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On the Eigensolutions of the One-Dimensional Kemmer Oscillator

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Abstract

In the present paper, we solve the one-dimensional Kemmer equation in the presence of the Dirac oscillator potential. Following Greiner in [23], we have shown that the eigensolutions are decoupled in two sets.

Key Words: Kemmer equation; Dirac equation, Dirac oscillator.
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1. Introduction

In relativistic quantum mechanics, exact solutions of the wave function are very important in the understanding of the physics that can be brought by such solutions.

The relativistic wave function for a massive spin-1 particle was initially derived by Kemmer in 1939 [1]. The Kemmer equation is a Dirac-type equation, which involves matrices obeying a different scheme of commutation rules [1–3]. The massive spin-1 particle, that we consider here, constitutes a two-particle system of spin-1/2 instead of a single spin-1 particle, and therefore the Kemmer equation is a two-body Dirac-like equation. Recently, this equation has particularly got more interest [4–17]. We review the Kemmer equation because of interest in the quark-anti-quark bound state problem.

The Dirac oscillator (DO) is one of the most important quantum systems, as it is one of the very few that can be solved exactly [7, 8, 15, 18, 19]. It was for the first time studied by Ito and Carriere [18]. On the other side, Moshinsky and Szczepaniak [19] were the first to introduce substitution in the free Dirac equation the momentum operator $\vec{p}$ like $\vec{p} - im\beta\omega\vec{x}$, with $\vec{x} = (x, y, z)$ being the position vector, m the mass of the particle and ω the frequency of the oscillator. They could obtain a system in which the positive energy states have a spectrum similar to the one of the non-relativistic harmonic oscillator. It can be shown that the Dirac oscillator interaction is a physical system, which can be interpreted as the interaction of the anomalous magnetic moment with a linear electric field [20, 21].

The Dirac oscillator has aroused a lot of interest both because it provides one of the few examples of Dirac equation exact solvability and because of its numerous physical applications. As a relativistic quantum mechanical problem, the DO has been studied from many viewpoints, including covariance properties, complete energy spectrum and corresponding wave functions, symmetry Lie algebra, shift operators, hidden super symmetry, conformal invariance properties, as well as completeness of wave functions (see [22]). Relativistic many-body problems with Dirac oscillator interactions have been extensively studied with special emphasis on the mass spectra of mesons (quark-anti quark systems) and baryons (three-quark systems). The dynamics of wave packets in a Dirac oscillator has been determined and a relation with the Jaynes-Cummings model established. The (2 + 1) space time has also been shown to be an interesting framework for discussing the DO in connection with new phenomena (such as the quantum Hall effect and fractional statistics) in condensed matter physics. Thermodynamic properties of the DO in (1+1) space time have been mentioned to be relevant to studies on quark-gluon plasma models (see [22] and references therein).

The aim of this paper is the explore the salient features of the Kemmer oscillator in the case of one dimensional. So, this article is planned as follows. In Section II, we calculate the eigenvalues and eigenfunctions of massive spin-1 particles by using the Kemmer equation. Section III will be the conclusion.

2. Eigensolutions of one-dimensional Kemmer oscillator

The Dirac-like relativistic Kemmer equation for spin-1 particles is [1–3]

$$(\beta^\mu p_\mu - Mc)\,\psi_K = 0, \tag{1}$$

where M is the total mass of two identical spin-$\frac{1}{2}$ particles. The 16×16 Kemmer matrices β^μ $(\mu = 0, 1, 2, 3)$ satisfy the relation

