn-Justin, *Phys. Rept.*, **254**, (1995), 1, `hep-th/9306153`.
     S. Alexandrov, `hep-th/0311273`.

[57] V. Kazakov, I. K. Kostov, and D. Kutasov, *Nucl. Phys.*, **B622**, (2002), 141, `hep-th/0101011`.

[58] L. Bergamin, D. Grumiller, W. Kummer, and D. V. Vassilevich, *Class. Quant. Grav.*, **22**, (2005), 1361, `hep-th/0412007`.

[59] I. R. Klebanov and A. A. Tseytlin, *Nucl. Phys.*, **B546**, (1999), 155, `hep-th/9811035`.
     A. Strominger, *JHEP*, **03**, (2004), 066, `hep-th/0312194`.
     J. L. Davis, L. A. Pando Zayas, and D. Vaman, *JHEP*, **03**, (2004), 007, `hep-th/0402152`.
     U. H. Danielsson, J. P. Gregory, M. E. Olsson, P. Rajan, and M. Vonk, *JHEP*, **04**, (2004), 065, `hep-th/0402192`.
     J. L. Davis and R. McNees, *JHEP*, **09**, (2005), 072, `hep-th/0411121`.

[60] A. A. Tseytlin, *Phys. Lett.*, **B268**, (1991), 175.
     I. Jack, D. R. T. Jones, and J. Panvel, *Nucl. Phys.*, **B393**, (1993), 95, `hep-th/9201039`.

[61] A. A. Tseytlin, *Nucl. Phys.*, **B399**, (1993), 601, `hep-th/9301015`.
     I. Bars and K. Sfetsos, *Phys. Rev.*, **D48**, (1993), 844, `hep-th/9301047`.

[62] K. Becker, `hep-th/9404157`.

[63] V. A. Kazakov and A. A. Tseytlin, *JHEP*, **06**, (2001), 021, `hep-th/0104138`.

[64] M. J. Perry and E. Teo, *Phys. Rev. Lett.*, **70**, (1993), 2669, `hep-th/9302037`.
     P. Yi, *Phys. Rev.*, **D48**, (1993), 2777, `hep-th/9302070`.

[65] D. Grumiller and D. V. Vassilevich, *JHEP*, **11**, (2002), 018, `hep-th/0210060`.

[66] W. Kummer and P. Widerin, *Phys. Rev.*, **D52**, (1995), 6965, `arXiv:gr-qc/9502031`.

[67] V. Frolov and I. Novikov, *Black Hole Physics*. Kluwer Academic Publishers, 1998.

[68] L. D. Faddeev, *Sov. Phys. Usp.*, **25**, (1982), 130.

[69] H. Liebl, D. V. Vassilevich, and S. Alexandrov, *Class. Quant. Grav.*, **14**, (1997), 889, `arXiv:gr-qc/9605044`.

[70] V. P. Frolov, *Phys. Rev.*, **D46**, (1992), 5383.

[71] R. B. Mann, *Phys. Rev.*, **D47**, (1993), 4438, `hep-th/9206044`.

[72] H. Grosse, W. Kummer, P. Presnajder, and D. J. Schwarz, *J. Math. Phys.*, **33**, (1992), 3892, `hep-th/9205071`.

[73] R. M. Wald, *Living Rev. Rel.*, **4**, (2001), 6, `gr-qc/9912119`.

[74] G. W. Gibbons and S. W. Hawking, eds., *Euclidean quantum gravity.* Singapore: World Scientific, 1993.

[75] W. Kummer and D. V. Vassilevich, *Annalen Phys.*, **8**, (1999), 801, `gr-qc/9907041`.

[76] R. M. Wald, *Phys. Rev.*, **D48**, (1993), 3427, `gr-qc/9307038`.

V. Iyer and R. M. Wald, *Phys. Rev.*, **D50**, (1994), 846, `gr-qc/9403028`.

[77] J. Gegenberg, G. Kunstatter, and D. Louis-Martinez, *Phys. Rev.*, **D51**, (1995), 1781, `gr-qc/9408015`.

[78] H. W. J. Bloete, J. L. Cardy, and M. P. Nightingale, *Phys. Rev. Lett.*, **56**, (1986), 742.

J. L. Cardy, *Nucl. Phys.*, **B270**, (1986), 186.

[79] A. Strominger and C. Vafa, *Phys. Lett.*, **B379**, (1996), 99, `hep-th/9601029`.

