Turk J Phys
31 (2007) , 123 – 135.

Unsteady MHD Flow Due to Eccentric Rotating Disks for Suction and Blowing

Muhammad R. MOHYUDDIN
Department of Science in Engineering, Kulliyyah of Engineering,
International Islamic University Malaysia, Jalan Gombak, 53100, Kuala Lumpur-MALAYSIA
e-mail: m.r.mohyuddin@gmail.com

Received 10.07.2006

Abstract

An exact solution of unsteady flow of a viscous fluid due to a sudden pull with constant velocities of non-coaxial rotations between two porous infinite disks in the presence of uniform transpose magnetic field is investigated. Two different methods are used to obtain the solution at large times and the solution at small times. The effects of magnetic field suction and injection on the velocity distributions are presented.

Key Words: MHD; Newtonian flow; rotating disks; suction/blowing.

1. Introduction

Most of the fundamental concepts of unsteady viscous flows have been known since the early part of the century. However, the past decade has seen an unprecedented number of publications in this area. Interest is concentrated here on the behavior of viscous fluids and, in particular, their response to time dependent external conditions. The solid boundaries of such flows as well as the conditions of the oncoming flow are time-independent, and yet, unsteadiness sets in by itself. Such phenomena have been predicted with some success by analytic methods. A straightforward approach to the problem has been possible via the solution of Navier-Stokes equations [1]. Typical problems are the instability and transition of boundary layers, free shear layers and jets, the shedding of vortices, the development of unsteady wakes, a boundary layer and, of course, all problems involving turbulence. In most of the problems, the body forces in the Navier-Stokes equations are neglected for simplicity and convenience. It is observed theoretically [2, 3, 4] and experimentally [5] that, when magnetohydrodynamics (MHD) forces acts as the body forces in the flow field phenomena, it controls the boundary layers. Also, MHD is the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field and it acts perpendicular to the velocity field. It has significant applications in many engineering problems, geophysics and astronomy.

In the present work, we assume the same geometry as that of Ersoy's [6] paper, in which he discussed the flow due to a pull arising from eccentric rotating disks with constant angular velocity. We have generalized the results when the disks are porous and an external uniform magnetic field acts perpendicular to them. The magnetic Reynold number is small so that the induced magnetic field is neglected [2, 3]. Solutions for large time and small time is obtained by the method of eigenfunction and Laplace transform, respectively.

Although the large time solution can be obtained from the small time solution, it converges effectively via the method we have used for large time [6].

The paper is organized as follows. In section 2, formulation of the problem and basic equations are derived. Section 3 deals with large time solution of the problem. Section 4 is devoted to the small time solution. In section 5, the case of blowing is discussed and finally concluding remarks are given in section 6.

2. Formulation of the Problem

The flow field of the problem is bounded by two infinite disks located at $z = h$ and $z = -h$. The top disk and the bottom disk rotate about the z'-axis and z''-axis with the same angular velocity Ω, respectively. The two non-coincident axes are separated by a distance $2l$. The disks initially rotate eccentrically and are assumed to be infinite. The upper and the lower disks are suddenly pulled along their common axis with constant velocities $\mathbf{U}$ and $-\mathbf{U}$, respectively. The velocity $\mathbf{U}$ has two components: U_1 in the x-direction and U_2 in the y-direction. Therefore, the appropriate initial and boundary conditions are given by

$$u = -\Omega y + \overline{f}(z), \quad v = \Omega x + \overline{g}(z), \quad w = -w_0 \;\text{ at }\; t = 0 \;\text{ for }\; -h \leq z \leq h,$$

$$u = -\Omega(y - l) + U_1, \quad v = \Omega x + U_2, \quad w = -w_0 \;\text{ at }\; z = h \;\text{ for }\; t > 0,$$

$$u = -\Omega(y + l) - U_1, \quad v = \Omega x - U_2, \quad w = -w_0 \;\text{ at }\; z = -h \;\text{ for }\; t > 0, \tag{1}$$

where $\overline{f}(z)$ and $\overline{g}(z)$ are the known functions and they represent the flow between eccentric rotating disks for a Newtonian fluid. Abbott and Walters [7] found that

$$\overline{f}(z) + i\overline{g}(z) = \Omega l \frac{\sinh kz}{\sinh kh},$$

where $k = \sqrt{\frac{\Omega}{2\nu}}(1 + i)$ and ν is the kinematic viscosity.

The components of the velocity field are

$$u = -\Omega y + f(z,t), \quad v = \Omega x + g(z,t), \quad w = -w_0, \tag{2}$$

where $w_0 > 0$ is the suction velocity and $w < 0$ is the injection velocity.

The appropriate generalization for arbitrary, time-dependent flows is [8, 9]

$$\begin{aligned} \mathbf{T} &= -p\delta + \tau \\ &= -p\delta + \mu\left[\nabla\mathbf{V} + (\nabla\mathbf{V})^{\top}\right] + \left(\frac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{V})\delta, \end{aligned}$$

where $(\nabla\mathbf{V})^{\top}$ is the transpose of the dyadic $(\nabla\mathbf{V})$ and δ is the unit tensor. This expression reduces to the hydrostatic pressure when there are no velocity gradients; it contains all possible combinations of first derivatives of velocity components that are allowed if one assumes that the fluid is isotropic and the momentum flux tensor is symmetric [10, 11]. The symbol p represents the thermodynamic pressure, which is related to the density ρ and the temperature T through a "thermodynamic equation of state," $p = p(\rho, T)$; that is, this is taken to be the same function that one uses in thermal equilibrium. The stress τ is the part of the momentum flux tensor or stress tensor that is associated with the viscosity of the fluid. An equation that assigns a value to τ is called a constitutive equation for the Newtonian fluid. Note that in generalizing Newton's law of viscosity to arbitrary flows an additional transport property κ, the dilatational viscosity, arises. The dilatational viscosity is identically zero for ideal, monotonic gases; for incompressible liquids $\nabla \cdot \mathbf{V} = 0$, and the term containing κ vanishes.