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\lambda\nu}\beta^\mu, \tag{2}$$

with

$$\beta^\mu = \gamma^\mu \otimes I + I \otimes \gamma^\mu. \tag{3}$$

In equation (3), I is a 4×4 identity matrix, γ^μ are the Dirac matrices, and $\otimes$ indicates a direct product. In the presence of the Dirac oscillator potential, the momentum operator $\vec{p}$, in the free Kemmer equation, could be substituted by $\vec{p} - iMB\omega\vec{x}$, where the additional term is linear in $|x|$. In this case, the Kemmer equation with a Dirac oscillator interaction is [15]

$$\left[\left(\gamma^0 \otimes I + I \otimes \gamma^0\right) E - c\left(\gamma^0 \otimes \vec{\alpha} + \vec{\alpha} \otimes \gamma^0\right) \cdot \left(\vec{p} - iM\omega B\vec{x}\right) - Mc^2\gamma^0 \otimes \gamma^0\right] \psi_K = 0, \tag{4}$$

where ω is the oscillator frequency, and the operator B is chosen as $B = \gamma^0 \otimes \gamma^0$,with $B^2 = I$. In (1+1) dimensions the standard Dirac γ matrices are replaced by Pauli σ matrices, and the equation (4) becomes

$$\left[\left(\gamma^0 \otimes I + I \otimes \gamma^0\right) E - c\left(\gamma^0 \otimes \sigma_x + \sigma_x \otimes \gamma^0\right) \cdot \left(p_x - iM\omega Bx\right) - Mc^2\gamma^0 \otimes \gamma^0\right] \psi_K = 0, \tag{5}$$

where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

The stationary state ψ_K of equation (5) is four-component wave function of the Kemmer equation, which can be written in the form

$$\psi_K = \psi_D \otimes \psi_D = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \end{pmatrix}^T, \tag{7}$$

where ψ_D is the solution of the Dirac equation. Putting ψ_K given in equation (7) into equation (5), we easily obtain four linear algebraic equations

$$\left(2E - Mc^2\right)\psi_1 - cp_+\psi_2 - cp_+\psi_3 = 0, \tag{8}$$

$$-cp_{-}\psi_1 + Mc^2\psi_2 + cp_{-}\psi_4 = 0, \tag{9}$$

$$-cp_{-}\psi_1 + Mc^2\psi_3 + cp_{-}\psi_4 = 0, \tag{10}$$

$$cp_{+}\psi_2 + cp_{+}\psi_3 - \left(2E + Mc^2\right)\psi_4 = 0, \tag{11}$$

where $p_{\pm} = p_x \pm iM\omega x$. From these equations we get the results:

$$\psi_2 = \psi_3, \psi_1 = \frac{2c}{2E - Mc^2} p_{+}\psi_2, \psi_4 = \frac{2c}{2E + Mc^2} p_{+}\psi_2. \tag{12}$$

Using (12), ψ_1, ψ_3 and ψ_4 are directly eliminated in favor of ψ_2, so one can get

$$\left[\left(\frac{2c^2}{2E + Mc^2} - \frac{2c^2}{2E - Mc^2}\right) p_{-} \cdot p_{+} + Mc^2\right]\psi_2 = 0, \tag{13}$$

where

$$p_{-} \cdot p_{+} = (p_x - iM\omega x) \cdot (p_x + iM\omega x) = p_x^2 + M^2\omega^2 x^2 + M\omega\hbar, \text{ with } [p_x, x] = -i\hbar. \tag{14}$$

After a simple calculation, the wave equation of ψ_2 appearing in equation (13) verifies

$$\left[\frac{d^2}{dx^2} + \xi^2 - \lambda^2 x^2\right]\psi_2(x) = 0, \tag{15}$$

where

$$\xi^2 = \frac{E^2 - \left(mc^2\right)^2}{\hbar^2 c^2} - \frac{M\omega}{\hbar}, \lambda = \frac{M\omega}{\hbar}, \tag{16}$$

and where

$$m = \frac{M}{2} \tag{17}$$

is the mass of the spin-$\frac{1}{2}$ particle. In introducing a new variable, $y = \lambda x^2$, and a new function $\varphi(y)$, linked to ψ_2 like

$$\psi_2(y) = e^{-\frac{y}{2}}\varphi(y), \tag{18}$$

one may simplify (15) into a new form:

$$y\frac{d^2\varphi(y)}{dy^2} + \left(\frac{1}{2} - y\right)\varphi(y) + \left(\frac{\kappa}{2} - \frac{1}{4}\right)\varphi(y) = 0. \tag{19}$$

where κ is

$$\kappa = \frac{\xi^2}{2\lambda} = \frac{1}{2}\left(\frac{E^2 - \left(mc^2\right)^2}{\hbar\omega mc^2} - 1\right). \tag{20}$$

