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The Asymptotic Iteration Method for the Eigenenergies of the Complex Potential $V(x) = \gamma x^4 + i\beta x^3 + i\alpha x$

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Abstract

Recently, three complex potentials $V(x) = ix^3$, $V(x) = ix^3 + i\alpha x$, and $V(x) = x^4 + i\alpha x$ have been studied in the literature. Here, we combine these potentials in one. With the aid of the asymptotic iteration method we have numerically calculated the eigenenergies of the new complex potential. The obtained numerical results are compared with those obtained by using the WKB, EMM, and MRF methods and found to be in an excellent agreement. We discuss how an adjustable parameter ζ can help to improve our results.

Key Words: on-Hermitian potential, Schrödinger equation, the asymptotic iteration method, an adjustable parameters.

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1. Introduction

Searching for eigenvalues E_n generated by the non-Hermitian potentials in one dimension has acquired much interest in recent years. Bender and Boettcher [1] have suggested that instead of Hermiticity, it might be enough to have PT-invariant potential so as to have real eigenenergies. As first potential model, Bender and Boettcher [1] have studied the Schrödinger eigenvalue differential equation

$$-\psi^{''}(x) + V(x)\psi(x) = E_n\psi(x), \tag{1}$$

where $V(x)$ is a complex potential of the form

$$V(x) = ix^3. \tag{2}$$

In general, when one make the transformation of $x \to -x$ and $i \to -i$, if the relation $V(-x) = V^*(x)$ exists, the potential $V(x)$ is said to be PT -symmetric. Bender and Boettcher [1] applied the Runge-Kutta technique and the WKB approximation in the complex plane to obtain the discrete energy eigenvalues of the PT-symmetric non-Hermitian potential (2). Bender and Boettcher [1] showed that the discrete spectrum generated by the potential (2) should be real. Using different methods, similar investigations [2–4] have agreed with the Runge-Kutta results. As second potential model, Delabaere and Trinh [5] applied asymptotic

method, while Handy et al [6–7] applied the eigenvalue moment method (EMM), and multiscale reference function (MRF) approach to calculate the eigenenergies of the potential

$$V(x) = ix^3 + i\alpha x, \tag{3}$$

where α is a real parameter and they studied PT-invariance of the potential (3), and have shown that such potential exhibits two types of behavior. The first type in the unbroken PT-symmetry phase, yields real and discrete eigenenergies so long as $\alpha > \alpha_{\text{critical}}$. While the second type, when PT-symmetry is spontaneously broken through the potential retains PT - symmetry, this type yields arranged eigenenergies as complex conjugate pairs so long as $\alpha < \alpha_{\text{critical}}$. As a third potential model, Bender et al [8], Delabaere and Pham [9] applied the WKB approximation method and WKB exact method to generate values for discrete eigenenergies states of the PT -symmetry non- Hermitian potential

$$V(x) = x^4 + i\alpha x, \tag{4}$$

where α is a parameter value. Bender et al [8] showed that at critical α-parameter values, various real eigenvalue curves intersect (i.e. the first two discrete starts at $\alpha \approx 3.169053$, the next two at $\alpha \approx 7.62595$), making the onset of complex-E discrete state formation. Moreover, further study of potential (4) has been done by Handy et al [10] who used the EMM to generate the low-lying bound states for the potential (4).

Recently, the asymptotic iteration method (AIM) has been introduced by Ciftci et al [11] for solving the eigenvalue problem. They showed that the AIM yields the correct answer for exactly solvable models and gives reasonable approximation results for some nontrivial one-dimensional problems such as anharmonic oscillators and singular potentials. Furthermore, the AIM handles E_n with large n which are used to pose many numerical instabilities to some of the previously mentioned methods. In this study, we will combine the above three potentials in a single new complex potential, for which the eigenvalues E_n of each one of the three potentials above can be calculated through new single complex potential for each one. Therefore, the main aim of the present paper is to use the AIM to formulate algebraic approach within the new complex potential to calculate the eigenenergies to the high degree of accuracy for the complex potentials (2), (3), and (4).