For all fluids the density ρdepends on the local thermodynamic state variables, such as pressure and temperature. However, for fluids it is often a very good assumption to take the density to be constant. Such an idealized fluid is often called an incompressible fluid, and the momentum flux tensor simplifies to [8]

$$\mathbf{T} = -p\delta + \tau = -p\delta + \mu\ \dot{\gamma},$$

in which $\dot{\gamma} = \nabla\mathbf{V} + (\nabla\mathbf{V})^{\top}$ is the rate of strain tensor or rate of deformation tensor. Thus the Navier-Stokes equation become

$$\rho\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = -p\delta + \mu\dot{\gamma} + \mathbf{J} \times \mathbf{B}, \tag{3}$$

where $\mathrm{d}/\mathrm{d}t$ is the usual material time derivative and $\mathbf{J} \times \mathbf{B}$ are MHD body forces arising from the Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_m \mathbf{J}, \quad \nabla \cdot \mathbf{E} = 0$$

in which $\mathbf{J}$ is the electric current density, $\mathbf{B}$ is the total magnetic field so that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$; and $\mathbf{b}$ is the induced magnetic field. The magnetic Reynolds number R_m is assumed to be small as is the case with most of conducting fluids, and hence the induced magnetic field is small in comparison with the applied magnetic field and therefore not taken into account [2, 3, 9, 12]. The magnetic body force $\mathbf{J} \times \mathbf{B}$ now becomes $\sigma(\mathbf{V} \times \mathbf{B}) \times \mathbf{B}$ (σ is the electrical conductivity of the fluid), that is

$$\sigma(\mathbf{V} \times \mathbf{B}) \times \mathbf{B} = -\sigma B_0^2 \mathbf{V}.$$

Substituting (2) in (3)and eliminating the pressure, we obtain

$$\nu\frac{\partial^2 f}{\partial z^2} - \frac{\partial f}{\partial t} + w_0\frac{\partial f}{\partial z} + \Omega g - \frac{\sigma B_0^2}{\rho} f = C_1(t), \tag{4}$$

$$\nu\frac{\partial^2 g}{\partial z^2} - \frac{\partial g}{\partial t} + w_0\frac{\partial g}{\partial z} - \Omega f - \frac{\sigma B_0^2}{\rho} g = C_2(t), \tag{5}$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions of time t. The corresponding boundary conditions (1) become

$$f(z,0) = \overline{f}(z), \quad g(z,0) = \overline{f}(z), \quad \text{for} \quad -h \le z \le h,$$

$$f(h,t) = \Omega l + U_1, \quad g(h,t) = U_2, \quad \text{for} \quad t > 0$$

$$f(-h,t) = -\Omega l - U_1, \quad g(-h,t) = -U_2, \quad \text{for} \quad t > 0. \tag{6}$$

To obtain a symmetry velocity distribution we have the following condition:

$$f(0,t) = 0, \quad g(0,t) = 0, \quad \text{for} \quad t \ge 0 \tag{7}$$

Applying (7) in (4) and (5) and then coupling, we get

$$\nu\frac{\partial^2 F}{\partial z^2} - \frac{\partial F}{\partial t} + w_0\frac{\partial F}{\partial z} - (i\Omega + \phi)F = 0, \tag{8}$$

and the conditions (6) and (7) takes the form

$$F(z,0) = \Omega l\frac{\sinh kz}{\sinh kh},$$

$$F(0,t) = 0, \quad F(\pm h,t) = \pm\left[(\Omega l + U_1) + U_2\right], \tag{9}$$

where

$$F(z,t) = \frac{f(z,t)}{\Omega l} + i\frac{g(z,t)}{\Omega l}.$$

3. Solution at Large Times

At long times, we anticipate that the flow will reach a steady state with a velocity profile given by

$$F_{ss}(z,t) = [(\Omega l + U_1) + U_2]\, e^{\frac{w_0}{2\nu}(h-z)} \frac{\sinh \xi z}{\sinh \xi h}.$$

In order to obtain the solution of (8) we take the form

$$F(z,t) = [(\Omega l + U_1) + U_2]\, e^{\frac{w_0}{2\nu}(h-z)} \frac{\sinh \xi z}{\sinh \xi h} - G(z,t), \tag{10}$$

where the first term on the right hand side is the steady-state solution and the second one is the deviation from it. Inserting (10) into (8) and (9), we obtain the system

$$\nu \frac{\partial^2 G}{\partial z^2} - \frac{\partial G}{\partial t} + w_0 \frac{\partial G}{\partial z} - (i\Omega + \phi) G = 0, \tag{11}$$

$$G(z,0) = [(\Omega l + U_1) + U_2]\, e^{\frac{w_0}{2\nu}(h-z)} \frac{\sinh \xi z}{\sinh \xi h} - \Omega l \frac{\sinh kz}{\sinh kh},$$

$$G(0,t) = 0, \quad G(\pm h, t) = 0, \tag{12}$$

where

$$\xi = \frac{\sqrt{w_0^2 + 4\nu(i\Omega + \phi)}}{2\nu}.$$

To obtain the solution of (11) we introduce the transformation

$$G(z,t) = e^{\left[z + \frac{w_0}{2} t\right]\frac{w_0}{2\nu}} w_1(z,t) \tag{13}$$

so that the equations (11) and (12) takes the form

$$\nu \frac{\partial^2 w_1}{\partial z^2} - \frac{\partial w_1}{\partial t} - (i\Omega + \phi) w_1 = 0, \tag{14}$$

$$w_1(z,0) = [(\Omega l + U_1) + U_2]\, e^{\frac{w_0}{2\nu}(h-z)} \frac{\sinh \xi z}{\sinh \xi h} - \Omega l \frac{\sinh kz}{\sinh kh},$$

$$w_1(0,t) = 0, \quad w_1(\pm h, t) = 0. \tag{15}$$

Equation (14) is solved by the method of separation of variables and its solution is given by

$$w_1(z,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{h} z \cdot e^{-\lambda_n t}, \tag{16}$$

where $\lambda_n = \nu \left(\frac{n\pi}{h}\right)^2 + (\phi + i\Omega)$. The coefficient C_n is determined from the initial condition and given as

$$C_n = -2\pi [(\Omega l + U_1) + U_2]\, e^{\frac{w_0}{2\nu} h} \frac{n(-1)^n}{h^2 \xi^2 + n^2 \pi^2}$$

$$+ i \frac{4\pi \Omega \ln (-1)^n}{\sinh kh} \left[\frac{\sinh (k+c) h}{h^2 (k+c)^2 + n^2 \pi^2} + \frac{\sinh (k-c) h}{h^2 (k-c)^2 + n^2 \pi^2} \right]. \tag{17}$$

