

Distribution of Circles on a Circle and Correlation Between Vortex Rings of Superfluids

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Abstract

Superfluids are characterized by absence of viscosity. When superfluids are rotated, differently from normal fluids, they form more than one vortex in the containers where they are placed. The number of vortices change as the rotation velocity changes, but this change is not linear. M. W. Zwierlein et al. observed the vortices in experiments, observing up to a number of 80. Experiments also showed that the vortex distributions cannot include large spaces. By using experimental data, we noticed that when we think of vortices as vortex rings, their centers are at the same geometric location and these geometric locations are concentric circles. We generalized the distribution of these geometric places and formulized it. Our formula includes the magic circle numbers. When the number of vortices reach these magic numbers, the number of geometric locations increase by 1.

Key Words: Superfluid, vortex, vortex rings, vortex pattern, circles on a circle.

1. Introduction

Superfluids are characterized by absence of viscosity. Superfluids placed in a closed container can flow without any friction. In other words, superfluids are a phase of matter that can flow infinitely without energy loss. Superfluid's properties are seen in cryogenic fluids. Only cryogenic hydrogen and helium can exhibit superfluid properties. Due to its chemical inertness, quantum mechanical effects, and occurrence in the nature, many experiments are conducted with helium. Although there are 8 known isotopes of Helium, only Helium-3 and Helium-4 are stable. The most common isotope of Helium is Helium-4. In Earth atmosphere, Helium-4 is 10^6 times more abundant than Helium-3, by mole fraction. Helium-4 contains 2 protons, 2 neutrons and 2 electrons and therefore it is a boson. On the other hand Helium-3 particle contains 2 protons, 1 neutron and 2 electrons and hence is a fermion [1,2]. Helium is in gaseous state at temperatures above 4.2 K. However, it liquefies below 4.2 K and above 25 atm pressure. Helium-I state is observed below 4.2 K which is the boiling point. At 2.1768 K, Helium-I becomes Helium-II. This point is called the lambda point [3, 4, 5].

The region of He-I/He-II transition is known as the lambda-line. Both Helium-I and Helium-II are colorless and transparent liquids. The conversion of Helium-I to Helium-II, and hence a superfluid, is related to Bose-Einstein condensation (BEC). Normal/Super fluid transition is not only related to temperature but also related to pressure. Helium-II is stable below the temperatures 2.1768 K and above the pressures 25 atm [1].

Helium-3 can also form a superfluid, but require conditions different from Helium-4, such as a temperature nearly 1000 times reduced. Helium-3 has two different superfluid phases known as the A-phase and B-phase, with the transition between them related to pressure. If the pressure is nearly 34 atm and temperature below 0.0025 K, Helium-3 transitions to the superfluid phase. BEC mechanism is not related to the superfluidity of Helium-3. However, but is instead explained by Bardeen-Cooper-Schrieffer (BCS) theory. According to BCS theory, when electrons pass through a crystal, it shows an introverted contortion and sound packets called phonons are formed. These phonons form a positively charged region in the deformed part and allows the electrons to pass from the same region. This event is called phonon-mediated coupling and allows electrons to pass as couples. This event also admits superconductivity [3]. Some important manifestations of superfluids are thermal superconductivity, superfluidity, the fountain effect, and supersurface films [6, 7].

Superfluid also have the ability of forming vortices. When superfluids are rotated, they display properties different from normal fluids. More than one vortex is formed when superfluids are rotated. Rotation is described by the phase gradient along the direction of the flow in an arbitrary wave function. The wave functions of superfluids are single valued. If the wave function is not single valued then the velocity field decreases and velocity cannot be rotational. Therefore, superfluids which do not have vortices are assumed to be irrotational. If we denote the distance from the center of the vortex by d , then the phase gradient is proportional to $1/d$ and it decreases as we get further from the center of the vortex. The vortices in rotating dilute gas Bose-Einstein condensate have many effects on the BEC system and these vortices are also related to the other supersystems. Examples of the connection of BEC systems with other supersystems are Tkachenko oscillations and sound waves in a vortex lattice [8–10]. Tkachenko [11] considers the lattices consistent with sound waves. Glaberson [12] is the first who observed the vortices in rotating Helium-II.

