

Equivariant Localization for SUSY Quantum Mechanics

Levent AKANT

*Feza Gursev Institute, Emek Mah. Rasathane Yolu No:68,
34684 Cengelkoy, İstanbul-TURKEY
e-mail: akant@gursev.gov.tr*

Received 14.04.2006

Abstract

We apply equivariant localization to supersymmetric quantum mechanics and show that the partition function localizes on the instantons of the theory. Our construction of equivariant cohomology for SUSY quantum mechanics is different than the ones that already exist in the literature. A hidden bosonic symmetry is made explicit and the supersymmetry is extended. New bosonic symmetry is the square of the new fermionic symmetry. The D term is now the parameter of the bosonic symmetry. This construction provides us with an equivariant complex together with a Cartan differential and makes the use of localization principle possible.

Key Words: Supersymmetric quantum mechanics; equivariant cohomology.

1. Introduction

The aim of this article is to apply equivariant localization to SUSY quantum mechanics [1] and show that the partition function of the theory localizes on the instantons of the action. This is a well known result that can also be derived by using arguments based on Nicolai map [2], [3]. The advantage of localization principle is that it is a very general result, almost as general as Stoke's theorem of ordinary de Rham theory, which can easily be applied whenever the right setting, namely the equivariant cohomology, is present. There are important instances in quantum field theory where such a setting is available [4–9]. We will show that SUSY quantum mechanics is another example which accommodates equivariant cohomology, and use localization principle to derive the above mentioned result. Our work will also make the symplectic structure of the theory more explicit.

The construction presented in this article is different than the one used by Morozov, Niemi and Palo in their work on SUSY theories [9] and the one used by Cordes, Moore and Ramgoolam in [10]. The main difference is that in our case the BRST like operator which gives rise to equivariant localization is not nilpotent; instead it squares to a hidden bosonic symmetry of the theory. So the BRST operator is nilpotent only on the gauge invariant observables of the theory. With this, our construction becomes a special abelian case of the general construction given by Witten in [6].

Equivariant cohomology is a generalization of de Rham theory to manifolds with group actions. If a group G acts freely on a manifold M then equivariant cohomology is defined to be the de Rham cohomology of the quotient manifold M/G . If the action is not free it is still possible to define equivariant cohomology as cohomology of certain complexes associated with M and G . Two commonly used complexes are the Weil complex and the Cartan complex. Resulting cohomologies are equivalent and reduce to de Rham cohomology of M/G in the case of free actions. In this article we will use the Cartan model of equivariant cohomology. For reviews of equivariant cohomology see [10–14]

The relevance of equivariant cohomology to quantum field theory is that the field content of theories with BRST symmetry fit into an equivariant complex (Cartan complex). Therefore one can hope to apply many interesting and useful results of equivariant cohomology to the analysis of these quantum field theories. One such result is localization.

In de Rham theory one can integrate differential forms. The theorems of classical integral calculus such as Stoke’s theorem and divergence theorem are naturally unified in exterior calculus and this provides us with great technical prowess and deep geometric insights. In equivariant theory one can integrate equivariant forms. The most interesting result in equivariant integration is localization. It means that the integral of a closed, but otherwise arbitrary, equivariant form over a large region is equal to the integral of the same form over a much smaller subregion. In quantum field theory this means that only a subset of field configurations contribute to the path integral. A particularly useful special case of localization arises when the manifold is symplectic and the action of the group is Hamiltonian. All the known applications of equivariant localization to field theory, including this one, involve infinite dimensional analogs of this case.

Localization was first proved by Duistermaat and Heckman in the case of torus actions on symplectic manifolds [15]. Nonabelian generalization was given by Witten in [6]. A more rigorous treatment is given in [16].

Our analysis of localization in SUSY quantum mechanics will proceed as follows.