Finally, substituting (16) in (13), and then the resulting equation in (10), we obtain

$$F(z,t) = [(\Omega l + U_1) + U_2]\, e^{\frac{w_0}{2\nu}(h-z)} \frac{\sinh \xi z}{\sinh \xi h}$$

$$-e^{-\frac{w_0}{2\nu}z - \frac{w_0^2}{4\nu}t} \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{h} z \cdot e^{-\lambda_n t}. \tag{18}$$

and

$$\frac{f}{\Omega l} = \frac{e^{\frac{\overline{w}_0 R}{2}(1-\eta)}}{\Delta} \left[\begin{array}{c} (1+V_1)\{P(1)P(\eta) + Q(1)Q(\eta)\} \\ -V_2\{P(1)Q(\eta) - Q(1)P(\eta)\} \end{array} \right]$$

$$+2\pi e^{-\frac{\overline{w}_0 R}{2}\left(1+\eta+\frac{1}{2}\overline{w}_0\tau\right) - \overline{\phi}\tau}$$

$$\times \sum_{n=1}^{\infty} \left[\begin{array}{c} \{(1+V_1)c + V_2 d\}\cos\tau \\ +\{V_2 c - (1+V_1)d\}\sin\tau \end{array} \right] n(-1)^n \sin n\pi\eta \cdot e^{-n^2\pi^2\tau/R}$$

$$-4\pi e^{-\frac{\overline{w}_0 R}{2}\left(\eta+\frac{1}{2}\overline{w}_0\tau\right) - \overline{\phi}\tau}$$

$$\times \sum_{n=1}^{\infty} \left[[J_{11}\sin\tau - J_{12}\cos\tau]\, n(-1)^n \sin n\pi\eta \cdot e^{-n^2\pi^2\tau/R} \right], \tag{19}$$

$$\frac{g}{\Omega l} = \frac{e^{\frac{\overline{w}_0 R}{2}(1-\eta)}}{\Delta} \left[\begin{array}{c} (1+V_1)\{P(1)Q(\eta) - Q(1)P(\eta)\} \\ +V_2\{P(1)P(\eta) + Q(1)Q(\eta)\} \end{array} \right]$$

$$+2\pi e^{-\frac{\overline{w}_0 R}{2}\left(1+\eta+\frac{1}{2}\overline{w}_0\tau\right) - \overline{\phi}\tau}$$

$$\times \sum_{n=1}^{\infty} \left[\begin{array}{c} \{cV_2 - d(1+V_1)\}\cos\tau \\ -\{c(1+V_1) + dV_2\}\sin\tau \end{array} \right] n(-1)^n \sin n\pi\eta \cdot e^{-n^2\pi^2\tau/R}$$

$$-4\pi e^{-\frac{\overline{w}_0 R}{2}\left(\eta+\frac{1}{2}\overline{w}_0\tau\right) - \overline{\phi}\tau}$$

$$\times \sum_{n=1}^{\infty} \left[[J_{11}\cos\tau + J_{12}\sin\tau]\, n(-1)^n \sin n\pi\eta \cdot e^{-n^2\pi^2\tau/R} \right], \tag{20}$$

where

$$\begin{aligned} R &= \frac{\Omega h^2}{\nu},\ \eta = \frac{z}{h},\ \tau = \Omega t,\ V_1 = \frac{U_1}{\Omega l},\ V_2 = \frac{U_2}{\Omega l},\ \overline{w}_0 = \frac{w_0}{\Omega h},\ \overline{\phi} = \frac{\phi}{\Omega}, \\ \Delta &= (\sinh a \cos b)^2 + (\cosh a \sin b)^2, \\ P(1) &= \sinh a \cos b, \quad Q(1) = \cosh a \sin b, \\ P(\eta) &= \sinh a\eta \cos b\eta, \quad Q(\eta) = \cosh a\eta \sin b\eta, \end{aligned}$$

$$
\begin{aligned}
a &= \pm\frac{1}{2}\sqrt{\sqrt{\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{\phi}\right)^2+4R^2}+\left(\frac{1}{2}R^2\overline{w}_0^2+2R\overline{\phi}\right)},\\
b &= \pm\frac{1}{2}\sqrt{\sqrt{\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{\phi}\right)^2+4R^2}-\left(\frac{1}{2}R^2\overline{w}_0^2+2R\overline{\phi}\right)},
\end{aligned}
$$

$$
\begin{aligned}
J_1 &= \sinh\sqrt{\frac{R}{2}}\cos\sqrt{\frac{R}{2}},\ J_2=\cosh\sqrt{\frac{R}{2}}\sin\sqrt{\frac{R}{2}},\\
\lambda_1 &= 1+\sqrt{\frac{R}{2}}\overline{w}_0,\ \lambda_2=1-\sqrt{\frac{R}{2}}\overline{w}_0,\\
J_3 &= \sinh\sqrt{\frac{R}{2}}\lambda_1\cos\sqrt{\frac{R}{2}},\ J_4=\cosh\sqrt{\frac{R}{2}}\lambda_1\sin\sqrt{\frac{R}{2}},\\
J_5 &= \sinh\sqrt{\frac{R}{2}}\lambda_2\cos\sqrt{\frac{R}{2}},\ J_6=\cosh\sqrt{\frac{R}{2}}\lambda_2\sin\sqrt{\frac{R}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\lambda_3 &= \frac{R}{2}\left(\lambda_1^2-1+2\frac{n^2\pi^2}{R}\right),\ \lambda_4=\lambda_1R,\\
\lambda_5 &= \frac{R}{2}\left(\lambda_2^2-1+2\frac{n^2\pi^2}{R}\right),\ \lambda_6=\lambda_2R,\\
c &= \frac{a^2-b^2+n^2\pi^2}{\left(a^2-b^2+n^2\pi^2\right)^2+4a^2b^2},\ d=\frac{2ab}{\left(a^2-b^2+n^2\pi^2\right)^2+4a^2b^2},
\end{aligned}
$$