We can identify (19) as Kummer's differential equation (see Greiner [23] and Andrews [24]). The solutions of the equation (19), according the y variable, is then

$$\varphi(y) = A\,{}_1F_1\left(a; \frac{1}{2}; y\right) + By^{\frac{1}{2}}\,{}_1F_1\left(a + \frac{1}{2}; \frac{3}{2}; y\right), \tag{21}$$

where

$$a = -\left(\frac{\kappa}{2} - \frac{1}{4}\right), \tag{22}$$

and where ${}_1F_1(\mu; \upsilon; y)$ is the confluent hyper-geometric function. In terms of the variable x, (21) becomes

$$\psi_2(x) = Ae^{-\frac{m\omega}{2\hbar}x^2} {}_1F_1\left(a; \frac{1}{2}; \frac{m\omega}{\hbar}x^2\right) + Be^{-\frac{m\omega}{2\hbar}x^2}\sqrt{\frac{m\omega}{\hbar}}x\, {}_1F_1\left(a+\frac{1}{2}; \frac{3}{2}; \frac{m\omega}{\hbar}x^2\right), \tag{23}$$

where A and B are a normalizing factors. The solutions of our physical problem follows is determined by the wave function in (23). Therefore the necessary square integrability of ψ implies that ψ_2 must vanish at infinity. This requirement is fulfilled only when the hyper-geometric functions terminate and become polynomials. In this case, the requirement for normalization leads to the quantization of energy.

Following Greiner [23], the solutions of equation (23) can be decoupled, according to parameter a, in two possible cases:

- For $a + \frac{1}{2} = -n$, where $A = 0$, we obtain

$$\frac{\kappa}{2} - \frac{1}{4} = n + \frac{1}{2}, \tag{24}$$

with the eigenfunction

$$\psi_2(x) = Be^{-\frac{m\omega}{2\hbar}x^2} x\, {}_1F_1\left(-n; \frac{3}{2}; \frac{m\omega}{\hbar}x^2\right), \tag{25}$$

and the energy

$$E_n = mc^2\left(1 + 4\frac{\hbar\omega}{mc^2}[n+1]\right)^{\frac{1}{2}}, \tag{26}$$

It is straightforward to check that equation (26) may be written as

$$E_n = mc^2(1 + 4r[n+1])^{\frac{1}{2}}, \tag{27}$$

where the parameter r, which controls the non relativistic limit, is defined by

$$r = \frac{\hbar\omega}{mc^2}. \tag{28}$$

In the non-relativistic limit with $E = \epsilon + mc^2$ and where $\epsilon << mc^2$, the Taylor expansion up to second order of the above would give us

$$E \backsimeq mc^2 + 2(n+1)\hbar\omega - 2(n+1)^2\frac{\hbar^2\omega^2}{mc^2}. \tag{29}$$

It is thus seen that the first term corresponds to the rest energy of the particle, the second term to the non relativistic harmonic oscillator and the third is the relativistic correction term.

- For $a = -n$, where $B = 0$, we have

$$\frac{\kappa}{2} - \frac{1}{4} = n, \tag{30}$$

with the eigenfunction

$$\psi_2(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}\,{}_1F_1\left(-n;\frac{1}{2};\frac{m\omega}{\hbar}x^2\right), \tag{31}$$

and the energy

$$E_n = mc^2\left(1 + r\left[4n+2\right]\right)^{\frac{1}{2}}. \tag{32}$$

In the case of the non-relativistic limit, equation (32) becomes

$$E \backsimeq mc^2 + (2n+1)\,\hbar\omega - \frac{1}{2}(2n+1)^2\frac{\hbar^2\omega^2}{mc^2}. \tag{33}$$

As in the first case, the first term corresponds to the rest energy of the particle, the second term to the non relativistic harmonic oscillator and the third is the relativistic correction term.