As a first step we show that the field content and symmetries of the theory allow us to embed the theory into an equivariant complex. At this point we encounter an obstruction. It seems that a bosonic field is missing from the theory. The cure is to add one extra field, modify the symmetry of the theory by making it act on the new variable nontrivially and do all this without changing the action. As a result of this construction we obtain a modified fermionic symmetry and a new bosonic symmetry with the crucial property that the latter is the square of the former. This allows us to interpret the fermionic symmetry operator as an equivariant differential operator and thus embed the theory into an equivariant complex. We note that the extra bosonic field we introduce is not the usual D term which is already present in the action. In fact in our construction the D term becomes the parameter of the bosonic symmetry. Next we show that the action of the theory is an equivariantly closed form. Using this fact, we apply localization principle to the partition function and show that the path integral localizes on the instantons of the action.

In the first few sections of the article we review the basics of equivariant cohomology and discuss the field theoretic generalizations. In Sec.4 and the first part of Sec.5, following [6], we derive localization formula and show how it can be applied to localize integrals on symplectic manifolds with Hamiltonian group action. The remainder of the article is devoted to the discussion of SUSY quantum mechanics in the context of equivariant cohomology and localization principle.

2. Equivariant Cohomology

Let M be a smooth manifold and G a simple, compact Lie group with a smooth action on M . We will denote the Lie algebra of G by \mathcal{G} . A typical element in \mathcal{G} will be written as $\phi^a T_a$ where T_a ’s are the generators of the Lie algebra. Killing-Cartan form will be denoted by $(,)$. The infinitesimal generator of the action corresponding to the element $\sum_a \phi^a T_a$ of the Lie algebra is $V(\phi) = \sum_a \phi^a V_a$ where V_a is the infinitesimal generator corresponding to T_a . From now on we will use summation convention.

We will also need the algebra of formal power series over \mathcal{G} ; it will be denoted by $P(\mathcal{G})$. This algebra is isomorphic to the symmetric algebra on the dual of the Lie algebra

$$P(\mathcal{G}) \cong S(\mathcal{G}^*). \tag{1}$$

As a consequence of this isomorphism G acts on $P(\mathcal{G})$ by the co-adjoint action whose infinitesimal version is

$$ad^*(\psi^b T_b)\phi^a = -\psi^b c_{bc}^a \phi^c. \tag{2}$$

Here c_{bc}^a are the structure constants of \mathcal{G} .

$\Omega_G^*(M)$ is defined as the G -invariant part of the tensor product of the exterior algebra on M with $P(\mathcal{G})$: $\Omega_G^*(M) = (\Omega^*(M) \otimes P(\mathcal{G}))_G$. A typical element of Ω_G^* is of the form $\sum_k \alpha^k P_k(\phi)$ and is invariant under the

group action whose infinitesimal version is

$$\mathcal{L}_{V(\psi)} \otimes 1 + 1 \otimes ad^*(\psi^b T_b). \tag{3}$$

Here α^k is a k-form on M , P_k is a polynomial in the Lie algebra and $\mathcal{L}_{V(\psi)}$ is the Lie derivative with respect to the infinitesimal generator $V(\psi)$. Multiplication in this algebra is given by

$$\alpha^k P_k(\phi) \cdot \beta^l R_l(\phi) = (\alpha^k \wedge \beta^l) P_k(\phi) R_l(\phi). \tag{4}$$

Alternatively one can represent elements of $\Omega_G^*(M)$ as polynomials in \mathcal{G} with differential forms as coefficients

$$\sum_I \sum_k \alpha_I^k \phi^I. \tag{5}$$

Here I is a multi-index, α_I^k a k-form on M and ϕ^I a monomial in $P(\mathcal{G})$. This representation allows us to define a grading on $(\Omega_G^*(M))$ by declaring the degree of $\alpha_I^k \phi^I$ to be $k + 2 | I |$.

$$\Omega_G^*(M) = \bigoplus_{i \geq 0} \Omega_G^i(M) \tag{6}$$

where

$$\Omega_G^i(M) = span\{\alpha_I^k \phi^I \in \Omega_G^*(M) : i = k + 2 | I |\}. \tag{7}$$