$$
\begin{aligned}
J_7 &= \frac{J_1\left(J_3\lambda_3+\lambda_4J_4\right)+J_2\left(\lambda_3J_4-\lambda_4J_3\right)}{\left(J_1^2+J_2^2\right)\left(\lambda_3^2+\lambda_4^2\right)},\\
J_8 &= \frac{J_1\left(\lambda_3J_4-\lambda_4J_3\right)-J_2\left(J_3\lambda_3+\lambda_4J_4\right)}{\left(J_1^2+J_2^2\right)\left(\lambda_3^2+\lambda_4^2\right)},\\
J_9 &= \frac{J_1\left(J_5\lambda_5+\lambda_6J_6\right)+J_2\left(\lambda_5J_6-\lambda_6J_5\right)}{\left(J_1^2+J_2^2\right)\left(\lambda_5^2+\lambda_6^2\right)},\\
J_{10} &= \frac{J_1\left(\lambda_5J_6-\lambda_6J_5\right)-J_2\left(J_5\lambda_5+\lambda_6J_6\right)}{\left(J_1^2+J_2^2\right)\left(\lambda_5^2+\lambda_6^2\right)},\\
J_{11} &= J_7+J_9,\quad J_{12}=J_8+J_{10}.
\end{aligned}
$$

4. Solution at Small Times

In order to obtain the solution at small times we use Laplace transform technique which converges rapidly at small times. Let us consider the function

$$
F(z,t)=H(z,t)e^{-i\Omega t} \tag{21}
$$

so that equation (8) and the initial and boundary conditions (9) take the form

$$
\nu\frac{\partial^2H}{\partial z^2}-\frac{\partial H}{\partial t}+w_0\frac{\partial H}{\partial z}-\phi H=0, \tag{22}
$$

$$
H\left(z,0\right)=\Omega l\frac{\sinh kz}{\sinh kh},
$$

$$H(0,t)=0,\quad H(\pm h,t)=\pm\left[(\Omega l+U_1)+U_2\right]e^{i\Omega t}. \tag{23}$$

Applying the Laplace transform to equation (22) and (23), we obtain

$$\nu\frac{d^2\bar{H}}{dz^2}+w_0\frac{d\bar{H}}{dz}-(s+\phi)\bar{H}=-\Omega l\frac{\sinh kz}{\sinh kh}, \tag{24}$$

$$\overline{H}\left(\pm h,s\right)=\pm\frac{\left[(\Omega l+U_1)+U_2\right]}{s-i\Omega},\ \ \bar{H}\left(0,s\right)=0. \tag{25}$$

The solution of (24) subject to boundary conditions (25) is given by

$$\bar{H}\left(z,s\right)=\frac{\left[(\Omega l+U_1)+U_2\right]}{s-i\Omega}\frac{\sinh mz}{\sinh mh}e^{\frac{w_0}{2\nu}(h-z)}-\frac{\Omega l}{s-\psi}\frac{\sinh mz}{\sinh mh}e^{\frac{w_0}{2\nu}(h-z)}$$

$$+\frac{\Omega l}{s-\psi}\frac{\sinh kz}{\sinh kh}, \tag{26}$$

where

$$m=\frac{\sqrt{w_0^2+4\nu\left(s+\phi\right)}}{2\nu}.$$

Laplace inversion of (26) is given by

$$H\left(z,t\right)=\left[(\Omega l+U_1)+U_2\right]e^{\frac{w_0}{2\nu}(h-z)}\cdot I_1-\Omega l e^{\frac{w_0}{2\nu}(h-z)}\cdot I_2+\Omega l\cdot I_3, \tag{27}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\sinh mz}{(s-i\Omega)\sinh mh}e^{st}ds,\\
I_2 &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\sinh mz}{(s-\psi)\sinh mh}e^{st}ds,\\
I_3 &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\sinh kz}{(s-\psi)\sinh kh}e^{st}ds.
\end{aligned}$$

In equation (27) we solve the integrals by residue method. Therefore in I_1 the residue at $s-i\Omega$ is

$$\mathrm{Res}\left(i\Omega\right)=\frac{\sinh\sigma_1 z}{\sinh\sigma_1 h}e^{i\Omega t},$$

where

$$\sigma_1=\frac{\sqrt{w_0^2+4\nu\left(i\Omega+\phi\right)}}{2\nu}.$$

The other singular points are the zeros of

$$\sinh mh=0.$$

Setting $m=i\alpha$, we find that

$$\sin\alpha h=0 \tag{28}$$

and $\alpha_n = \frac{n\pi}{h}$, $n = 1, 2, 3, \ldots, \infty$ are the zeros of equation (28); then

$$s_n = -\frac{w_0^2 + 4\alpha_n^2\nu^2 + 4\nu\phi}{4\nu},\ n = 1, 2, 3, \ldots, \infty$$

are the poles. Since all α_n, $n = 1, 2, 3, \ldots, \infty$ are symmetrically placed about origin on the real axis, all poles s_n lie on the negative real axis. These are the simple poles and the residue at all these poles can be obtained as

$$\mathrm{Res}\,(s_n) = \frac{2\,(-1)^{n+1}\,n\pi\nu e^{s_n t}}{h^2\,(s_n - i\Omega)} \sin\frac{n\pi}{h}z.$$

So that the complete solution of I_1 is given as

$$I_1 = \mathrm{Res}\,(i\Omega) + \mathrm{Res}\,(s_n) = \frac{\sinh\sigma_1 z}{\sinh\sigma_1 h}e^{i\Omega t} - \frac{2\pi\nu}{h^2}\sum_{n=1}^{\infty}\frac{(-1)^n\,ne^{s_n t}}{(s_n - i\Omega)}\sin\frac{n\pi}{h}z. \tag{29}$$

Similarly,

$$I_2 = \frac{\sinh\sigma_2 z}{\sinh\sigma_2 h}e^{\psi t} - \frac{2\pi\nu}{h^2}\sum_{n=1}^{\infty}\frac{(-1)^n\,ne^{s_n t}}{(s_n - \psi)}\sin\frac{n\pi}{h}z, \tag{30}$$

and

$$I_3 = \frac{\sinh kz}{\sinh kh}\sum_{n=1}^{\infty}\psi^n\frac{t^n}{n!}. \tag{31}$$

