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Bianchi Type VI_1 Viscous Fluid Cosmological Model in Wesson's Theory of Gravitation

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Abstract

Field equations of a scale invariant theory of gravitation proposed by Wesson [1, 2] are obtained in the presence of viscous fluid with the aid of Bianchi type VI_h space-time with the time dependent gauge function (Dirac gauge). It is found that Bianchi type VI_h ($h = 1$) space-time with viscous fluid is feasible in this theory, whereas Bianchi type VI_h (h = -1, 0) space-times are not feasible in this theory, even in the presence of viscosity. For the feasible case, by assuming a relation connecting viscosity and metric coefficient, we have obtained a nonsingular-radiating model. We have discussed some physical and kinematical properties of the models.

Key Words: Gauge function, viscous fluid, scale invariant theory.

1. Introduction

In recent years there has been much interest in alternative theories of gravitation. Among them the scalar tensor theory of gravitation, the scale invariant and scale covariant theories of gravitation are noteworthy. Wesson [1, 2] proposed a scale invariant theory of gravitation incorporating the Dirac gauge function $\beta\ (x^i)$, where x^i are coordinates in the four dimensional space time and the tensor field is identified with Riemannian tensor g_{ij}. Scale invariant theory [1, 2] has been shown to agree with observations involving general relativity conducted thus far. It is said [3, 4] that Wesson's formulation of scale invariant theory of gravitation is so far the best theory to describe all interactions between the matter field and gravitation.

The field equations for scale invariant theory formulated by Wesson with Dirac gauge function $\beta\ (x^i)$ are

$$G_{ij} + 2\frac{\beta_{;ij}}{\beta} - 4\frac{\beta_{,i}\beta_{,j}}{\beta^2} + \left(g^{ab}\frac{\beta_{,a}\beta_{,b}}{\beta^2} - 2g^{ab}\frac{\beta_{;ab}}{\beta}\right)g_{ij} + \Lambda_0\beta^2 g_{ij} = -\kappa T_{ij}, \tag{1}$$

with

$$G_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij}. \tag{2}$$

Here, G_{ij} is the usual Einstein tensor, T_{ij} is the energy momentum tensor, R_{ij} is the Ricci tensor and R is the Ricci scalar. Also, the coma (,) and semicolon (;) notation in the subscripts, respectively, denote partial and conventional covariant differentiation. The cosmological term Λg_{ij} of Einstein theory is now transformed to $\Lambda_0\beta^2 g_{ij}$ in scale invariant theory with the dimensionless constant Λ_0. G and κ are the Newtonian gravitational constant and Wesson gravitational constant, respectively.

Dirac [3, 4], Hoyle and Naralikar [5], Canuto et al. [6, 7], Mohanty and Daud [8], Mohanty and Mishra [9, 10] and Mishra [11] are among some authors who have investigated several aspects of scale invariant theory of gravitation. In particular, Mohanty and Mishra [10] studied the feasibility of Bianchi type-VI$_h$ space-time in this theory with perfect fluid as a source. They showed Bianchi type VI$_h$ ($h = 1$) space time is the only possibility in this theory, and space-time with $h = -1$ and $h = 0$ are not feasible. It is well known that cosmological models with bulk viscosity in the presence of perfect fluid source are important in the study of astrophysical problems. It is evident from the literature that the investigations in this direction have not been taken up.

In this paper, we have investigated Bianchi type VI$_h$ space-time in scale invariant theory of gravitation with bulk viscosity in the presence of perfect fluid. It is shown that, even with bulk viscosity the Bianchi type VI$_h$ (h = 1), only the cosmological model is feasible. We have obtained the particular solution of a cosmological model filled with disordered radiation. We have also studied the physical behaviour of the model.