Now let us consider the operator

$$D_\phi = d - i\phi^a \iota_{V_a} \tag{8}$$

Here ι is the contraction. The action on $\Omega_G^*(M)$ is given by

$$D_\phi \sum_I \sum_k \alpha_I^k \phi^I = [d\alpha_I^k + i\phi^a \iota_{V_a} \alpha_I^k] \phi^I. \tag{9}$$

Note that

$$D_\phi : \Omega_G^i \rightarrow \Omega_G^{i+1} \tag{10}$$

and

$$D_\phi^2 = -i\mathcal{L}_{V(\phi)} \tag{11}$$

the last statement implies that $D_\phi^2 = 0$ on $\Omega_G^*(M)$.

It is also easy to show that for $\alpha \in A_i$ and $\beta \in A_j$

$$D_\phi(\alpha\beta) = (D_\phi\alpha)\beta + (-1)^i \alpha D_\phi\beta \tag{12}$$

therefore D is a graded differential of degree 1.

Now we have all the ingredients needed for a cohomology theory, namely, a graded algebra and a graded differential operator of degree 1. The resulting cohomology is called the Cartan model of equivariant cohomology and is denoted by $H_G^*(M)$.

3. Field Theoretic Generalization

Let us now show how equivariant cohomology arises in quantum field theories with BRST like symmetries. More precisely we will consider field theories which involve three types of fields: dynamical bosonic fields A_a , anticommuting ghosts ψ^a and auxiliary bosonic fields ϕ^a . We will require the theory to possess two symmetries; one bosonic and one fermionic. The bosonic symmetry, which from now on will be referred to as gauge symmetry, will effect only A_a 's by mixing them with ϕ^a 's. The fermionic symmetry will be called BRST symmetry and the infinitesimal action will be denoted by δ_{BRST} . The crucial point is that these two symmetries are related to each other by the following relations

$$\delta_{BRST} A^a = \psi^a \tag{13}$$

$$\delta_{BRST} \psi^a = -i\delta_{gauge} A^a \tag{14}$$

$$\delta_{BRST} \phi^a = 0 \tag{15}$$

which implies

$$\delta_{BRST}^2 = -i\delta_{gauge}. \tag{16}$$

Roughly speaking one can define the BRST symmetry as the square root of the gauge symmetry. Note that it is the inclusion of the ghosts to the theory which makes it possible to take the square root of the gauge transformation. Incidentally this is very similar to the definition of imaginary numbers as square roots of negative numbers. In order to take square roots of negative numbers we have to enlarge \mathbf{R} to $\mathbf{C} = \mathbf{R} \oplus \mathbf{R}$.

A prototypical example of a BRST like theory is of course the gauge fixed Yang-Mills theory in the first order formalism where A_a 's are the gauge fields, ψ^a 's are the Faddeev-Popov ghosts and ϕ^a 's are the Lie algebra valued auxiliary fields.

Note that the BRST operator is nilpotent (of order 2) on the gauge invariant sector of the theory. This is again the familiar situation encountered in gauge theories.

In order to see the relevance of equivariant cohomology to field theory let us go back to the finite dimensional case and consider the action of the equivariant differential on the generators of $\Omega_G(M)$

$$D_\phi x^i = dx^i \tag{17}$$

$$D_\phi dx^i = -i\phi^a V_a^i \tag{18}$$

$$D_\phi \phi^a = 0. \tag{19}$$

Comparing this with the relation between the gauge and BRST transformations, and using the following identifications in passing from finite dimensional case to field theory

$$x^i \rightarrow A^a \tag{20}$$

$$dx^i \rightarrow \psi^a \tag{21}$$

$$\phi^a \rightarrow \phi^a \tag{22}$$

we conclude that δ_{BRST} and δ_{gauge} are the field theory analogs of D_ϕ and $\mathcal{L}_{V(\phi)}$ respectively.

Note that this identification also implies the interpretation of ghost fields as infinite dimensional differential forms.