K. V. Krasnov, *Gen. Rel. Grav.*, **30**, (1998), 53, `gr-qc/9605047`.

A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, *Phys. Rev. Lett.*, **80**, (1998), 904, `gr-qc/9710007`.

M. Cadoni and S. Mignemi, *Phys. Rev.*, **D59**, (1999), 081501, `hep-th/9810251`.

S. Carlip, *Phys. Rev. Lett.*, **82**, (1999), 2828, `hep-th/9812013`.

S. N. Solodukhin, *Phys. Lett.*, **B454**, (1999), 213, `hep-th/9812056`.

M.-I. Park and J. H. Yee, *Phys. Rev.*, **D61**, (2000), 088501, `hep-th/9910213`.

S. Carlip, *Phys. Rev. Lett.*, **83**, (1999), 5596, `hep-th/9910247`.

M.-I. Park and J. Ho, *Phys. Rev. Lett.*, **83**, (1999), 5595, `hep-th/9910158`.

I. Sachs and S. N. Solodukhin, *Phys. Rev.*, **D64**, (2001), 124023, `hep-th/0107173`.

M.-I. Park, *Nucl. Phys.*, **B634**, (2002), 339, `hep-th/0111224`.

N. Pinamonti and L. Vanzo, *Phys. Rev.*, **D69**, (2004), 084012, `hep-th/0312065`.

G. Kang, J.-I. Koga, and M.-I. Park, *Phys. Rev.*, **D70**, (2004), 024005, `hep-th/0402113`.

[80] D. Grumiller, in *Path Integrals from Quantum Information to Cosmology*, C. Burdik, N. Navratil, and S. Posta, eds. JINR Publishing Department, Prague, June, 2005. `hep-th/0506175`.

[81] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Phys. Rept.*, **323**, (2000), 183, `hep-th/9905111`.

[82] J. D. Brown, J. Creighton, and R. B. Mann, *Phys. Rev.*, **D50**, (1994), 6394, `gr-qc/9405007`.

[83] M. E. Olsson, `hep-th/0511106`.

[84] J. A. Harvey and A. Strominger, in *Recent directions in particle theory: from superstrings and black holes to the standard model (TASI - 92),*. 1992. `hep-th/9209055`.

A. Strominger, `arXiv:hep-th/9501071`. Talk given at NATO Advanced Study Institute.

L. Thorlacius, *Nucl. Phys. Proc. Suppl.*, **41**, (1995), 245, `hep-th/9411020`.

S. B. Giddings, *Trieste HEP Cosmology*, (1994), 0530, `arXiv:hep-th/9412138`.

[85] M. J. Duff, *Class. Quant. Grav.*, **11**, (1994), 1387, `hep-th/9308075`.

[86] S. M. Christensen and S. A. Fulling, *Phys. Rev.*, **D15**, (1977), 2088.

[87] V. Mukhanov, A. Wipf, and A. Zelnikov, *Phys. Lett.*, **B332**, (1994), 283, `hep-th/9403018`.

[88] T. Chiba and M. Siino, *Mod. Phys. Lett.*, **A12**, (1997), 709.

S. Ichinose, *Phys. Rev.*, **D57**, (1998), 6224, `hep-th/9707025`.

W. Kummer, H. Liebl, and D. V. Vassilevich, *Mod. Phys. Lett.*, **A12**, (1997), 2683, `hep-th/9707041`.

[89] K. I. Izawa, *Prog. Theor. Phys.*, **103**, (2000), 225, `hep-th/9910133`.

[90] O. Lauscher and M. Reuter, *Phys. Rev.*, **D65**, (2002), 025013, `hep-th/0108040`.

[91] A. Bonanno and M. Reuter, `hep-th/0602159`.

[92] M. Niedermaier, *Nucl. Phys.*, **B673**, (2003), 131, `hep-th/0304117`.

[93] V. P. Frolov and G. A. Vilkovisky, *Phys. Lett.*, **B106**, (1981), 307.

[94] M. K. Parikh and F. Wilczek, *Phys. Lett.*, **B449**, (1999), 24, `gr-qc/9807031`.

[95] A. Ashtekar and M. Bojowald, *Class. Quant. Grav.*, **22**, (2005), 3349, `gr-qc/0504029`.