The organization of the present article is as follows. In Sect 2, we define our complex potential which includes the aforesaid three potentials. In section 3, we give a brief review of AIM method to obtain numerically the eigenenergies. In section 4, numerical results and discussion of ground state and excited states are given. Finally, in section 5, the respective discussion and summary of our numerical results are presented.

2. The Formalism of a New Complex Potential

In this section we will build a complex potential which includes the potentials (2), (3), and (4). Under some constraints on the parameters values of the new complex potential, the use of the new potential representation will be able to combine the three complex potentials. To achieve this, we introduce the new complex potential as:

$$V(x) = \gamma x^4 + i\beta x^3 + i\alpha x, \tag{5}$$

where, at least one of the parameters α, β, and γ must equal zero, and, only one of them must equal one.

Thus, the formalism of potential (5) allows us to obtain the potentials (2), (3), and (4). We shall now determine the potentials corresponding to various values of the parameters α, β, and γ as follows:

Case 1: Let $\alpha = \gamma = 0$, and $\beta = 1$. In this case we have a potential similar to the potential (2).

Case 2: Let $\gamma = 0$, and $\beta = 1$. In this case we have a potential similar to the potential (3).

Case 3: Let $\beta = 0$, and $\gamma = 1$. In this case we have a potential similar to the potential (4).

Therefore, by applying the AIM we will show that potential (5) can be used to calculate the eigenenergies of the potentials (2), (3), and (4).

3. Asymptotic Iteration Solution to the Eigenvalue Problem

In this section we shall outline the general procedure of the AIM for determining numerical solvability of the eigenvalue differential equation. Let us now return to the potential (5). The general form for the eigenvalue equation with (5) is (we work in unit where $\hbar = 1$)

$$\frac{-1}{2m}\psi^{''}(x) + V(x)\psi(x) = E_n\psi(x). \tag{6}$$

Equation (6) is defined on a certain complex curve $x = x(t) \in \mathcal{C}$, $t \in (-\infty, \infty)$, and $\{\alpha, \beta, \gamma\} \in \boldsymbol{R}$. The full illustration for this property is available in the literature [12, 13]. For the sake of general solutions of equation (6), we make the ansatz

$$\psi(x) = e^{-\zeta\frac{x^2}{2}}\ y(x), \tag{7}$$

where ζ is an adjustable parameter introduced to improve the rate of convergence of the AIM [14]. Substitution of (7) into (6) gives the homogeneous linear second-order differential equation

$$y^{''}(x) = k_0(x)y^{'}(x) + z_0(x)y(x), \tag{8}$$

with functions $k_0(x)$ and $z_0(x)$ given by

$$\begin{aligned} k_0(x) &= 2\,\zeta\,x \\ z_0(x) &= \zeta - \zeta^2x^2 + 2\,im\alpha x + 2\,im\beta x^3 + 2\,m\gamma x^4 - 2\,mE. \end{aligned} \tag{9}$$

The eigenfunctions and eigenenergies of the potential (5) may now be explicitly formulated by using the techniques of AIM. We do not give the details of the AIM which could be found in references [11, 14-20]. Thus for some $j > 0$ if one can obtain a representation for $\varphi(x)$ as

$$\varphi_j(x) = \frac{z_j(x)}{k_j(x)} \equiv \frac{z_{j-1}(x)}{k_{j-1}(x)}, \tag{10}$$

with

$$\begin{aligned} k_j(x) &= k_{j-1}^{'}(x) + z_{j-1}(x) + k_0(x)k_{j-1}(x), \\ z_j(x) &= z_{j-1}^{'}(x) + z_0(x)k_{j-1}(x). \end{aligned} \tag{11}$$

Equation (10) leads to exact results after a finite number of iterations j for some trivial cases and approximate results for nontrivial cases [11, 14]. In this case, with the help of the equations (9) and (11), one can speculate that in some suitable large j the equation for approximate eigenenergies E_n can be obtain from the roots of

$$\delta_j(x) = k_{j-1}(x)z_j(x) - z_{j-1}(x)k_j(x). \tag{12}$$

To complete this section, it is important to mention that, the general solution of equation (8) is given by combination of two independent fundamental solutions

$$y(x) = \exp\left(-\int \varphi(x)dx\right)\left[C_2 + C_1\int\left(\exp\left[\int\{k_0(x) + 2\varphi(x)\}\,dx\right]\right)dx\right]. \tag{13}$$