2. Metric and Field Equations

We consider the Bianchi type VI$_h$ line element with Dirac gauge function $\beta = \beta\ (ct)$ as Ellis and Mac Callum [12],

$$dS_w^2 = \beta^2 dS_E^2, \tag{3}$$

with

$$dS_E^2 = -c^2dt^2 + A^2dx^2 + B^2\mathrm{e}^{2x}dy^2 + C^2\mathrm{e}^{2hx}dz^2, \tag{4}$$

where $A = A(t)$, $B = B(t)$, $C = C(t)$, h =constant and dS_W and dS_E are the intervals in Wesson and Einstein theories, respectively.

Here we consider the energy momentum tensor for perfect fluid with bulk viscosity in the form

$$T_{ij}(m) = (\bar{p}_m + \rho_m c^2)u_i u_j + \bar{p}_m g_{ij}, \tag{5}$$

together with

$$g_{ij}u^i u^j = -1, \tag{6}$$

and

$$\bar{p}_m = p_m - \xi\theta, \tag{7}$$

where $\theta = u^i_{;i}$, with u^i, is the four velocity vector of the fluid. ρ_m, p_m and ξ are energy density, proper isotropic pressure and bulk viscosity of the matter, respectively.

The non-vanishing components of conventional Einstein tensors for the metric (4) are

$$G_{11} = \frac{A^2}{c^2}\left[\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} - h\frac{c^2}{A^2}\right] \tag{8}$$

$$G_{22} = \frac{B^2\mathrm{e}^{2x}}{c^2}\left[\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC} - h^2\frac{c^2}{A^2}\right] \tag{9}$$

$$G_{33} = \frac{C^2 e^{2hx}}{c^2}\left[\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{c^2}{A^2}\right] \tag{10}$$

$$G_{44} = -\left[\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC} - (1+h+h^2)\frac{c^2}{A^2}\right] \tag{11}$$

$$G_{14} = -(1+h)\frac{A_4}{A} + \frac{B_4}{B} + h\frac{C_4}{C} \tag{12}$$

and

$$\theta = u^i_{;i} = \left[\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right]. \tag{13}$$

The suffix 4 after a field variable denotes differentiation with respect to time t only.

Field equations (1) for the metric (3) can be expressed as

$$G_{11} = -\kappa \bar{p}_m A^2 - \frac{A^2}{c^2}\left[\frac{2\beta_4}{\beta}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) - \frac{\beta_4^2}{\beta^2} - \frac{2\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2\right], \tag{14}$$

$$G_{22} = -\kappa \bar{p}_m B^2 e^{2x} - \frac{B^2 e^{2x}}{c^2}\left[\frac{2\beta_4}{\beta}\left(\frac{A_4}{A} + \frac{C_4}{C}\right) - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2\right], \tag{15}$$

$$G_{33} = -\kappa \bar{p}_m C^2 e^{2hx} - \frac{C^2 e^{2hx}}{c^2}\left[\frac{2\beta_4}{\beta}\left(\frac{A_4}{A} + \frac{B_4}{B}\right) - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2\right], \tag{16}$$

$$G_{44} = -\kappa \rho_m c^4 + \left[\frac{2\beta_4}{\beta}\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) + 3\frac{\beta_4^2}{\beta^2} - 4\frac{\beta_{44}}{\beta} + \Lambda_0 \beta^2 c^2\right], \tag{17}$$

$$G_{14} = 0 \Rightarrow \frac{B_4}{B} + h\frac{C_4}{C} = (1+h)\frac{A_4}{A}, \ \textit{i.e.}\ A^{1+h} = k_1 B C^h, \tag{18}$$

where k_1 is a constant of integration.