E. J. Yarmchuk et al. [13] observed the stationary vortex lines in rotating superfluid Helium-II via experiment. They also showed that the formation of vortices is symmetric and their configuration size is certainly related to quantum theory. They observed vortices with vortex numbers from $N = 1$ to $N = 11$ and showed the quantized vortex configurations obey the relation $\sum_{j=1}^N r_j^2 = \frac{\hbar}{m} \frac{N(N-1)}{4\pi\omega}$, where r_j is the distance of j th vortex from the axis of the cylindrical container and ω is the angular velocity.

M. W. Zwierlein et al. [14] observed that Fermi gases allows one to study strongly interacting fermions. The observations of BEC molecules in fermionic atoms has encouraged research into the similarities between BEC and BCS superfluids. M. W. Zwierlein et al. used strongly interacting fermi gases in their experiments and to form these gases fermionic ${}^6\text{Li}$ atoms were used. They observed 80 vortices and their stable patterns. They also searched long-lived lattices in strongly interacting fermi gases and observed BEC-BCS cross interactions.

D. Stauffer et al. [15,16] observed distribution of vortices in a rotating container filled by superfluid Helium-II. They find that an equilibrium distribution of vortices in a rotating container necessarily represents a stationary value of the free energy $F = E - M\Omega$, where E and M are energy and angular momentum of the liquid, respectively. The most important result was that the vortices, in a sense, form concentric circles about the center of the rotating cylindrical container. They find, in rotating Helium-II, vortex arrangement must not contain large separations. They also find that vortex distributions may have different patterns: some can have shapes with 3 vertices, while others have lattices which destroy regularity.

2. Relation Between Vortex Configurations and Circles on a Circle

The observed locations of the vortices at experiments give us an idea about generalizing the locations of vortex rings. If we think the cross-section of vortices of the rotated superfluid as perfect rings, is it possible to place them symmetrically on a circle without contradicting the experimental results? Without loss of generality, we can think the cylindrical container of the superfluid as a unit circle. This is shown in Figure 1. (We used here circle and ring in the same meaning.)

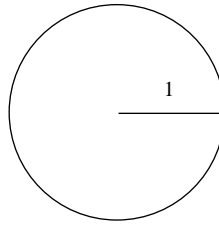


Figure 1. Cylindrical container, Radius = 1.

We employ the following rules for placing the circles. These are:

1. $\pi R^2 - N\pi r^2 > 0$, where R is the radius of the cylindrical container (taken as 1), N is the number of observed vortices, and r is the radius of the vortices
2. r is the same for all vortices.
3. When the centers of the circles placed in the unit circle are on a circle, the circles corresponding to these centers should be tangent to one another. An example for this property is shown in Figure 2.

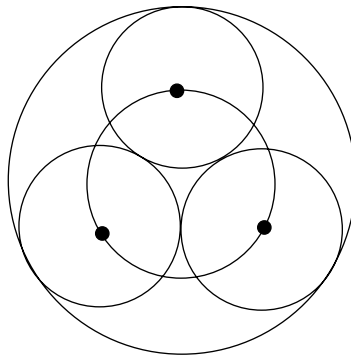


Figure 2. The centers of the circles for 3 vortices form a circle. Corresponding circles are tangent to one another.

By using these properties, for $N = 1$, it is obvious the vortex radius will be equal to 1, and the radius will start to decrease as N increases. We will try to find a relation between the centers of these circles. The first 20 pictures are shown in Figures 3 and 4.

Observations-I

1. Form $N = 1$ to $N = 6$: the centers of the circles are on one circle.
2. From $N = 7$ to $N = 18$: the centers of the circles are placed on two different circles.
3. At $N = 19$ and $N = 20$: the centers of the circles are placed on three different circles.

These observations induce us to ask the following. What are the critical values of the number N such that the centers of the circles start form a new circle?

Observations-II

1. For $N = 7$, when the centers of the circles are connected, triangular patterns are formed. This is shown in Figure 5.
2. For $N = 19$, when the centers of the circles are connected, one rhombus and one triangular (equilateral triangle) patterns are formed, respectively, as shown in Figure 6.