Thus we see that our BRST like field theory provides us with an equivariant cohomology where the BRST operator plays the role of the Cartan differential D_ϕ .

4. Equivariant Localization

Let us go back to the finite dimensional case and see how one can integrate equivariant forms. The field theoretic generalization will involve path integrals.

The integral of an equivariant form is defined as

$$\oint \sum_k \alpha^k P_k(\phi) = \sum_k \int_M \alpha^k \int [d\phi] P_k(\phi). \tag{23}$$

Here the ϕ integration is over all \mathcal{G} and $[d\phi]$ is a measure on the Lie algebra which includes a Gaussian convergence factor

$$[d\phi] = \frac{d\phi^1 \dots d\phi^n}{VolG} e^{-\frac{\epsilon}{2}(\phi, \phi)} \tag{24}$$

A very useful property of this integral is that if M is without boundary then the integral of an equivariantly exact form vanishes. In other words integration by parts produces a vanishing surface term

$$\oint_M D\alpha = 0 \tag{25}$$

provided ∂M is empty.

Localization Principle. Let α be an equivariantly closed form and λ an equivariant 1-form. Then it is easy to show that the following form is equivariantly exact

$$\alpha(1 - e^{tD_\phi\lambda}). \tag{26}$$

Integrating this exact form over M we obtain the very important formula which forms the basis of equivariant localization

$$\oint \alpha = \oint \alpha e^{tD_\phi\lambda}. \tag{27}$$

So the integral on the right hand side is in fact independent of the value of the parameter t and is equal to the integral of α . In particular $\oint \alpha$ can be calculated by evaluating $\oint \alpha e^{tD_\phi\lambda}$ in the asymptotic limit $t \rightarrow \infty$. So let us take a closer look at this integral

$$\oint \alpha e^{tD_\phi\lambda} = \oint \alpha e^{td\lambda + i \sum_a \phi^a \lambda(V_a)}. \tag{28}$$

Since the two terms appearing in the argument of the exponential commute with each other we can separate the exponential as $e^{td\lambda} e^{i \sum_a \phi^a \lambda(V_a)}$. The contribution of the first factor to the integral is just a polynomial in t . Therefore in the limit $t \rightarrow \infty$ the integral is localized on the critical points of the function $\sum_a \phi^a \lambda(V_a)$:

$$\lambda(V_a) = 0 \tag{29}$$

and

$$\phi^a d\lambda(V_a) = 0. \tag{30}$$

From now on we will assume that the only solution of the last equation is $\phi^a = 0$ for all a .

5. Localization on Symplectic Manifolds

In this section we want to localize the integral of a specific equivariantly closed form on a symplectic manifold (M, ω) . Here ω is the symplectic form on M . We assume that a compact Lie group G with a simple Lie algebra \mathcal{G} acts on M . Moreover, the action will be assumed to be symplectic and Hamiltonian. The moment map μ_a corresponding to the action of the generator T_a of \mathcal{G} is given by

$$d\mu_a = -\iota_{V_a}\omega. \tag{31}$$

We will also define

$$I = \sum_a \mu_a^2. \tag{32}$$

The equivariant form that want to integrate is $e^{\bar{\omega}}$ where

$$\bar{\omega} = \omega - i\phi^a \mu_a. \tag{33}$$

is the so called equivariant symplectic form. An easy calculation [12, Chapter 9] shows that $\bar{\omega}$ is G -invariant and equivariantly closed.

The explicit form of the integral in question is

$$\int_M \int \frac{d\phi^1 \dots d\phi^n}{VolG} \exp \left[\omega - i\phi^a \mu_a - \frac{\epsilon}{2}(\phi, \phi) \right] \tag{34}$$

In order to apply localization to this integral we have to choose a convenient λ . This can be done if we assume that there exists a G -invariant almost complex structure J on M whose associated metric $g(X, Y) = \omega(JX, Y)$ is positive definite. Then

$$\lambda = JdI \tag{35}$$

is a suitable choice.