Using the values from (29)–(31) in (27), we obtain the complete solution

$$H\,(z,t) = [(\Omega l + U_1) + U_2]\,e^{\frac{w_0}{2\nu}(h-z)}\left[\frac{\sinh\sigma_1 z}{\sinh\sigma_1 h}e^{i\Omega t} - \frac{2\pi\nu}{h^2}\sum_{n=1}^{\infty}\frac{(-1)^n\,ne^{s_n t}}{(s_n - i\Omega)}\sin\frac{n\pi}{h}z\right]$$

$$-\Omega l e^{\frac{w_0}{2\nu}(h-z)}\left[\frac{\sinh\sigma_2 z}{\sinh\sigma_2 h}e^{\psi t} - \frac{2\pi\nu}{h^2}\sum_{n=1}^{\infty}\frac{(-1)^n\,ne^{s_n t}}{(s_n - \psi)}\sin\frac{n\pi}{h}z\right]$$

$$+\Omega l\left[\frac{\sinh kz}{\sinh kh}\sum_{n=1}^{\infty}\psi^n\frac{t^n}{n!}\right], \tag{32}$$

where

$$\sigma_2 = \frac{\sqrt{w_0^2 + 4\nu\,(\psi + \phi)}}{2\nu}.$$

Finally, from (21) we have

$$\frac{F\,(\eta,\tau)}{\Omega l} = [(1 + V_1) + V_2]\,e^{\frac{\overline{w}_0 R}{2}(1-\eta)}\left[\frac{\sinh\sqrt{\frac{\overline{w}_0^2 R^2}{4} + \left(\overline{\phi} + i\right)R}\eta}{\sinh\sqrt{\frac{\overline{w}_0^2 R^2}{4} + \left(\overline{\phi} + i\right)R}}\right]$$

$$+\sum_{n=1}^{\infty}\frac{8n\pi\,(-1)^n\sin n\pi\eta}{\overline{w}_0^2 R^2 + 4n^2\pi^2 + 4R\overline{\phi} + 4iR}e^{-\left(\frac{1}{4}\overline{w}_0^2 R + \frac{n^2\pi^2}{R} + \overline{\phi}\right)\tau - i\tau}$$

$$-e^{\frac{\overline{w}_0 R}{2}(1-\eta)} \frac{\sinh\sqrt{\frac{\overline{w}_0^2 R^2}{4}+R\overline{w}_0\sqrt{\frac{R}{2}}+iR\left(1+\overline{w}_0\sqrt{\frac{R}{2}}\right)}\eta}{\sinh\sqrt{\frac{\overline{w}_0^2 R^2}{4}+R\overline{w}_0\sqrt{\frac{R}{2}}+iR\left(1+\overline{w}_0\sqrt{\frac{R}{2}}\right)}}$$

$$\times e^{\left[\left(\overline{w}_0\sqrt{\frac{R}{2}}-\overline{\phi}\right)+i\left(\overline{w}_0\sqrt{\frac{R}{2}}\right)\right]\tau}$$

$$+\sum_{n=1}^{\infty}\frac{8n\pi\left(-1\right)^n\sin n\pi\eta}{\overline{w}_0^2 R^2+4n^2\pi^2+4iR+4R\overline{w}_0\sqrt{\frac{R}{2}}\left(1+i\right)}e^{-\left(\frac{1}{4}\overline{w}_0^2 R+\frac{n^2\pi^2}{R}+\overline{\phi}\right)\tau-i\tau}$$

$$+\frac{\sinh\sqrt{\frac{R}{2}}\left(1+i\right)\eta}{\sinh\sqrt{\frac{R}{2}}\left(1+i\right)}e^{\left[\left(\overline{w}_0\sqrt{\frac{R}{2}}-\overline{\phi}\right)+i\left(\overline{w}_0\sqrt{\frac{R}{2}}\right)\right]\tau}; \tag{33}$$

and

$$\frac{f}{\Omega l}=\frac{e^{\frac{\overline{w}_0 R}{2}(1-\eta)}}{\delta_2}\left[\begin{array}{c}(1+V_1)\left\{X_2(1)X_2(\eta)+Y_2(1)Y_2(\eta)\right\}\\ -V_2\left\{X_2(1)Y_2(\eta)-Y_2(1)X_2(\eta)\right\}\end{array}\right]$$

$$+e^{\frac{\overline{w}_0 R}{2}(1-\eta)}\sum_{n=1}^{\infty}\frac{8n\pi\left(-1\right)^n\sin n\pi\eta}{R_5^2+R_6^2}e^{-\left(\frac{1}{4}\overline{w}_0^2 R+\frac{n^2\pi^2}{R}+\overline{\phi}\right)\tau}$$

$$\times\left[R_5\left\{(1+V_1)\cos\tau+V_2\sin\tau\right\}+R_6\left\{V_2\cos\tau-(1+V_1)\sin\tau\right\}\right]$$

$$-\frac{e^{\frac{\overline{w}_0 R}{2}(1-\eta)+\left(\overline{w}_0\sqrt{\frac{R}{2}}-\overline{\phi}\right)\tau}}{\delta_1}\left[\begin{array}{c}\cos\left(\overline{w}_0\sqrt{\frac{R}{2}}\tau\right)\left\{X_1(1)X_1(\eta)+Y_1(1)Y_1(\eta)\right\}\\ -\sin\left(\overline{w}_0\sqrt{\frac{R}{2}}\tau\right)\left\{X_1(1)Y_1(\eta)-Y_1(1)X_1(\eta)\right\}\end{array}\right]$$

$$-e^{\frac{\overline{w}_0 R}{2}(1-\eta)}\sum_{n=1}^{\infty}\frac{8n\pi\left(-1\right)^n\sin n\pi\eta}{R_1^2+R_2^2}e^{-\left(\frac{1}{4}\overline{w}_0^2 R+\frac{n^2\pi^2}{R}+\overline{\phi}\right)\tau}\left(R_1\cos\tau-R_2\sin\tau\right)$$

$$+\frac{e^{\left(\overline{w}_0\sqrt{\frac{R}{2}}-\overline{\phi}\right)\tau}}{\delta}\left[\begin{array}{c}\cos\left(\overline{w}_0\sqrt{\frac{R}{2}}\tau\right)\left\{X(1)X(\eta)+Y(1)Y(\eta)\right\}\\ -\sin\left(\overline{w}_0\sqrt{\frac{R}{2}}\tau\right)\left\{X(1)Y(\eta)-Y(1)X(\eta)\right\}\end{array}\right], \tag{34}$$