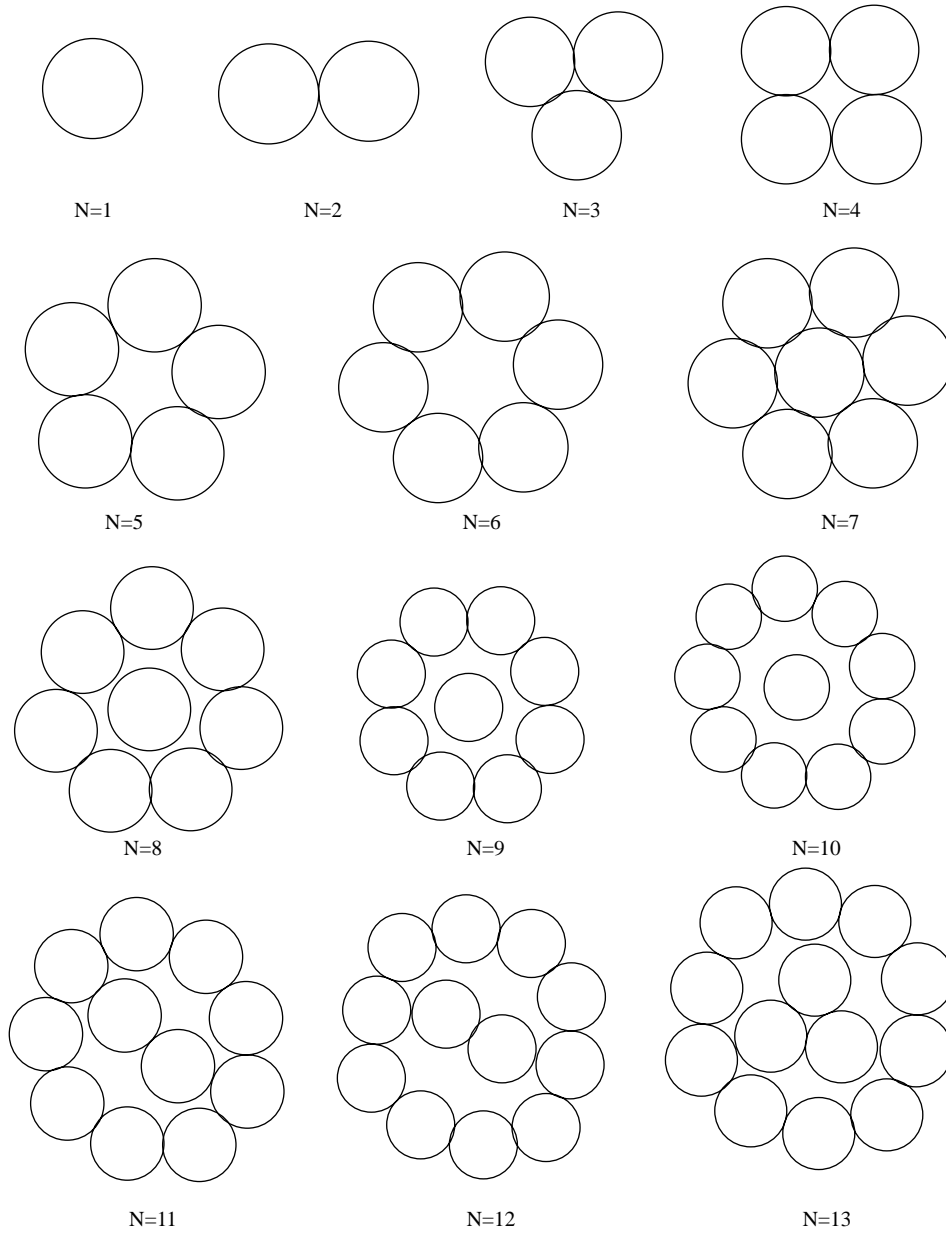


Figure 3. The first 15 circles in a unit circle. The outer unit circles are not drawn.

By using these patterns it is possible to obtain $N = 37$ as follows. Take the figure for $N = 19$ and form patterns (2 rhombus and 1 triangular) around the outer circles. This is shown in Figure 7.