[96] S. A. Hayward, *Phys. Rev. Lett.*, **96**, (2006), 031103, `gr-qc/0506126`.

[97] S. Carlip, *Class. Quant. Grav.*, **22**, (2005), 1303, `hep-th/0408123`; `gr-qc/0508071`; `gr-qc/0601041`.

[98] L. Bergamin, D. Grumiller, W. Kummer, and D. V. Vassilevich, *Class. Quant. Grav.*, **23**, (2006), 3075, `hep-th/0512230`.

[99] G. 't Hooft, `gr-qc/0401027`.

[100] K. V. Kuchař, *Phys. Rev.*, **D50**, (1994), 3961, `gr-qc/9403003`.

[101] M. W. Choptuik, *Phys. Rev. Lett.*, **70**, (1993), 9.

[102] C. Gundlach, *Adv. Theor. Math. Phys.*, **2**, (1998), 1, `arXiv:gr-qc/9712084`.

[103] A. Strominger and L. Thorlacius, *Phys. Rev. Lett.*, **72**, (1994), 1584, `hep-th/9312017`.

[104] J. G. Russo, L. Susskind, and L. Thorlacius, *Phys. Rev.*, **D47**, (1993), 533, `hep-th/9209012`.

[105] Y. Peleg, S. Bose, and L. Parker, *Phys. Rev.*, **D55**, (1997), 4525, `gr-qc/9608040`.

[106] M. Birukou, V. Husain, G. Kunstatter, E. Vaz, and M. Olivier, *Phys. Rev.*, **D65**, (2002), 104036.

[107] E. Sorkin and Y. Oren, *Phys. Rev.*, **D71**, (2005), 124005, `hep-th/0502034`.

[108] J. Bland, B. Preston, M. Becker, G. Kunstatter, and V. Husain, *Class. Quant. Grav.*, **22**, (2005), 5355.

[109] S. Husa, C. Lechner, M. Purrer, J. Thornburg, and P. C. Aichelburg, *Phys. Rev.*, **D62**, (2000), 104007, `gr-qc/0002067`.

[110] M. Purrer, S. Husa, and P. C. Aichelburg, *Phys. Rev.*, **D71**, (2005), 104005, `gr-qc/0411078`.

[111] J. Kettner, G. Kunstatter, and A. J. M. Medved, *Class. Quant. Grav.*, **21**, (2004), 5317, `gr-qc/0408042`.

[112] H.-P. Nollert, *Phys. Rev.*, **D47**, (1993), 5253.

N. Andersson, *Class. Quant. Grav.*, **L10**, (1993), 61.

[113] L. Motl, *Adv. Theor. Math. Phys.*, **6**, (2003), 1135, `gr-qc/0212096`.

[114] L. Motl and A. Neitzke, *Adv. Theor. Math. Phys.*, **7**, (2003), 307, `hep-th/0301173`.

[115] W. G. Unruh, *Phys. Rev. Lett.*, **46**, (1981), 1351.

[116] M. Novello, M. Visser, and G. Volovik, eds., *Artificial black holes.* World Scientific, River Edge, USA, 2002.

G. E. Volovik, *The universe in a helium droplet.* Clarendon, Oxford, UK, 2003.

C. Barcelo, S. Liberati, and M. Visser, *Living Rev. Rel.*, **8**, (2005), 12, `gr-qc/0505065`.

[117] P. O. Fedichev and U. R. Fischer, *Phys. Rev. Lett.*, **91**, (2003), 240407, `cond-mat/0304342`.

[118] R. Balbinot, S. Fagnocchi, A. Fabbri, and G. P. Procopio, *Phys. Rev. Lett.*, **94**, (2005), 161302, `gr-qc/0405096`.

M. Cadoni, *Class. Quant. Grav.*, **22**, (2005), 409, `gr-qc/0410138`.

M. Cadoni and S. Mignemi, *Phys. Rev.*, **D72**, (2005), 084012, `gr-qc/0504143`.

[119] R. Schutzhold, M. Uhlmann, Y. Xu, and U. R. Fischer, *Phys. Rev.*, **D72**, (2005), 105005, `cond-mat/0503581`.

[120] D. Grumiller, "Black holes and analogues in two dimensions." talk presented at QUASIM05 in Dresden, July, 2005.