$$\frac{g}{\Omega l}=\frac{e^{\frac{\overline{w}_0 R}{2}(1-\eta)}}{\delta_2}\left[\begin{array}{c}(1+V_1)\left\{X_2(1)Y_2(\eta)-Y_2(1)X_2(\eta)\right\}\\ +V_2\left\{X_2(1)X_2(\eta)+Y_2(1)Y_2(\eta)\right\}\end{array}\right]$$

$$+e^{\frac{\overline{w}_0 R}{2}(1-\eta)}\sum_{n=1}^{\infty}\frac{8n\pi\left(-1\right)^n\sin n\pi\eta}{R_5^2+R_6^2}e^{-\left(\frac{1}{4}\overline{w}_0^2 R+\frac{n^2\pi^2}{R}+\overline{\phi}\right)\tau}$$

$$\times\left[R_5\left\{V_2\cos\tau-(1+V_1)\sin\tau\right\}-R_6\left\{(1+V_1)\cos\tau+V_2\sin\tau\right\}\right]$$

$$-\frac{e^{\frac{\overline{w}_0 R}{2}(1-\eta)+\left(\overline{w}_0\sqrt{\frac{R}{2}}-\overline{\phi}\right)\tau}}{\delta_1}\left[\begin{array}{c}\cos\left(\overline{w}_0\sqrt{\frac{R}{2}}\tau\right)\{X_1(1)\,Y_1(\eta)-Y_1(1)\,X_1(\eta)\}\\+\sin\left(\overline{w}_0\sqrt{\frac{R}{2}}\tau\right)\{X_1(1)\,X_1(\eta)+Y_1(1)\,Y_1(\eta)\}\end{array}\right]$$

$$+e^{\frac{\overline{w}_0 R}{2}(1-\eta)}\sum_{n=1}^{\infty}\frac{8n\pi(-1)^n\sin n\pi\eta}{R_1^2+R_2^2}e^{-\left(\frac{1}{4}\overline{w}_0^2R+\frac{n^2\pi^2}{R}+\overline{\phi}\right)\tau}\left(R_1\sin\tau+R_2\cos\tau\right)$$

$$+\frac{e^{\left(\overline{w}_0\sqrt{\frac{R}{2}}-\overline{\phi}\right)\tau}}{\delta}\left[\begin{array}{c}\cos\left(\overline{w}_0\sqrt{\frac{R}{2}}\tau\right)\{X(1)\,Y(\eta)-Y(1)\,X(\eta)\}\\+\sin\left(\overline{w}_0\sqrt{\frac{R}{2}}\tau\right)\{X(1)\,X(\eta)+Y(1)\,Y(\eta)\}\end{array}\right]. \tag{35}$$

Here,

$$\begin{aligned}
X(\eta) &= \sinh\sqrt{\frac{R}{2}}\eta\cos\sqrt{\frac{R}{2}},\ Y(\eta)=\cosh\sqrt{\frac{R}{2}}\eta\sin\sqrt{\frac{R}{2}},\\
X(1) &= \sinh\sqrt{\frac{R}{2}}\cos\sqrt{\frac{R}{2}},\ Y(1)=\cosh\sqrt{\frac{R}{2}}\sin\sqrt{\frac{R}{2}},\\
\delta &= \left[\sinh\sqrt{\frac{R}{2}}\cos\sqrt{\frac{R}{2}}\right]^2+\left[\cosh\sqrt{\frac{R}{2}}\sin\sqrt{\frac{R}{2}}\right]^2,\\
R_1 &= R^2\overline{w}_0+4n^2\pi^2+\frac{4\overline{w}_0R^{\frac{3}{2}}}{\sqrt{2}},\ R_2=4R+\frac{4\overline{w}_0R^{\frac{3}{2}}}{\sqrt{2}},
\end{aligned}$$

$$\begin{aligned}
R_3 &= \pm\frac{1}{2}\sqrt{\sqrt{\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{w}_0\sqrt{\frac{R}{2}}\right)^2+R^2\left(1+\overline{w}_0\sqrt{\frac{R}{2}}\right)^2}+\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{w}_0\sqrt{\frac{R}{2}}\right)},\\
R_4 &= \pm\frac{1}{2}\sqrt{\sqrt{\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{w}_0\sqrt{\frac{R}{2}}\right)^2+R^2\left(1+\overline{w}_0\sqrt{\frac{R}{2}}\right)^2}-\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{w}_0\sqrt{\frac{R}{2}}\right)},
\end{aligned}$$

$$\begin{aligned}
X_1(\eta) &= \sinh R_3\eta\cos R_4\eta,\ Y_1(\eta)=\cosh R_3\eta\sin R_4\eta,\\
X_1(1) &= \sinh R_3\cos R_4,\ Y_1(1)=\cosh R_3\sin R_4,\\
\delta_1 &= [\sinh R_3\cos R_4]^2+[\cosh R_3\sin R_4]^2,\\
R_5 &= R^2\overline{w}_0^2+4n^2\pi^2+4R\overline{\phi},\ R_6=4R,\\
X_2(\eta) &= \sinh R_7\eta\cos R_8\eta,\ Y_2(\eta)=\cosh R_7\eta\sin R_8\eta,\\
X_2(1) &= \sinh R_7\cos R_8,\ Y_2(1)=\cosh R_7\sin R_8,\\
\delta_2 &= [\sinh R_7\cos R_8]^2+[\cosh R_7\sin R_8]^2,
\end{aligned}$$

$$\begin{aligned}
R_7 &= \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{\phi}\right)^2+R^2}+\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{\phi}\right)},\\
R_8 &= \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{\phi}\right)^2+R^2}-\left(\frac{1}{4}R^2\overline{w}_0^2+R\overline{\phi}\right)}.
\end{aligned}$$