It is possible to continue in this manner to form other circles. For example, $N = 61$ can be obtained from $N = 37$ by putting circles to the outermost circles as forming 3 rhombus and 1 triangular patterns. The next regular arranged one is $N = 91$. Again, $N = 91$ can be formed by putting 4 rhombus and 1 triangular patterns to $N = 61$. These critical N numbers can be obtained up to infinity by using the following rules.

Rule-1: If the centers of the circles form n circles for critical N values, then our shape will have consecutive $(n - 2)$ rhombus patterns (as shown in Figures 6 and 7).

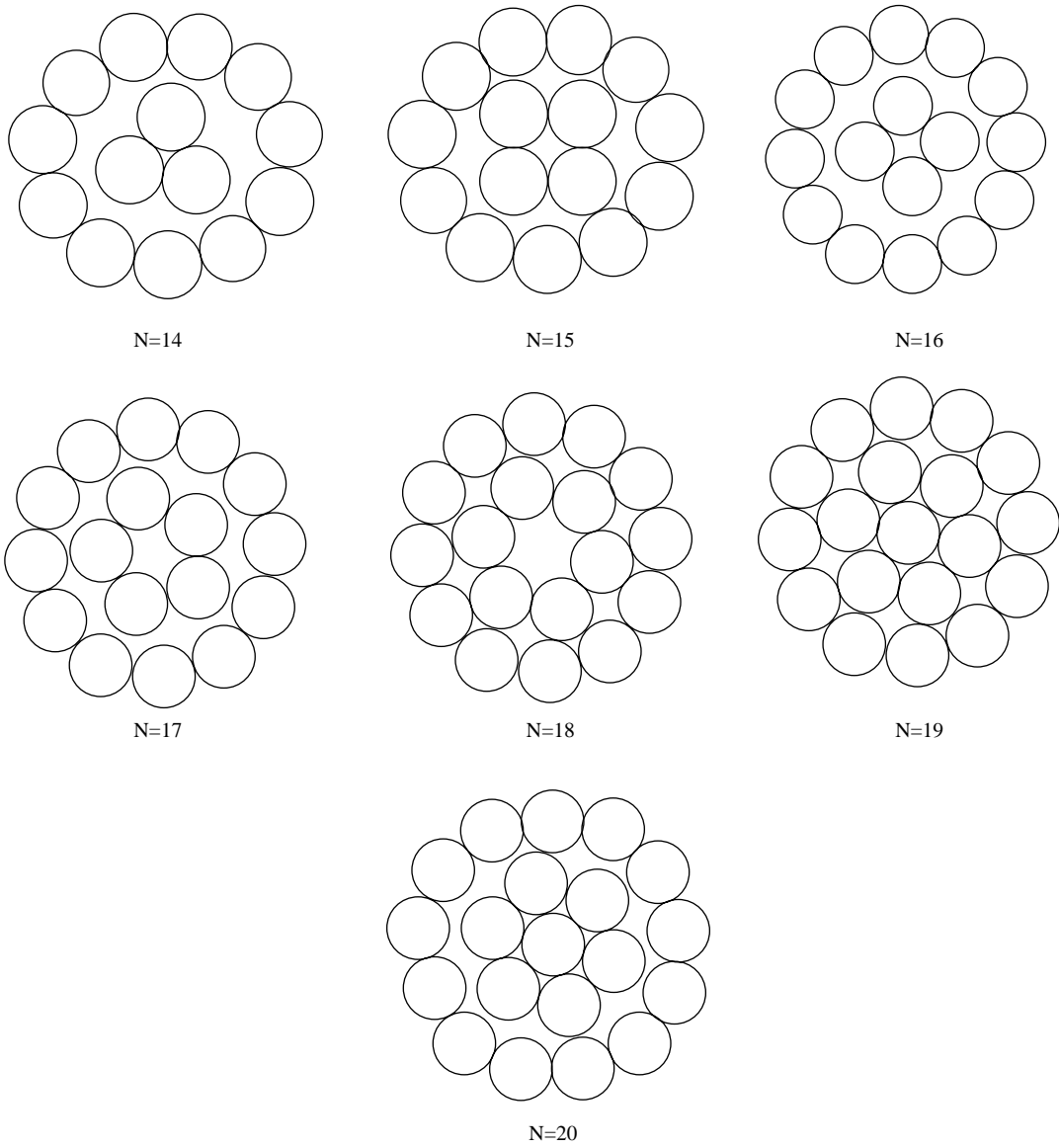


Figure 4. The 16–20 circles in a unit circle. The outer unit circles are not drawn.