4. Results and Discussion

Based on the clear results in the previous section, it is possible to numerically determine the eigenenergies E_n by the condition given in equation (12). It should also be pointed out that, to obtain the eigenenergies E_n, the condition of the iterations should be terminated at the point by putting $\delta_j(x)$ terms to be zero. As it is explicitly seen, in each iteration, the expression $\delta_j(x) = k_{j-1}(x)z_j(x) - z_{j-1}(x)k_j(x)$ will depend

on two variables E_n and x. The calculated eigenenergies E_n by means of this condition should, however, be dependent on the choice of x. The chosen value of x is arbitrary in principle, and affects the rate of convergence of the method [14]. It is observed that the optimal choice of x is when $x = 0$. In Table 1. Test computations were successfully performed on the potential (5) between two values of an adjustable parameter ζ with $\gamma = 1$, $\beta = 0$, $\alpha = \frac{315}{100}$. The numerical results confirm that the rate of convergence of the method yield to best convergence rate when $\zeta = 4$ because of a faster convergence rate of $\zeta = 4$. We have truncated the iterations at $j = 70$. Factually, convergence was achieved after $j = 60$ iterations. The calculated results of $\triangle\ E_0^j = E_0^j - E_0^{j-1}$, shows the rate of convergence. A calculation of full perfect convergence gives $\triangle E_0^j = 0$ when $j = 60$. Furthermore, Table 1 shows that our value for E_0 is in perfect agreement with the results of Handy et al [10] within the first 8 digits, and our value for E_0 is inside the bounds provided by the EMM. In Tables 2–5 we give the eigenenergies E_n for the first four states for the potential (5) with $\gamma = 0$ and $\beta = 1$ obtained using the AIM by means of 45 iterations, and compare them with the results of Handy et al [6] obtained by the eigenvalue moment method (EMM), and multiscale reference function (MRF) approach. It is clear from these tables that our calculated eigenenergies E_n are in a good agreement with eigenenergies

Table 1. AIM, MRF, and EMM eigenenergies E_1 for the potential for $V(x) = \gamma x^4 + i\beta x^3 + i\alpha x$ with $\gamma = 0$ and $\beta = 1,\ \ 2m = 1$, $\zeta = 2$.

α	$E_{\rm AIM}$	$E_{\rm MRF}$ [7]	$E_{\rm EMM}$ [7]
-5	1.343343304 - 2.907390643 i	1.3433409 ± 2.9073602 i	1.343311 (1.343354)+ 2.9073 (2.9075) i
$-\frac{9}{2}$	1.29924235822265 - 2.31251547361173 i	1.2992519 ± 2.3124924 i	1.299242 (1.299252) + 2.3124 (2.3126) i
-4	1.248656742 - 1.761719301 i	1,2486637 ± 1.7617076 i	1.248637 (1.248666) + 1.761688 (1.761742) i
$-\frac{7}{2}$	1.212436734 - 1.260909973 i	1.2124399 ± 1.2609114 i	1.212421 (1.212448) + 1.26088 (1.26094) i
-3	1.225847578 - .7600224709 i	1.2258438± .7600296 i	1.225837 (1.225864) + .76000 (.76004) i
$-\frac{5}{2}$	.9280003422	.9280136	.92799980 (.92800101)
-2	.6209135740	.6209137	.62091347 (.62091386)
$-\frac{3}{2}$	.5964933841	.5964936	.59649326 (.59649351)
-1	.6999599208	.6999615	.69995977 (.69995978)
$-\frac{1}{2}$	.8926684336	.8926699	.89266872 (.89266849)
0	1.156267072	1.1562673	1.15626695 (1.15626718)
$\frac{1}{2}$	1.479851861	1.4798519	1.47985179 (1.47985206)
1	1.856110766	1.8561128	1.85611065 (1.85611108)
$\frac{3}{2}$	2.279752048	2.2797563	2.27975185 (2.27975232)
2	2.746739981	.2.7467434	2.74673952 (2.74674023)
$\frac{5}{2}$	3.253876926	3.2538767	3.25387596 (3.25387723)
3	3.798554701	3.7985559	3.79855387 (3.79855395)
$\frac{7}{2}$	4.378596946	4.3786140	4.37859645 (4.37859736)
4	4.992154083	4.9921974	4.99215436 (4.99215504)
$\frac{9}{2}$	5.637630446	5.6376822	5.63763149 (5.63763200)
5	6.313632040	6.3136428	6.31359739 (6.31360665)