It is shown in [6] that the condition $\lambda(V_a) = 0$ for all a is equivalent to $dI = 0$. Thus the integral in question localizes on the critical set of the function I . Writing down the condition $dI = 0$ explicitly

$$\sum_a \mu_a d\mu_a = 0 \tag{36}$$

we see that there are two types of critical points: those coming from the simultaneous vanishing of moment maps and those that don't (higher critical points). In what follows we will ignore the higher critical points.

Example. Let us consider \mathbf{R}^{2n} with a symplectic form

$$\omega = \omega_{ij}(q) dp^i \wedge dq^j. \tag{37}$$

Here the matrix $\omega_{ij}(q)$ is a function of q coordinates only. The closedness of ω implies

$$\frac{\partial \omega_{jk}}{\partial q^i} - \frac{\partial \omega_{ji}}{\partial q^k} = 0 \tag{38}$$

Now the action of the additive group \mathbf{R}^n given by

$$q^i \rightarrow q^i \tag{39}$$

$$p^j \rightarrow p^j + a^j \tag{40}$$

is symplectic and Hamiltonian. Infinitesimal generators of this action are

$$V_k = \frac{\partial}{\partial p^k} \tag{41}$$

The moments are given implicitly by the solutions of

$$\frac{\partial \mu_k}{\partial q^i} = -\omega_{ki} \tag{42}$$

$$\frac{\partial \mu_k}{\partial p^i} = 0. \tag{43}$$

In this example we have relaxed the compactness condition on the group $G = \mathbf{R}^n$. This is not harmful in the case of \mathbf{R}^n which admits a bi-invariant measure anyway. In any case, it is a simple matter to put periodicity conditions on the coordinates and replace \mathbf{R}^n by an appropriate toric group.

6. Localization for SUSY Quantum Mechanics

Here we have a 0+1 dimensional theory on the circle with the action

$$S = \int dt \left[\frac{dx}{dt} + \frac{\partial V(x)}{\partial x} \right] (iD) + \frac{1}{2} F^2 - i\bar{\psi} \left[\frac{d}{dt} + \frac{\partial^2 V(x)}{\partial x^2} \right] \psi. \tag{44}$$

This theory is invariant under a SUSY transformation of the form

$$\delta_{SUSY} x = \psi \tag{45}$$

$$\delta_{SUSY} \psi = 0 \tag{46}$$

$$\delta_{SUSY} \bar{\psi} = -i(iD) \tag{47}$$

$$\delta_{SUSY} (iD) = 0 \tag{48}$$

$$\delta_{SUSY}^2 = 0. \tag{49}$$

We see that this SUSY transformation is nilpotent of degree 2. However δ_{SUSY}^2 vanishes identically and therefore our theory does not seem to be a BRST like theory as defined above. The problem is that there

is no bosonic gauge symmetry at sight. The solution is to introduce an extra variable without changing the theory and extend the SUSY transformation in such a way that its square gives us a new bosonic symmetry. Thus we introduce a new bosonic field π and replace the SUSY transformation by the BRST transformation

$$\delta_{BRST} x = \psi \quad (50)$$

$$\delta_{BRST} \psi = 0 \quad (51)$$

$$\delta_{BRST} \pi = \bar{\psi} \quad (52)$$

$$\delta_{BRST} \bar{\psi} = -i(iD) \quad (53)$$

$$\delta_{BRST}(iD) = 0. \quad (54)$$

or

$$\delta_{BRST} = \int dt \psi(t) \frac{\delta}{\delta x(t)} + \bar{\psi}(t) \frac{\delta}{\delta \pi(t)} - i(iD)(t) \frac{\delta}{\delta \bar{\psi}(t)}. \quad (55)$$

Now we have

$$\delta_{BRST}^2 = -i\delta_{gauge} \quad (56)$$

where

$$\delta_{gauge} \pi = iD \quad (57)$$

$$\delta_{gauge}(x, \psi, \bar{\psi}, iD) = 0. \quad (58)$$

or

$$\delta_{gauge} = \int dt (iD)(t) \frac{\delta}{\delta \pi(t)} \quad (59)$$

Note that δ_{BRST} agrees with δ_{SUSY} on the original variables so it is a symmetry of the theory. Moreover δ_{gauge} is a symmetry since it effects only the extra variable π which does not even appear in the action. In this way we turn our theory into a BRST like theory.