The polynomials occurring in (25) and (31) are known as the Hermite polynomials. They are defined by

$$H_{2n}(\xi) = (-1)^n\frac{(2n)!}{n!}\,{}_1F_1\left(-n;\frac{1}{2};\xi^2\right), \tag{34}$$

$$H_{2n-1}(\xi) = (-1)^n\frac{2\,(2n+1)!}{n!}\xi\,{}_1F_1\left(-n;\frac{3}{2};\xi^2\right), \tag{35}$$

where $\xi = \sqrt{\lambda}x$. In this case, and from equations (34) and (35), the eigenfunctions of the massive spin-1 particles can be rewritten into another form as:

- For $a + \frac{1}{2} = -n$, we write

$$\psi_2(x) = N_n e^{-\frac{\lambda}{4}x^2}H_{2n-1}(\xi), \tag{36}$$

and the total associated wave function is

$$(\psi_K)_n(x) = \begin{pmatrix} \frac{2c}{2E-Mc^2}(p_x + iM\omega x) \\ 1 \\ 1 \\ \frac{2c}{2E+Mc^2}(p_x + iM\omega x) \end{pmatrix} N_{\text{norm}}H_{2n-1}(\xi)\,e^{-\frac{\lambda}{4}x^2}. \tag{37}$$

- For $a = -n$, we have

$$\psi_2(x) = N'_n e^{-\frac{\lambda}{4}x^2}H_{2n}(\xi), \tag{38}$$

and the total corresponding associated wave function is

$$(\psi_K)_n(x) = \begin{pmatrix} \frac{2c}{2E-Mc^2}(p_x + iM\omega x) \\ 1 \\ 1 \\ \frac{2c}{2E+Mc^2}(p_x + iM\omega x) \end{pmatrix} N'_{\text{norm}}H_{2n}(\xi)\,e^{-\frac{\lambda}{4}x^2}. \tag{39}$$

The both normalized factors N_{norm} and N'_{norm} are given by

$$N_{\text{norm}} = \sqrt{\frac{1}{\left[\hbar^2c^2\left(a^2-b^2\right)\lambda\left\{2^{2n}\left(2n-\frac{1}{\lambda}\right)-2\left(2n-1\right)\right\}\right]}\frac{1}{(2n-1)!\sqrt{\pi}}}, \tag{40}$$

$$N'_{\text{norm}} = \sqrt{\frac{1}{\left[\hbar^2 c^2 \left(a^2 - b^2\right) \lambda \left\{2^{n+1} \left(n + 1 - \frac{1}{\lambda}\right) - 2n\right\}\right]} \frac{1}{n!\sqrt{\pi}}}, \tag{41}$$

where

$$a = \frac{1}{E - mc^2},\ b = \frac{1}{E + mc^2}. \tag{42}$$

Equations (40) and (41) are obtained by using the fundamental formula [24]

$$(\psi_K, \psi_K) = \int_{-\infty}^{+\infty} \psi_K^\dagger \left(\gamma^0 \otimes \gamma^0\right) \psi_K dx = 1, \tag{43}$$

$$\int_{-\infty}^{+\infty} H_n(y)\, H_m(y)\, e^{-y^2} dy = 2^n n! \sqrt{\pi} \delta_{mn}, \tag{44}$$

$$\int_{-\infty}^{+\infty} H_l(y)\, H_m(y)\, H_n(y)\, e^{-y^2} dy = \frac{2^{\frac{(l+m+n)}{2}} l!m!n!\sqrt{\pi}}{\left(\frac{l+m-n}{2}\right)! \left(\frac{n+l-m}{2}\right)! \left(\frac{n+m-l}{2}\right)!}. \tag{45}$$

3. Conclusion

In this article, and following the prescription of Moshinsky and Szczepaniak [19], the form of the Dirac oscillator potential is included in the Kemmer equation, in order to explore the salient features of the Kemmer oscillator in the case of one dimensional, and to obtain the form of the energy spectrum. Interestingly, this prescription yields the relativistic eigenvalues having unequal spacing. From Tables (1) and (2), we show a comparison between the relativistic energy levels with those non-relativistic, and that in three region according to value's of parameters r: here we comment only on the case where $a + \frac{1}{2} = -n$ and the results are extended to the second case where $a = -n$.