Table 2. AIM, MRF, and EMM eigenenergies E_2 for the potential for $V(x) = \gamma x^4 + i\beta x^3 + i\alpha x$ with $\gamma = 0$ and $\beta = 1,\ \ 2m = 1, \zeta = 2$.

α	$E_{\rm AIM}$	$E_{\rm MRF}$ [7]	$E_{\rm EMM}$ [7]
-5	1.343343304 + 2.907390643 i	1.3433409 ± 2.9073602 i	1.343311 (1.343354)+ 2.9073 (2.9075) i
$-\frac{9}{2}$	1.299242358 + 2.312515474 i	1.2992519 ± 2.3124924 i	1.299243 (1.299252) + 2.3142 (2.3126) i
-4	1.248656742 + 1.761719301 i	1,2486637 ± 1.7617076 i	1.248637 (1.248666) + 1.761688 (1.761742) i
$-\frac{7}{2}$	1.212436734 + 1.260909973 i	1.2124399 ± 1.2609114 i	1.212421 (1.212448) + 1.26088 (1.26094) i
-3	1.225847578 + .7600224709 i	1.2258438± .7600296 i	1.225837 (1.225864) + .76000 (.76004) i
$-\frac{5}{2}$	1.685979343	1.6859358	1.68597765 (1.68598087)
-2	2.292292501	2.2922626	2.29229055 (2.29229333)
$-\frac{3}{2}$	2.742529394	2.7425268	2.74272667 (2.74253034)
-1	3.179715777	3.1797220	3.17971312 (3.17971750)
$-\frac{1}{2}$	3.632074462	3.6320373	3.63207237 (3.63207767)
0	4.109228753	4.1091279	4.10922704 (4.10923558)
$\frac{1}{2}$	4.614838727	4.6147402	4.61483391 (4.61484633)
1	5.150168955	5.1501688	5.15016059 (5.15017640)
$\frac{3}{2}$	5.715408707	5.7154576	5.71538438 (5.71541649)
2	6.310238361	6.3100192	6.31020527 (6.31025282)
$\frac{5}{2}$	6.934096041	6.9332376	6.93405453 (6.93412954)
3	7.586310989	7.5850094	7.58627841 (7.58638028)
$\frac{7}{2}$	8.266172869	8.2656580	8.26613065 (8.26633647)
4	8.972968434	8.9745543	8.97272640 (8.97332836)
$\frac{9}{2}$	9.706000860	9.7077326	9.70210565 (9.70789436)
5	10.46459958	10.4575130	10.45227656 (10.48006875)

E_n calculated with other methods [6]. As special case in Tables $2-5$, it is important to note that in the case of $\alpha = 0$ the results are actually in reasonable agreement with the results in [1, 4, 6] for the potential (2). In the preceding tables we have calculated the eigenenergies only for the first four states. This was in order to make comparison between the AIM and the results of [1, 4, 6]. Actually by using this method we can extend our calculated eigenenergies for the highest excited states. Finally, by trial and error, we found that the faster rate of convergence in Tables 2-5 is that when $\zeta = 2$. Moreover, if we use α_{crit} directly from [7], the AIM can be used to calculate the eigenenergies for $\alpha_{\rm crit}$. It gives a good approximation for the eigenenergies $E_{cri_{1,2}}$. At these critical points given by [7] the energy goes from being real (PT- invariant solutions) to complex (PT- breaking solutions). They are

$\alpha_{cr_1} = -\frac{2611809356}{1000000000}\ \ (\gamma = 0, \beta = 1)$ corresponding to $E_{cr_1} = 1.28274561544057$, and

$\alpha_{cr_2} = -\frac{5375879629}{1000000000}\ \ (\gamma = 0, \beta = 1)$ corresponding to $E_{cr_2} = 4.18128166000600$.