5. Injection

In case of injection $w_0 < 0$, we take $w_0 = -\epsilon$ so that $\epsilon > 0$. The solution is given by

$$
\begin{aligned}
\frac{f}{\Omega l} = {} & \frac{e^{-\frac{\epsilon R}{2}(1-\eta)}}{\delta_2}\left[\begin{array}{c} (1+V_1)\left\{\widehat{X}_2(1)\,\widehat{X}_2(\eta)+\widehat{Y}_2(1)\,\widehat{Y}_2(\eta)\right\} \\ -V_2\left\{\widehat{X}_2(1)\,\widehat{Y}_2(\eta)-\widehat{Y}_2(1)\,\widehat{X}_2(\eta)\right\} \end{array}\right] \\
& +e^{-\frac{\epsilon R}{2}(1-\eta)}\sum_{n=1}^{\infty}\frac{8n\pi\,(-1)^n\sin n\pi\eta}{\widehat{R}_5^2+\widehat{R}_6^2}e^{-\left(\frac{1}{4}\epsilon^2 R+\frac{n^2\pi^2}{R}+\overline{\phi}\right)\tau} \\
& \times\left[\widehat{R}_5\left\{(1+V_1)\cos\tau+V_2\sin\tau\right\}+\widehat{R}_6\left\{V_2\cos\tau-(1+V_1)\sin\tau\right\}\right] \\
& -\frac{e^{-\frac{\epsilon R}{2}(1-\eta)-\left(\epsilon\sqrt{\frac{R}{2}}+\overline{\phi}\right)\tau}}{\widehat{\delta}_1}\left[\begin{array}{c} \cos\left(\epsilon\sqrt{\frac{R}{2}}\tau\right)\left\{\widehat{X}_1(1)\,\widehat{X}_1(\eta)+\widehat{Y}_1(1)\,\widehat{Y}_1(\eta)\right\} \\ +\sin\left(\epsilon\sqrt{\frac{R}{2}}\tau\right)\left\{\widehat{X}_1(1)\,\widehat{Y}_1(\eta)-\widehat{Y}_1(1)\,\widehat{X}_1(\eta)\right\} \end{array}\right] \\
& -e^{-\frac{\epsilon R}{2}(1-\eta)}\sum_{n=1}^{\infty}\frac{8n\pi\,(-1)^n\sin n\pi\eta}{\widehat{R}_1^2+\widehat{R}_2^2}e^{-\left(\frac{1}{4}\epsilon^2 R+\frac{n^2\pi^2}{R}+\overline{\phi}\right)\tau}\left(R_1\cos\tau-R_2\sin\tau\right) \\
& +\frac{e^{-\left(\epsilon\sqrt{\frac{R}{2}}+\overline{\phi}\right)\tau}}{\delta}\left[\begin{array}{c} \cos\left(\epsilon\sqrt{\frac{R}{2}}\tau\right)\left\{X(1)\,X(\eta)+Y(1)\,Y(\eta)\right\} \\ +\sin\left(\epsilon\sqrt{\frac{R}{2}}\tau\right)\left\{X(1)\,Y(\eta)-Y(1)\,X(\eta)\right\} \end{array}\right],
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
\frac{g}{\Omega l} = {} & \frac{e^{-\frac{\epsilon R}{2}(1-\eta)}}{\widehat{\delta}_2}\left[\begin{array}{c} (1+V_1)\left\{\widehat{X}_2(1)\,\widehat{Y}_2(\eta)-\widehat{Y}_2(1)\,\widehat{X}_2(\eta)\right\} \\ +V_2\left\{\widehat{X}_2(1)\,\widehat{X}_2(\eta)+\widehat{Y}_2(1)\,\widehat{Y}_2(\eta)\right\} \end{array}\right] \\
& +e^{-\frac{\epsilon R}{2}(1-\eta)}\sum_{n=1}^{\infty}\frac{8n\pi\,(-1)^n\sin n\pi\eta}{\widehat{R}_5^2+\widehat{R}_6^2}e^{-\left(\frac{1}{4}\epsilon^2 R+\frac{n^2\pi^2}{R}+\overline{\phi}\right)\tau} \\
& \times\left[\widehat{R}_5\left\{V_2\cos\tau-(1+V_1)\sin\tau\right\}-\widehat{R}_6\left\{(1+V_1)\cos\tau+V_2\sin\tau\right\}\right] \\
& -\frac{e^{-\frac{\epsilon R}{2}(1-\eta)-\left(\epsilon\sqrt{\frac{R}{2}}+\overline{\phi}\right)\tau}}{\widehat{\delta}_1}\left[\begin{array}{c} \cos\left(\epsilon\sqrt{\frac{R}{2}}\tau\right)\left\{\widehat{X}_1(1)\,\widehat{Y}_1(\eta)-\widehat{Y}_1(1)\,\widehat{X}_1(\eta)\right\} \\ -\sin\left(\epsilon\sqrt{\frac{R}{2}}\tau\right)\left\{\widehat{X}_1(1)\,\widehat{X}_1(\eta)+\widehat{Y}_1(1)\,\widehat{Y}_1(\eta)\right\} \end{array}\right] \\
& +e^{-\frac{\epsilon R}{2}(1-\eta)}\sum_{n=1}^{\infty}\frac{8n\pi\,(-1)^n\sin n\pi\eta}{\widehat{R}_1^2+\widehat{R}_2^2}e^{-\left(\frac{1}{4}\epsilon^2 R+\frac{n^2\pi^2}{R}+\overline{\phi}\right)\tau}\left(\widehat{R}_1\sin\tau+\widehat{R}_2\cos\tau\right) \\
& +\frac{e^{-\left(\epsilon\sqrt{\frac{R}{2}}+\overline{\phi}\right)\tau}}{\delta}\left[\begin{array}{c} \cos\left(\epsilon\sqrt{\frac{R}{2}}\tau\right)\left\{X(1)\,Y(\eta)-Y(1)\,X(\eta)\right\} \\ -\sin\left(\epsilon\sqrt{\frac{R}{2}}\tau\right)\left\{X(1)\,X(\eta)+Y(1)\,Y(\eta)\right\} \end{array}\right].
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
X(\eta) &= \sinh\sqrt{\frac{R}{2}}\eta\cos\sqrt{\frac{R}{2}},\ Y(\eta) = \cosh\sqrt{\frac{R}{2}}\eta\sin\sqrt{\frac{R}{2}},\\
X(1) &= \sinh\sqrt{\frac{R}{2}}\cos\sqrt{\frac{R}{2}},\ Y(1) = \cosh\sqrt{\frac{R}{2}}\sin\sqrt{\frac{R}{2}},\\
\delta &= \left[\sinh\sqrt{\frac{R}{2}}\cos\sqrt{\frac{R}{2}}\right]^2 + \left[\cosh\sqrt{\frac{R}{2}}\sin\sqrt{\frac{R}{2}}\right]^2,\\
\widehat{R}_1 &= -R^2\epsilon + 4n^2\pi^2 - \frac{4\epsilon R^{\frac{3}{2}}}{\sqrt{2}},\ \widehat{R}_2 = 4R - \frac{4\epsilon R^{\frac{3}{2}}}{\sqrt{2}},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{R}_3 &= \pm\frac{1}{2}\sqrt{\sqrt{\left(\frac{1}{4}R^2\epsilon^2 - R\epsilon\sqrt{\frac{R}{2}}\right)^2 + R^2\left(1-\epsilon\sqrt{\frac{R}{2}}\right)^2} + \left(\frac{1}{4}R^2\epsilon^2 - R\epsilon\sqrt{\frac{R}{2}}\right)},\\
\widehat{R}_4 &= \pm\frac{1}{2}\sqrt{\sqrt{\left(\frac{1}{4}R^2\epsilon^2 - R\epsilon\sqrt{\frac{R}{2}}\right)^2 + R^2\left(1-\epsilon\sqrt{\frac{R}{2}}\right)^2} - \left(\frac{1}{4}R^2\epsilon^2 - R\epsilon\sqrt{\frac{R}{2}}\right)},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{X}_1(\eta) &= \sinh\widehat{R}_3\eta\cos\widehat{R}_4\eta,\ \widehat{Y}_1(\eta) = \cosh\widehat{R}_3\eta\sin\widehat{R}_4\eta,\\
\widehat{X}_1(1) &= \sinh\widehat{R}_3\cos\widehat{R}_4,\ \widehat{Y}_1(1) = \cosh\widehat{R}_3\sin\widehat{R}_4,\\
\widehat{\delta}_1 &= \left[\sinh\widehat{R}_3\cos\widehat{R}_4\right]^2 + \left[\cosh\widehat{R}_3\sin\widehat{R}_4\right]^2,\\
\widehat{R}_5 &= \widehat{R}^2\overline{w}_0^2 + 4n^2\pi^2 + 4\widehat{R}\overline{\phi},\ \widehat{R}_6 = 4\widehat{R},\\
\widehat{X}_2(\eta) &= \sinh\widehat{R}_7\eta\cos\widehat{R}_8\eta,\ \widehat{Y}_2(\eta) = \cosh\widehat{R}_7\eta\sin\widehat{R}_8\eta,\\
\widehat{X}_2(1) &= \sinh\widehat{R}_7\cos\widehat{R}_8,\ \widehat{Y}_2(1) = \cosh\widehat{R}_7\sin\widehat{R}_8,\\
\widehat{\delta}_2 &= \left[\sinh\widehat{R}_7\cos\widehat{R}_8\right]^2 + \left[\cosh\widehat{R}_7\sin\widehat{R}_8\right]^2,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{R}_7 &= \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\frac{1}{4}R^2\epsilon^2 + R\overline{\phi}\right)^2 + R^2} + \left(\frac{1}{4}R^2\epsilon^2 + R\overline{\phi}\right)},\\
\widehat{R}_8 &= \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\frac{1}{4}R^2\epsilon^2 + R\overline{\phi}\right)^2 + R^2} - \left(\frac{1}{4}R^2\epsilon^2 + R\overline{\phi}\right)}.
\end{aligned}
$$