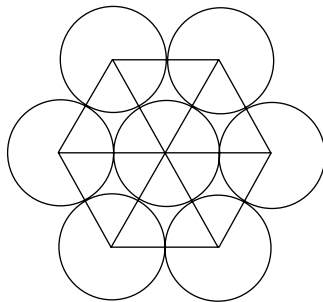


Figure 5. Triangular patterns for $N = 7$.

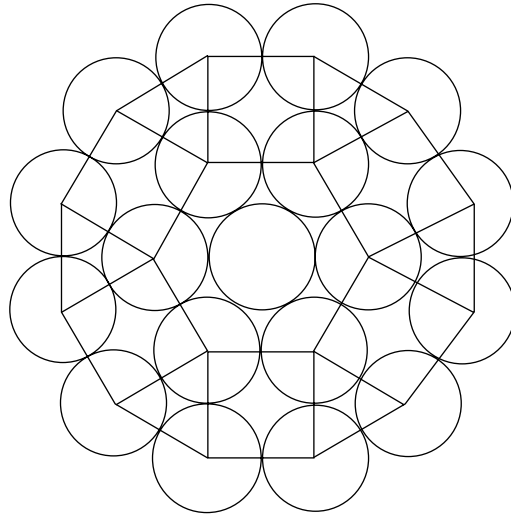


Figure 6. 1 Rhombus and 1 triangular patterns for $N = 19$.

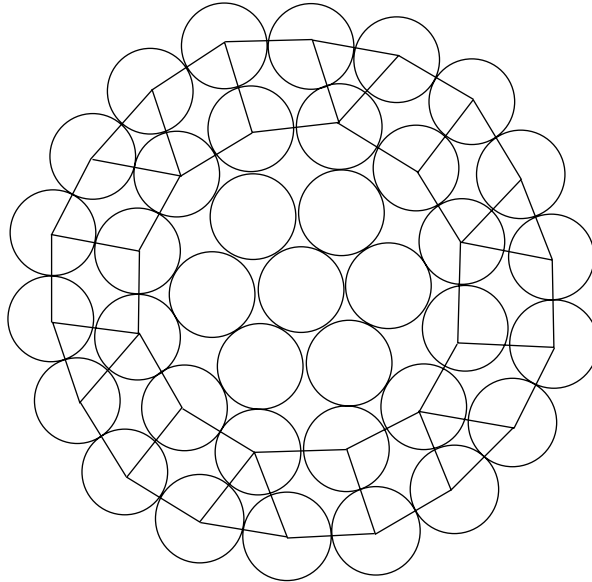


Figure 7. Formation of $N = 37$ from $N = 19$. 2 rhombus and 1 triangular patterns are formed, respectively.

Rule-2: Number of triangular patterns is 1 for all $(n - 2)$ rhombus patterns at critical N values (as shown in Figures 6 and 7).

Important: The innermost circle in these shapes is included in the number n . For example, for $N = 7$, the centers of the circles form 2 circles. So, $n = 2$ for $N = 7$. Hence, number of rhombus patterns is $n - 2 = 0$. Number of triangular patterns is 1.

Observations-III

1. $N = 6$ is obtained from $N = 7$ by removing the central circle. (Figure 3)
2. $N = 18$ is obtained from $N = 19$ by removing the central circle. (Figure 4)
3. $N = 7$ and $N = 19$ have triangular and rhombus patterns.

Hence, we can use $N = 37$ to obtain $N = 36$, $N = 61$ to obtain $N = 60$, $N = 91$ to obtain $N = 90$ and so on. The only thing that has to be done is removing the innermost circle. Formation of $N = 36$ from $N = 37$ is shown in Figure 8.