These eigenenergies result which calculated by means of 70 iterations, and with $\zeta = 2$ are in general agreement with the earlier calculation [7]. In Tables 6 and 7, we present the first four energy levels for potential (4) (i.e. Potential (5) with $\beta = 0$, and $\gamma = 1$) for different values of α, and with adjustable parameter $\zeta = 4$. These results are in agreement with previously reported results [10]. For the critical points ($\alpha_{\rm crit1,2}$) it is found that the AIM works very well and gives results of good accuracy. We have calculated the eigenenergies for the potential (5) by means of 60 iterations. Other eigenenergies for higher excited states

Table 3. AIM, MRF, and EMM eigenenergies E_3 for the potential for $V(x) = \gamma x^4 + i\beta x^3 + i\alpha x$ with $\gamma = 0$ and $\beta = 1,\ \ 2m = 1,\ \zeta = 2$.

α	$E_{\rm AIM}$	$E_{\rm MRF}$ [7]	$E_{\rm mEMM}$ [7]
$-\frac{15}{2}$	4.410337852 - 3.266538423 i	4.4103093 ± 3.2664292 i	4.408662 (4.411426) + 3.2645 (3.2700) i
-7	4.312023756 - 2.563558451	4.3120349 ± 2.5634751 i	4.310530 (4.313107) + 2.5618 (2.5674) i
$-\frac{11}{2}$	4.157476964 - .5313530951 i	4.1575135 ± .5313291 i	4.155648 (4.158235) + .5300 (.5370) i
-5	3.431382846	3.4314015	3.43136741 (3.43139540)
-4	3.508765600	3.5087615	3.50876099 (3.50877544)
$-\frac{7}{2}$	3.877693006	3.8776981	3.87768485 (3.87770178)
-3	4.333439839	4.3334536	4.33342275 (4.33344654)
$-\frac{5}{2}$	4.822975397	4.8229806	4.82294849 (4.82298427)
$-\frac{3}{2}$	5.857622002	5.8576148	5.85759578 (5.85766576)
$-\frac{1}{2}$	6.971403910	6.9714613	6.97133951 (6.97152763)
0	7.562273855	7.5622889	7.56215901 (7.56242355)
$\frac{1}{2}$	8.177082532	8.1770143	8.17687201 (8.17720644)
$\frac{3}{2}$	9.479889209	9.4802011	9.47913594 (9.48044230)
$\frac{5}{2}$	10.88022600	10.8807993	10.87896180 (10.88253390)
3	11.61644566	11.6159802	11.61535000 (11.62075000)
$\frac{7}{2}$	12.37627694	12.3755952	12.37090000 (12.38980000)

Table 4. AIM, MRF, and EMM eigenenergies E_4 for the potential for $V(x) = \gamma x^4 + i\beta x^3 + i\alpha x$ with $\gamma = 0$ and $\beta = 1,\ \ 2m = 1,\ \zeta = 2$.