In the usual way (see Wesson [1, 2]) equation (1) and equations (14)–(18) suggest the definition of quantities $\bar{p}_v$ (vacuum pressure with bulk viscosity) and ρ_v (vacuum density) that involves neither the Einstein tensor of conventional theory nor the properties of conventional matter. These two quantities can be obtained as

$$\frac{2\beta_4}{\beta}\left[\left(\frac{h+1}{h}\right)\frac{A_4}{A} + \left(\frac{h-1}{h}\right)\frac{B_4}{B}\right] - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2 c^2 = \kappa \bar{p}_v c^2, \tag{19}$$

$$\frac{2\beta_4}{\beta}\left[\left(\frac{2h+1}{h}\right)\frac{A_4}{A} - \frac{1}{h}\frac{B_4}{B}\right] - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2 c^2 = \kappa \bar{p}_v c^2, \tag{20}$$

$$\frac{2\beta_4}{\beta}\left[\frac{A_4}{A} + \frac{B_4}{B}\right] - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2 c^2 = \kappa \bar{p}_v c^2, \tag{21}$$

$$\frac{2\beta_4}{\beta}\left[\left(\frac{2h+1}{h}\right)\frac{A_4}{A} + \left(\frac{h-1}{h}\right)\frac{B_4}{B}\right] + 3\frac{\beta_4^2}{\beta^2} - 4\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2 c^2 = -\kappa \rho_v c^4, \tag{22}$$

where

$$\bar{p}_v = p_v - \xi\theta. \tag{23}$$

When there is no matter and gauge function β is constant, equations (19)–(21) give the relation

$$c^2\rho_v = \frac{-c^4\lambda_{GR}}{8\pi G} = -\bar{p}_v, \text{ i.e. } c^2\rho_v + \bar{p}_v = 0, \tag{24}$$

where $\lambda_{GR} = \lambda_0\beta^2$= constant is the cosmological constant in general relativity and $\kappa = 8\pi G$. Here, $\bar{p}_v$ is dependent on constant λ_{GR}, G and c, hence uniform in all directions. Thus $\bar{p}_v$ is isotropic in nature and consistent only when

$$A = k_2 B, \tag{25}$$

where k_2 is a constant of integration.

Using equation (25) in (19)–(22), the pressure and energy density for the vacuum case can be obtained as

$$\bar{p}_v = p_v - \varepsilon\theta = \frac{1}{\kappa c^2}\left[\left(\frac{2\beta_4}{\beta}\right)\left(\frac{2A_4}{A}\right) - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2c^2\right], \tag{26}$$

$$\rho_v = -\frac{1}{\kappa c^4}\left[\left(\frac{2\beta_4}{\beta}\right)\left(\frac{3A_4}{A}\right) + 3\frac{\beta_4^2}{\beta^2} - 4\frac{\beta_{44}}{\beta} + \Lambda_0\beta^2c^2\right], \tag{27}$$

where $\bar{p}_v$ and ρ_v relates to the properties of vacuum only in conventional physics. Following Wesson [1, 2], the total pressure and energy density can be defined as

$$\bar{p}_t = \bar{p}_m + \bar{p}_v \Rightarrow p_t - \varepsilon\theta = (p_m - \varepsilon\theta) + (p_v - \varepsilon\theta) \Rightarrow p_t = p_m + p_v - \varepsilon\theta \tag{28}$$

$$\rho_t = \rho_m + \rho_v. \tag{29}$$

Using the aforesaid definition of p_t and ρ_t, components of Einstein tensor [equations (8)–(13)] and consistency condition (25), field equations (14)–(18) can be written in the following explicit form:

$$2\frac{A_{44}}{A} + \frac{A_4^2}{A^2} - h\frac{c^2}{A^2} = -\kappa p_t c^2 + 3\kappa\xi c^2\left(\frac{A_4}{A}\right), \tag{30}$$

$$2\frac{A_{44}}{A} + \frac{A_4^2}{A^2} - h^2\frac{c^2}{A^2} = -\kappa p_t c^2 + 3\kappa\xi c^2\left(\frac{A_4}{A}\right), \tag{31}$$

$$2\frac{A_{44}}{A} + \frac{A_4^2}{A^2} - \frac{c^2}{A^2} = -\kappa p_t c^2 + 3\kappa\xi c^2\left(\frac{A_4}{A}\right), \tag{32}$$

$$3\frac{A_4^2}{A^2} - (1 + h + h^2)\frac{c^2}{A^2} = \kappa\rho_t c^4. \tag{33}$$