[121] W. Kummer, H. Liebl, and D. V. Vassilevich, *Nucl. Phys.*, **B544**, (1999), 403, `hep-th/9809168`.

[122] D. Grumiller, W. Kummer, and D. V. Vassilevich, *Nucl. Phys.*, **B580**, (2000), 438, `gr-qc/0001038`.

[123] D. Grumiller, *Class. Quant. Grav.*, **19**, (2002), 997, `gr-qc/0111097`; *Int. J. Mod. Phys.*, **D13**, (2004), 1973, `hep-th/0409231`.

[124] W. Kummer, H. Liebl, and D. V. Vassilevich, *Nucl. Phys.*, **B493**, (1997), 491, `gr-qc/9612012`; "Exact path integral quantization of 2-d dilaton gravity," `gr-qc/9710033`.

[125] W. Kummer, "Progress and problems in quantum gravity," `gr-qc/0512010`.

D. Grumiller and W. Kummer, "How to approach quantum gravity: Background independence in 1+1 dimensions," in *What comes beyond the Standard Model? Symmetries beyond the standard model*, N. M. Borstnik, H. B. Nielsen, C. D. Froggatt, and D. Lukman, eds., vol. 4 of *Bled Workshops in Physics*, p. 184, EURESCO. Portoroz, Slovenia, July, 2003. `gr-qc/0310068`. based upon two talks.

[126] W. Kummer, *Eur. Phys. J.*, **C21**, (2001), 175, `hep-th/0104123`.

[127] D. Grumiller, W. Kummer, and D. V. Vassilevich, *JHEP*, **07** (2003), 009, `hep-th/0305036`.

[128] P. Fischer, D. Grumiller, W. Kummer, and D. V. Vassilevich, *Phys. Lett.*, **B521**, (2001), 357, `gr-qc/0105034`. Erratum ibid. **B532**, (2002), 373.

D. Grumiller, *Quantum dilaton gravity in two dimensions with matter.* PhD thesis, Technische Universität Wien, 2001. `gr-qc/0105078`.

[129] H. Balasin, W. Kummer, O. Piguet, and M. Schweda, *Phys. Lett.*, **B287**, (1992), 138.

[130] A. M. Polyakov, *Phys. Lett.*, **B103**, (1981), 207.

[131] S. Nojiri and I. Oda, *Phys. Lett.*, **B294**, (1992), 317, `hep-th/9206087`.

A. Ori, *Phys. Rev.*, **D63**, (2001), 104016, `gr-qc/0102067`.

[132] J. Wess and B. Zumino, *Phys. Lett.*, **B37**, (1971), 95.

[133] S. R. Coleman, *Phys. Rev.*, **D11**, (1975), 2088.

S. Mandelstam, *Phys. Rev.*, **D11**, (1975), 3026.

[134] A. V. Frolov, K. R. Kristjansson, and L. Thorlacius, *Phys. Rev.*, **D72**, (2005), 021501, `hep-th/0504073`.

[135] V. Guillemin and A. Pollack, *Differential Topology.* Englewood Cliffs, NJ: Prentice-Hall, 1974.

[136] J. Ambjorn, R. Loll, J. L. Nielsen, and J. Rolf, *Chaos Solitons Fractals*, **10**, (1999), 177, `hep-th/9806241`.

[137] R. Gastmans, R. Kallosh, and C. Truffin, *Nucl. Phys.*, **B133**, (1978), 417.

S. M. Christensen and M. J. Duff, *Phys. Lett.*, **B79**, (1978), 213.

S. Weinberg in *General Relativity, an Einstein Centenary Survey*, S. Hawking and W. Israel, eds. Cambridge University Press, 1979.

[138] R. B. Mann and S. F. Ross, *Class. Quant. Grav.*, **10**, (1993), 345, `gr-qc/9208004`.

[139] R. Jackiw, `hep-th/0511065`.

[140] S. Deser, R. Jackiw, and S. Templeton, *Ann. Phys.*, **140**, (1982), 372; *Erratum-ibid.*, **185**, (1988), 406; *Phys. Rev. Lett.*, **48**, (1982), 975.

[141] M. Banados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.*, **69**, (1992), 1849, `hep-th/9204099`.

M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev.*, **D48**, (1993), 1506, `gr-qc/9302012`.