α	$E_{\rm AIM}$	$E_{\rm MRF}$ [7]	$E_{\rm EMM}$ [7]
$-\frac{15}{2}$	4.410337852 + 3.266538423 i	4.4103093 ± 3.2664292 i	4.408662 (4.411426) + 3.2645 (3.2700) i
-7	4.312023756 + 2.563558451 i	4.3120349 ± 2.5634751 i	4.310530 (4.313107) + 2.5618 (2.5674) i
$-\frac{11}{2}$	4.157476964 + .5313530951 i	4.1575135 ± .5313291 i	4.155648 (4.158235) + .5300 (.5370) i
-5	5.167890381	5.1678291	5.16784214 (5.16798149)
-4	6.379805337	6.3796826	6.3796997 (6.37985480)
$-\frac{7}{2}$	6.949052345	6.9490904	6.94880880 (6.94915872)
-3	7.525191974	7.5252398	7.52489539 (7.52536568)
$-\frac{5}{2}$	8.113377228	8.1130836	8.11302413 (8.11368768)
$-\frac{3}{2}$	9.337028960	9.3364444	9.33658547 (9.33774669)
$-\frac{1}{2}$	10.63501627	10.6352870	10.63343056 (10.63660576)
0	11.31442183	11.3120046	11.31165120 (11.31588480)
$\frac{1}{2}$	12.01482436	12.0072541	12.01004000 (12.01676000)
$\frac{3}{2}$	13.47956320	13.4761039	13.47000000 (13.51500000)
$\frac{5}{2}$	15.02983115	15.0088412	14.96800000 (15.22000000)
3	15.83682846	15.8347839	15.71250000 (16.12500000)
$\frac{7}{2}$	16.66484822	16.4020563	16.40000000 (17.00000000)

can be obtained by AIM. The accuracy of the results for higher excited states can be increased if the number of iterations are increased.

Table 5. The comparison of the AIM results (present work) with the eigenvalue moment method [10] for the first four eigenenergies states E_n of the potential $V(x) = \gamma x^4 + i\beta x^3 + i\alpha x$, with $\gamma = 1$ and $\beta = 0$, $2m = 1$, $\zeta = 4$.

α	n	E_{AIM}	E_{EMM} [10]
0	0	1.06036209048429	$1.06036209048168 < E_R < 1.06036209049133$
0	1	3.79967302979615	$3.79967302969810 < E_R < 3.79967303009943$
0	2	7.45569793797516	$7.45569793646236 < E_R < 7.45569794450762$
0	3	11.6447455106200	$11.6447454124944 < E_R < 11.6447455916097$
$\frac{1}{2}$	0	1.09346613918847	$1.09346613915868 < E_R < 1.09346613919601$
$\frac{1}{2}$	1	3.80350288034512	$3.80350288028026 < E_R < 3.80350288067461$
$\frac{1}{2}$	2	7.46085427096371	$7.46085426922425 < E_R < 7.46085427717100$
$\frac{1}{2}$	3	11.6488362016166	$11.6488361053757 < E_R < 11.6488362844910$
1	0	1.19448994170113	$1.19448994169622 < E_R < 1.19448994171063$
1	1	3.81335726488546	$3.81335726478537 < E_R < 3.81335726521582$
1	2	7.47632956052414	$7.47632955885712 < E_R < 7.47632956659295$
1	3	11.6610744929874	$11.6610743806709 < E_R < 11.6610745841871$
2	0	1.63073079429846	$1.63073079428893 < E_R < 1.63073079431949$
2	1	3.82146752822032	$3.82146752813636 < E_R < 3.82146752871907$
2	2	7.53864646294889	$7.53864645991553 < E_R < 7.53864647040870$
2	3	11.7095061015968	$11.7095059093736 < E_R < 11.7095062316998$
3	0	2.62269905717768	$2.62269905710335 < E_R < 2.62269905727884$
3	1	3.57016001805762	$3.57016001789298 < E_R < 3.57016001884370$
3	2	7.64703040600171	$7.64703040200832 < E_R < 7.64703041484659$
3	3	11.7882106757446	$11.7882103641407 < E_R < 11.7882108662897$
$\frac{31}{10}$	0	2.83473212726125	$2.83473212710682 < E_R < 2.83473212740229$
$\frac{31}{10}$	1	3.44820508453984	$3.44820508436344 < E_R < 3.44820508538680$
$\frac{31}{10}$	2	7.66094518361486	$7.66094518003906 < E_R < 7.66094519223550$
$\frac{31}{10}$	3	11.7976221025666	$11.7976217616381 < E_R < 11.7976223168970$
$\frac{315}{100}$	0	3.00238802243326	$3.00238802194582 < E_R < 3.00238802269777$
$\frac{315}{100}$	1	3.32665289150960	$3.32665289127798 < E_R < 3.32665289269236$
$\frac{3169035}{1000000}$	0	3.17213028319439	$3.17213027251438 < E_R < 3.17213028512914$
$\frac{3169035}{1000000}$	1	3.17464778299582	$3.17464778126365 < E_R < 3.17464779415344$
4	3	7.82259326851421	$7.82259326098411 < E_R < 7.82259328224012$
4	4	11.8932109128598	$11.8932103468105 < E_R < 11.8932112547331$
$\frac{15}{2}$	3	10.6834000768055	$10.6833991858115 < E_R < 10.6834037179701$
$\frac{15}{2}$	4	11.7968384964128	$11.7968288611545 < E_R < 11.7968421504399$
$\frac{762595}{100000}$	3	11.3225647499793	$11.3225381250000 < E_R < 11.322588750000$
$\frac{762595}{100000}$	4	11.3326538497117	$11.3326241250000 < E_R < 11.3326815000000$