Equations (30)–(32) give

$$h(h-1)\frac{c^2}{A^2} = 0 \text{ and } (h^2-1)\frac{c^2}{A^2} = 0. \tag{34}$$

These equations hold good simultaneously for $h = 1$. For $h = 1$ and $h = 0$, we get the unphysical situation, i.e. either $c = 0$ or A is infinitely large. Hence this theory is not feasible for Bianchi type $\text{VI}_h(h = -1)$ and $\text{VI}_h(h = 0)$ metrics, but it is feasible only for Bianchi type $\text{VI}_h(h = 1)$ metric.

3. Solutions

Field equations (30)–(33) reduces to two field equations with four unknowns p_t, ρ_t, A and ξ for $h = 1$. For complete determinacy, two extra conditions are needed. We therefore consider two equations: the equation of state

$$p_t = \frac{1}{3}\rho_t c^2 \tag{35}$$

and the equation [13]

$$\xi = \xi_0 \frac{A_4}{A}, \tag{36}$$

where ξ_0 is a constant.

From equations (30)–(33), for $h = 1$, we get

$$A = \frac{c}{\sqrt{\alpha}}\left(t + t_0\right), \tag{37}$$

where $\alpha = 1 - \frac{3}{2}\xi_0 \kappa c^2 =$ constant and t_0 is constant of integration.

Without loss of generality, we take $k_1 = k_2 = 1$ in equations (18) and (25) to get

$$A = B = C = \frac{c}{\sqrt{\alpha}}(t + t_0). \tag{38}$$

The total pressure p_t and energy density ρ_t can be obtained as,

$$p_t = \frac{\rho_t c^2}{3} = \frac{1}{\kappa c^2 \left(t + t_0\right)^2}\left[1 - \alpha\right], \tag{39}$$

where the reality condition demands $\alpha < 1$.

Considering Dirac gauge function in the form $\beta = 1/ct$, the vacuum pressure and vacuum density can be obtained as

$$p_v = -\frac{1}{\kappa c^2}\left[\frac{4}{t(t+t_0)} - \frac{\Lambda_0 - 5}{t^2} - \frac{3\xi_0 \kappa c^2}{(t+t_0)^2}\right], \tag{40}$$

$$\rho_v = \frac{1}{\kappa c^4}\left[\frac{6}{t(t+t_0)} - \frac{\Lambda_0 - 5}{t^2}\right], \tag{41}$$

and the matter pressure and density can be obtained as

$$p_m = \frac{1}{\kappa c^2}\left[\frac{1-\alpha}{(t+t_0)^2} + \frac{4}{t(t+t_0)} - \frac{\Lambda_0 - 5}{t^2}\right] \tag{42}$$

$$\rho_m = \frac{1}{\kappa c^4}\left[\frac{3(1-\alpha)}{(t+t_0)^2} - \frac{6}{t(t+t_0)} + \frac{\Lambda_0 - 5}{t^2}\right]. \tag{43}$$

Thus the Bianchi type VI_1 model in scale invariant theory is given by

$$dS_w^2 = \frac{1}{c^2 t^2}\left[-c^2 dt^2 + \frac{c^2}{\alpha}(t+t_0)^2[dx^2 + \mathrm{e}^{2x}(dy^2 + dz^2)]\right]. \tag{44}$$

Using the transformation as $t = \mathrm{e}^T$, the above metric can be written as

$$dS_W^2 = -dT^2 + R^2(T)[dx^2 + \mathrm{e}^{2x}(dy^2 + dz^2)], \tag{45}$$

where

$$R(T) = \frac{1}{\sqrt{\alpha}}(1 + t_0 \mathrm{e}^{-T}) \tag{46}$$

4. Some Physical Properties

In this section, we study some of the physical properties of the model obtained in the last section in the transformed coordinate system, in which the behaviour of the physical quantities remain same even though there is a shift i.e. $t(0,\ 1,\ \infty) \to T(-\infty,\ 0,\ \infty)$. The new time coordinate being stretched covers the time region from past to future completely. So one can have a clear picture of the model in the new time coordinate T.