- The first region where the energy of oscillation $\hbar\omega$ is equal to the rest energy mc^2 ($r = 1$): in this case, the non-relativistic energy levels are larger than those in the relativistic case.
- The second region where the energy of oscillation $\hbar\omega$ is comparable to the rest energy mc^2 ($r = 0.1$, $r = 0.01$): the Kemmer oscillator has an appropriate non-relativistic limit.
- Finally, the third region where the energy of oscillation $\hbar\omega$ is smaller than that at the rest mc^2 ($r = 0.00001$): in this case the non-relativistic spectrum of energy is unimportant compared to the relativistic case.

From these three remarks, we can note that the levels of the Kemmer oscillator accumulate when the values of the parameter r decreases (see Figure 1).

To conclude, let us note that the form of the energy spectrum of the 1D Kemmer oscillator can be used in the study of the thermal properties of this oscillator like in the case of the Dirac oscillator for spin-$\frac{1}{2}$ particle [25].

Table 1. The energy spectrum $\overline{E} = \frac{E}{mc^2}$ of the 1D Kemmer oscillator for different values of r where $a + \frac{1}{2} = -n$. Here, we have used the non relativistic limit $\overline{E}_{nr} = 2r\,(n+1)$.

	$r=1$		$r=0.1$		$r=0.01$		$r=0.00001$	
n	$\overline{E}_r$	$\overline{E}_{nr}$	$\overline{E}_r$	$\overline{E}_{nr}$	$\overline{E}_r$	$\overline{E}_{nr}$	$\overline{E}_r$	$\overline{E}_{nr}$
0	2.236	2	1.183	0.2	1.019	0.02	1.00002	0.000002
1	3	4	1.341	0.4	1.039	0.04	1.00004	0.000004
2	3.605	6	1.483	0.6	1.058	0.06	1.00006	0.000006
3	4.123	8	1.612	0.8	1.077	0.08	1.00008	0.000008
4	4.582	10	1.732	1	1.095	0.1	1.0001	0.0001
5	5	12	1.843	1.2	1.113	0.12	1.00012	0.00012
6	5.385	14	1.949	1.4	1.131	0.14	1.00014	0.00014
7	5.744	16	2.049	1.6	1.148	0.16	1.00016	0.00016
8	6.082	18	2.144	1.8	1.166	0.18	1.00018	0.00018
9	6.403	20	2.236	2	1.183	0.2	1.0002	0.0002
10	6.708	22	2.323	2.2	1.2	0.22	1.00022	0.00022
20	9.129	42	3.065	4.2	1.356	0.42	1.00042	0.00042
30	11.180	62	3.66	6.2	1.496	0.62	1.00062	0.00062
40	12.845	82	4.171	8.2	1.624	0.82	1.00082	0.00082
50	14.317	102	4.626	10.2	1.743	1.02	1.00102	0.00102
60	15.652	122	5.039	12.2	1.854	1.22	1.00122	0.00122
70	16.881	142	5.422	14.2	1.959	1.42	1.00142	0.00142
80	18.027	162	5.779	16.2	2.059	1.62	1.00162	0.00162
90	19.105	182	6.115	18.2	2.154	1.82	1.00182	0.00182
100	20.124	202	6.434	20.2	2.244	2.02	1.00202	0.00202