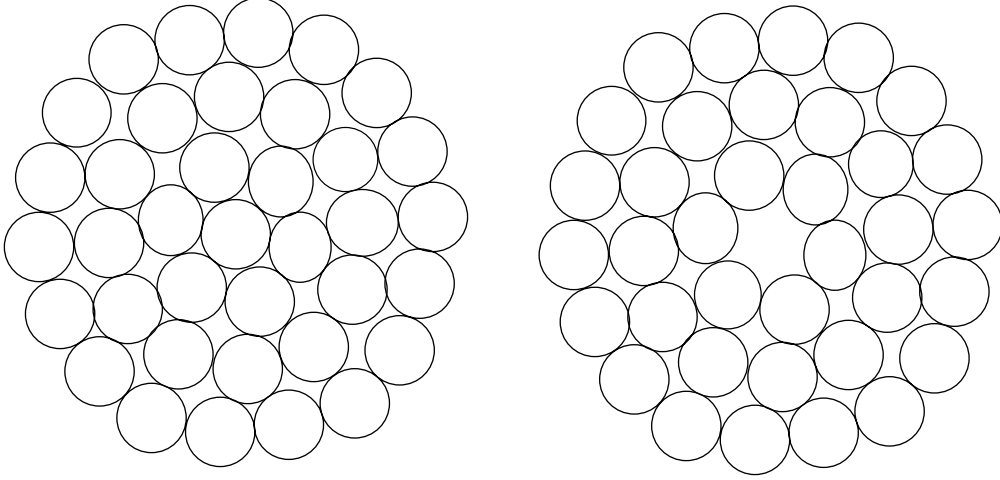


Figure 8. Formation of $N = 36$ from $N = 37$. The innermost circle of $N = 37$ is removed to form $N = 36$.

Observations-IV

The number of circles formed by the centers of the small circles increases by 1 for all critical N values. (Critical N values are 1, 7, 19, 37, 61, 91, ...) So we have the following recursion relation:

Rule-3: Critical values are $N_n = N_{n-1} + 6(n - 1)$ with initial condition $N_1 = 1$.

Notation: N_n as a critical number denotes the number of circles in the unit circle whose centers forms n circles.

For example, for $N = 37$, $n = 4$. Hence, denote this state by N_4 . Another example is $N = 1$. Here $n = 1$, so denote this state by N_1 . Therefore, from now on

- $N = 1 \rightarrow N_1$
- $N = 7 \rightarrow N_2$
- $N = 19 \rightarrow N_3$
- $N = 37 \rightarrow N_4$
- $N = 61 \rightarrow N_5$

and so on.

Example for Rule 3: How many circles in state N_5 ?

Solution:

$$N_2 = N_1 + 6(2 - 1) = 1 + 6 = 7 \text{ (satisfies our observations)}$$

$$N_3 = N_2 + 6(3 - 1) = 7 + 12 = 19 \text{ (satisfies our observations)}$$

$$N_4 = N_3 + 6(4 - 1) = 19 + 18 = 37 \text{ (satisfies our observations)}$$

$$N_5 = N_4 + 6(5 - 1) = 37 + 24 = 61$$

By using this rule we can find all critical N values up to infinity, and this will generalize our problem.

Generalization: Notation: $[i, j]$ will denote the number of circles between i and j . For example, $[1, 6]$ denotes circles with $N = 1, 2, 3, 4, 5, 6$.

By using our rules and observations we obtain the following:

- $[1, 6] \rightarrow 1$ circle ($n = 1$) (satisfies our observations)
- $[7, 18] \rightarrow 2$ circles ($n = 2$) (satisfies our observations)
- $[19, 36] \rightarrow 3$ circles ($n = 3$) (satisfies our observations)
- $[37, 60] \rightarrow 4$ circles ($n = 4$)
- $[61, 90] \rightarrow 5$ circles ($n = 5$)
- $[91, (N_7 - 1)] \rightarrow 6$ circles ($n = 6$)
- $[N_7, (N_8 - 1)] \rightarrow 7$ circles ($n = 7$)

and so on.