Our next task is to look for possible applications of localization. We will show that the partition function localizes on the instantons of the theory.

Symplectic Geometry of the Field Space and Localization. The bosonic fields of the theory are the coordinates of the field space $M = (x(t), \pi(t))$. We will assume periodic boundary conditions for these variables. Ghost fields are interpreted as differentials of the coordinates $\delta x(t) = \psi(t)$ and $\delta \pi(t) = \bar{\psi}(t)$.

The (pre)symplectic structure on the field space is defined as

$$\int dt dt' \bar{\psi}(t') \omega[t', t; x] \psi(t) \quad (60)$$

where

$$\omega[t', t; x] = i \left[-\frac{d}{dt} + \frac{\partial^2 V(x)}{\partial x^2} \right] \delta(t - t'). \quad (61)$$

This form is closed since

$$\frac{\delta \omega[t', t; x]}{\delta x(t'')} = V'''(x(t')) \delta(t' - t'') \delta(t' - t) \quad (62)$$

$$\frac{\delta \omega[t', t''; x]}{\delta x(t)} = V'''(x(t')) \delta(t' - t) \delta(t' - t'') \quad (63)$$

$$\frac{\delta \omega[t', t; x]}{\delta x(t'')} - \frac{\delta \omega[t', t''; x]}{\delta x(t)} = 0. \quad (64)$$

The action of the gauge transformation is symplectic since

$$\delta_{gauge}(x, \bar{\psi}, \psi) = (0, 0, 0) \quad (65)$$

$$\delta_{gauge} \int dt dt' \bar{\psi}(t') \omega[t', t; x] \psi(t) = 0. \quad (66)$$

In order to show that the action is Hamiltonian we have to solve the moment equations

$$\frac{\delta\mu[t';x]}{\delta x(t)} = -\omega[t',t;x] \tag{67}$$

$$\frac{\delta\mu[t';x]}{\delta\pi(t)} = 0. \tag{68}$$

These can be integrated to give

$$\mu[t';x] = i \left[\frac{dx(t')}{dt'} + V'(x(t')) \right]. \tag{69}$$

So the action is

$$S = \bar{\omega}[x] + \int dt \frac{1}{2} D^2. \tag{70}$$

where

$$\bar{\omega}[x] = \omega[x] - i \int dt \mu[t;x](iD)(t) \tag{71}$$

is the equivariant symplectic form on the field space.

The partition function is

$$Z = \int \mathcal{D}D \mathcal{D}x \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp i \left[\omega[x] - i \int dt \mu[t;x](iD)(t) + \int dt \frac{1}{2} D^2 \right]. \tag{72}$$

This is almost in the form of equation (34). All we need to do to arrive at the correct form (up to a harmless factor of i) is to integrate over the new variable π and divide by the volume of the gauge group. Since $VolG = \int \mathcal{D}\pi$ these operations do not change the value of the partition function. So we conclude that the partition function localizes on the zeroes of the moment map, that is, the instantons of the theory

$$\left[\frac{dx(t)}{dt} + V'(x(t)) \right] = 0. \tag{73}$$

Integrating the square of this expression and using periodicity to discard the surface term yields:

$$\frac{dx(t)}{dt} = 0 \tag{74}$$

$$V'(x(t)) = 0, \tag{75}$$

so localization is on the critical points of the potential V . Moreover it is possible to show that for these configurations the (pre)symplectic form ω is invertible and thus in fact is a symplectic form [17]. The effect of higher critical points will also be discussed in the cited reference.

Acknowledgement

I would like to thank O.T. Turgut for helpful discussions.

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