6. Conclusion

In the present work the magnetohydrodynamics effects are applied to the flow of a Newtonian fluid between eccentric rotating porous disks. The flow equations are solved analytically by two different methods. To obtain the large time solution, we applied separation of variable method; and for small time solution,,the Laplace transform method is used. Suction and blowing cases are discussed separately. It is found that with an increase in V_1 and V_2,,the velocity increases;;and with decrease in V_1 and V_2,,the velocity decreases [13]. The results in this paper is similar to the results of [6] if there are no MHD effects and disks are non-porous. It is in general observed that the boundary layers are controlled by the MHD effects [13]. However when the disks are porous, the boundary layer decreases for the suction case (similar to MHD effects) and the boundary layer increases for the injection or blowing case [14].

References

[1] H. Schlichting, *Grenzchichttheorie*, 8th Ed. (Karlsruhe: Braun,1982)

[2] J. A. Shercliff, *A textbook of magnetohydrodynamics*, (Pergamon Press, Oxford, 1965).

[3] P. A. Davidson, *An introduction to magnetohydrodynamics*, (Cambridge University Press, 2001).

[4] Julian Hartmann, *Theory of the Laminar Flow of an Electrically Conductive Liquid in a Homogeneous Magnetic Field,* 1937.

[5] A. Tsinober, *MHD Flow Drag Reduction," in Viscous Drag Reduction in Boundary Layers",* (American Institute of Astronautics and Aeronautics, 1990).

[6] H. V. Ersoy, *Int. J. Engng. Sci.*, **39,** (2001), 343.

[7] T. N. G. Abbott and K. Walters, *J. Fluid Mech.*, **40,** (1970), 205.

[8] R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of polymer liquids*, (Vol. 1. Wiley, New York, 1977).

[9] Muhammad R. Mohyuddin and T. Gotz, *Internat. J. Numer. Methods Fluids*, **49**(8), (2005), 837.

[10] L. Landau and E. M. Liftshitz, *Fluid Mechanics,* (Addison-Wesley, Reading, MA, 1959), pp. $47-48$ $187-188$.

[11] G. K. Batchelor, *An introduction to Fluid Mechanics,* (Cambridge University Press, Cambridge, 1967), Sects. 3.3 and 3.4.

[12] Muhammad R. Mohyuddin, *J. Porous Media*, **9** (3), (2006), 195.

[13] S. Asghar, Muhammad R. Mohyuddin and T. Hayat, *Int. J. Heat Mass transfer*, **48**, (2005), 599.

[14] T. Hayat, M. R. Mohyuddin, S. Asghar and A.M. Siddiqui, *ZAMM*, **84,** (2004), 65.