So we have the following rule:

Rule-4: $[N_\eta, (N_{\eta+1} - 1)] \rightarrow \eta$ circles ($n = \eta$).

This means that, if we have α circles in the unit circle and if the number α is in the interval $[N_\eta, (N_{\eta+1} - 1)]$ then the centers of these α circles will form η circles. For example, we have $\alpha = 100$: 100 is in the interval $[N_6(= 91), N_7 - 1(= N_6 + 6(7 - 1) - 1 = 126)]$. So the centers of 100 circles form 6 circles. The general distribution of circles are shown in Figure 9. This pattern looks like the distribution of point charges on a disk [17].

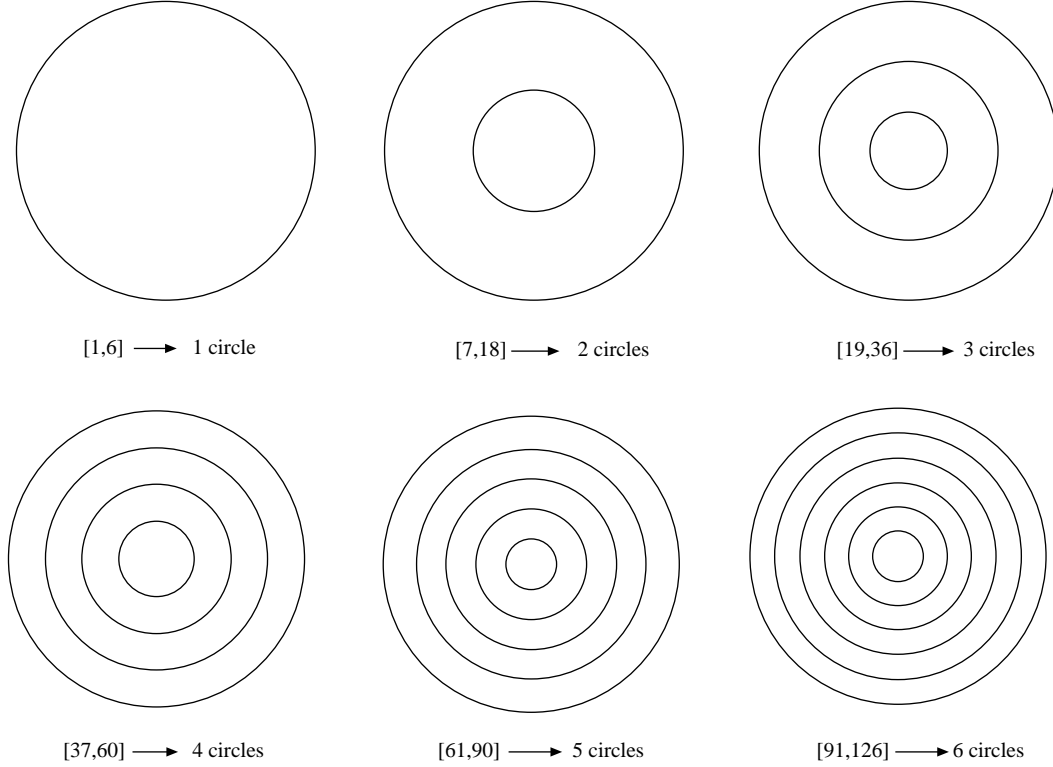


Figure 9. Distribution of vortices on the circles (for first 6 circles only) formed by the vortices' centers.

3. Conclusions

The vortices obtained by experiments are similar to circles whose centers are on the same circles in our study. As the velocity of the rotating superfluid increases, the numbers of vortices increase, in experimental results. This increase is not linear. By using our formulas, the places of the vortices and the shape of their distribution can be obtained for a given number of vortices in a particular rotation velocity. For the magic numbers N_n (critical values) of circles (like 1, 7, 19, ...) an increase by 1 in the number of circles, obtained by connecting the centers of the small circles, is observed. For these magic values we also observed rhombus and triangular patterns. Rhombus patterns show a regular increase as the magic number increases, but triangular patterns are always 1 for all $n - 2$ rhombus patterns. Our results are all consistent with Yarmchuk's, Zwierlein's and Stauffer's studies.

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