The scalar expansion of the model (45) can be obtained as

$$\theta(T) = U^i{}_{;i} = 3\frac{R_T}{R} = \frac{-3t_0}{e^T + t_0},\ \text{where } R_T = \frac{dR}{dT}. \tag{47}$$

On the mathematical ground, if t_0 is non zero, the model ceases contraction and the rate of contractions remains constant at infinite future. Thus model contracts during evolution.

The shear scalar $\sigma = 0$ indicates that the shape of the universe is unchanged during the evolution. Also, since $\frac{\sigma^2}{\theta^2} = 0$, spacetime is isotropized during the evolution in scale invariant theory. As the acceleration is found to be zero, particles of matter follow geodesic path in this theory. The vorticity w of the model vanishes, which indicates that U^i is hyper surface orthogonal.

From equation (43), with proper choice of parameters, we get $\rho_m(0) =$ positive constant and $\rho_m \to 0$ as $T \to \infty$. Thus the universe starts evolving with constant matter density at initial epoch.

Also it has been observed that

$$\frac{\rho_m}{\theta^2} = \text{ const. at } T = 0 \text{ and } \frac{\rho_m}{\theta^2} = 0 \text{ at } T = \infty, \tag{48}$$

which confirms the homogeneity nature of the space-time during the evolution.

The spatial volume of model (45) is found to be

$$V\ (T,\ x) = \frac{1}{\alpha^{3/2}}(1 + t_0 \mathrm{e}^{-T})^3 \mathrm{e}^{2x}, \tag{49}$$

which gives

$$V(0,\ 0) \to \frac{1}{\alpha^{3/2}}(1 + t_0)^3. \tag{50}$$

Further, we note that $V \to \frac{1}{\alpha^{3/2}}\mathrm{e}^{2x}$ as $T \to \infty$ and $V \to \infty$ as $x \to \infty$. Also, $V \to \infty$ as $(T,\ x) \to (\infty,\ \infty)$.

Thus the model is spatially open and temporary closed and expands uniformly in spatial direction but contracts uniformly in time direction till infinite future.

The Hubble parameter H for the model (45) is given by

$$\frac{R_T}{R} = \frac{-t_0}{\mathrm{e}^T + t_0}, \tag{51}$$

which determines the present rate of expansion of the universe. Also, $H(0)$ =const. and $H \to 0$ as $q = (0) =$ which indicates that the rate of expansion is accelerated or decelerated depending on the signature of the parameters.

Also the deceleration parameter q for model (45) can be calculated as

$$q = -\frac{R_{TT}R}{R_T^2} = \frac{1}{t_0}\left[\mathrm{e}^T + t_0\right]. \tag{52}$$

It can be seen that $q(0)$ =const., and at infinity q is not defined. Thus the model does not represent a steady state model.

5. Conclusion

In this paper we have studied a Bianchi type VI_1 cosmological model in the scale invariant theory of gravitation formulated by Wesson (1981). The model in this case starts evolving at initial epoch with a constant volume and ends at an infinite future. Also, the matter density ρ_m vanishes for $\Lambda_0 = 3\left(1 - \frac{1}{3}\xi_0 \kappa c^2\right)$ but $p_m \neq 0$ for $t_0 = 0$. This leads to an unphysical situation. Thus for a viable physical situation one should have $\Lambda_0 \neq 3\left(1 - \frac{1}{3}\xi_0 \kappa c^2\right)$. Also, the model in this case appears to be steady state.

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