Table 6. The rate of convergence of the AIM for the ground state for the potential $V(x) = \gamma x^4 + i\beta x^3 + i\alpha x$ with $\gamma = 1$ and $\beta = 0$, $\alpha = \frac{315}{100}$, $2m = 1$, computed with the number of iterations j, and adjustable parameters ζ.

j	$\zeta = 4$	$\zeta = 10$
10	2.61684744410	2.25842726026
15	3.03674827652	2.09217288918
20	2.99968354661	2.06321508736
25	3.00277094849	2.341847450149
30	3.00237215980	2.55750350450
35	3.00239129801	2.84082979465
40	3.00238787452	2.93920420595
45	3.00238805578	3.01184012366
50	3.00238802113	3.00453029145
55	3.00238802273	3.00592377706
60	3.00238802243	3.00314374817
65	3.00238802243	3.00276595513
70	3.00238802243	3.00245768578

Table 7. AIM analysis of PT-breaking , eigenenergies states for the potential $V(x) = \gamma x^4 + i\beta x^3 + i\alpha x$ with $\gamma = 1$ and $\beta = 0$, $2m = 1$, $\zeta = 4$.

α	n	E_{AIM}
$\frac{317}{100}$	0	3.17383999378883 - .0365788783495440 i
$\frac{317}{100}$	1	3.17383999378883 + .0365788783495440 i
4	0	3.60823547436812 - 1.20126056694566 i
4	1	3.60823547436812 + 1.20126056694566 i
$\frac{9}{2}$	0	3.91030340672228 - 1.63057737830328 i
$\frac{9}{2}$	1	3.91030340672228 + 1.63057737830328 i
5	0	4.23609820377638 - 2.05375358249030 i
5	1	4.23609820377638 + 2.05375358249030 i
$\frac{11}{2}$	0	4.57883695710143 - 2.48824405330136 i
$\frac{11}{2}$	1	4.57883695710143 + 2.48824405330136 i
6	0	4.93227254413298 - 2.93964641030454 i
6	1	4.93227254413298 + 2.93964641030454 i
$\frac{13}{2}$	0	5.29209457646892 - 3.40834468881720 i
$\frac{13}{2}$	1	5.29209457646892 + 3.40834468881720 i
7	0	5.65617719444392 - 3.89285630348179 i
7	1	5.65617719444392 + 3.89285630348179 i
$\frac{15}{2}$	0	6.02387797390366 - 4.39156838389473 i
$\frac{15}{2}$	1	6.02387797390366 + 4.39156838389473 i
$\frac{763}{100}$	0	6.12006858536678 - 4.52340252719549 i
$\frac{763}{100}$	1	6.12006858536678 + 4.52340252719549 i
$\frac{763}{100}$	2	11.3304673162269 - .101034029059563 i
$\frac{763}{100}$	3	11.3304673162269 - .101034029059563 i

5. Conclusion

In the present paper we combined three known complex potentials in one, then we use the new formed complex to calculate the eigenenergies of the original three potentials by using AIM. The new potential

can be used to calculate the eigenenergies of the original three complex potentials separately, but certain constraint relations must be taken in consideration. The calculated eigenenergies by means of a new complex potential are in good agreement with the results obtained by the other methods.

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