[142] B. Sahoo and A. Sen, `hep-th/0601228`.

[143] W. Kummer and D. J. Schwarz, *Nucl. Phys.*, **B382**, (1992), 171; *Phys. Rev.*, **D45**, (1992), 3628.

[144] A. Kotov, P. Schaller, and T. Strobl, **260**, (2005), 455, `hep-th/0411112`.

[145] H. Nicolai, D. Korotkin, and H. Samtleben, `hep-th/9612065`.

D. Korotkin and H. Samtleben, *Phys. Rev. Lett.*, **80**, (1998), 14, `gr-qc/9705013`; *Nucl. Phys.*, **B527**, (1998), 657, `hep-th/9710210`.

D. Bernard and N. Regnault, *Commun. Math. Phys.*, **210** (2000), 177, `solv-int/9902017`.

L. D. Faddeev, R. M. Kashaev, and A. Y. Volkov, *Commun. Math. Phys.*, **219**, (2001), 199, `hep-th/0006156`.

J. Teschner, *Class. Quant. Grav.*, **18**, (2001), R153, `hep-th/0104158`.

[146] M. Gurses and S. Tek, "KdV Surfaces," `nlin.si/0511049`.

[147] H.-J. Schmidt, *Gen. Rel. Grav.*, **31**, (1999), 1187, `gr-qc/9905051`.

[148] Y. N. Obukhov and F. W. Hehl, `hep-th/9807101`.

[149] T. Dereli and R. W. Tucker, *Class. Quant. Grav.*, **11**, (1994), 2575.

Y. N. Obukhov, *Phys. Rev.*, **D69**, (2004), 064009, `gr-qc/0311091`.

M. Adak, `gr-qc/0509010`.

[150] M. R. Douglas and N. A. Nekrasov, *Rev. Mod. Phys.*, **73**, (2001), 977, `hep-th/0106048`.

R. J. Szabo, *Phys. Rept.*, **378**, (2003), 207, `hep-th/0109162`.

[151] M. Buric and J. Madore, `hep-th/0406232`; *Phys. Lett.*, **B622**, (2005), 183, `hep-th/0507064`.

[152] S. Cacciatori *et al.*, *Class. Quant. Grav.*, **19**, (2002), 4029, `hep-th/0203038`.

[153] D. V. Vassilevich, *Nucl. Phys.*, **B715**, (2005), 695, `hep-th/0406163`.

[154] D. V. Vassilevich, "Constraints, gauge symmetries, and noncommutative gravity in two dimensions," `hep-th/0502120`.

[155] D. V. Vassilevich, R. Fresneda, and D. M. Gitman, "Stability of a noncommutative Jackiw-Teitelboim gravity," `hep-th/0602095`.

[156] D. Grumiller, W. Kummer, and D. V. Vassilevich, *Ukr. J. Phys.*, **48**, (2003), 329, `hep-th/0301061`.

[157] P. Aschieri *et al.*, *Class. Quant. Grav.*, **22**, (2005), 3511, `hep-th/0504183`.

[158] B. M. Zupnik, `hep-th/0512231`.

[159] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, *Phys. Lett.*, **B604**, (2004), 98, `hep-th/0408069`.
J. Wess, "Deformed coordinate spaces: Derivatives," `hep-th/0408080`.

[160] D. V. Vassilevich, "Twist to close," `hep-th/0602185`.
P. Aschieri, M. Dimitrijevic, F. Meyer, S. Schraml, and J. Wess, "Twisted gauge theories," `hep-th/0603024`.

[161] M. Chaichian, P. Presnajder, and A. Tureanu, *Phys. Rev. Lett.*, **94**, (2005), 151602, `hep-th/0409096`.
P. Aschieri, M. Dimitrijevic, F. Meyer, and J. Wess, *Class. Quant. Grav.*, **23**, (2006), 1883, `hep-th/0510059`.
A. Kobakhidze, "Theta-twisted gravity," `hep-th/0603132`.

[162] R. Oeckl, *Nucl. Phys.*, **B581**, (2000), 559, `hep-th/0003018`.

[163] M. Bojowald, A. Kotov, and T. Strobl, *J. Geom. Phys.*, **54**, (2005), 400, `math.dg/0406445`.

[164] D. Grumiller and R. Jackiw, *Phys. Lett., B*, (2006), in press, hep-th/0609197.