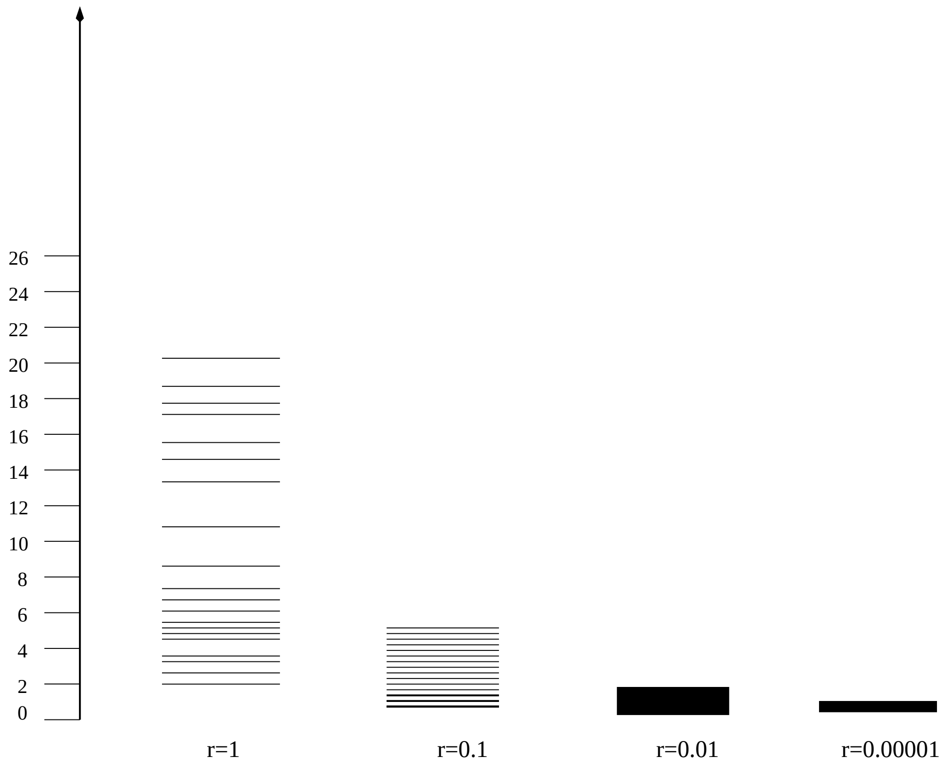

Figure 1. The diagramm of energy $\overline{E}$ of the 1D Kemmer oscillator for different values of r in both cases.

Table 2. The energy spectrum $\overline{E} = \frac{E}{mc^2}$ of the 1D Kemmer oscillator for different values of r where $a = -n$. Here, we have used the non relativistic limit $\overline{E}_{nr} = r\,(2n+1)$.

	$r = 1$		$r = 0.1$		$r = 0.01$		$r = 0.00001$	
n	$\overline{E}_r$	$\overline{E}_{nr}$	$\overline{E}_r$	$\overline{E}_{nr}$	$\overline{E}_r$	$\overline{E}_{nr}$	$\overline{E}_r$	$\overline{E}_{nr}$
0	1.732	1	1,095	0,1	1,009	0.01	1,00000	0.00001
1	2.645	3	1,264	0.3	1,029	0.03	1,00002	0.00003
2	3,316	5	1,414	0.5	1,048	0.05	1,00004	0.00005
3	3,872	7	1,549	0.7	1,067	0.07	1,00006	0.00007
4	4,358	9	1,673	0.9	1,086	0.09	1,00008	0.00009
5	4,795	11	1,788	1.1	1,104	0.11	1,00010	0.00011
6	5,196	13	1,897	1.3	1,122	0.13	1,00012	0.00013
7	5,567	15	2	1.5	1,140	0.15	1,00014	0.00015
8	5,916	17	2,097	1.7	1,157	0.17	1,00016	0.00017
9	6,244	19	2,190	1.9	1,174	0.19	1,00018	0.00019
10	6,557	21	2,280	2.1	1,191	0.21	1,00020	0.00021
20	9,11	41	3,033	4.1	1,349	0.41	1,00040	0.00041
30	11,09	61	3,633	6.1	1,489	0.61	1,00060	0.00061
40	12,767	81	4,147	8.1	1,618	0.81	1,00080	0.00081
50	14,247	101	4,604	10.1	1,737	1.01	1,00100	0.00101
60	15,588	121	5,019	12.1	1,849	1.21	1,00120	0.00121
70	16,822	141	5,403	14.1	1,954	1.41	1,00140	0.00141
80	17,972	161	5,761	16.1	2,054	1.61	1,00160	0.00161
90	19,052	181	6,099	18.1	2,149	1.81	1,00180	0.00181
100	20,074	201	6,418	20.1	2,24	2.01	1,00200	0.00201

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