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Path Integral Evaluation of a Time-Dependent Oscillator in an External Field

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Abstract

The Lagrangian of a system describing the dynamical behaviour of a time-dependent harmonic oscillator is modified and then used to evaluate the Feynman path integral of the oscillator. The path integral of the time-dependent oscillator is shown to reduce to the time-independent within certain limits.

Key Words: Path Integral Propagator, Green functions and harmonic oscillator.

1. Introduction

Approximation methods such as perturbation theory have been used to evaluate exact solutions of time-dependent Schrödinger equation [1–9]. Khandekar and Lawande [10] had evaluated the exact quantum theory of a classical force oscillator with a time dependent frequency and a velocity-dependent damping term using path integral approach. Feynman path integral in quantum mechanics, however, involves a kinetic and potential energies term in the Lagrangian. The kinetic term which is quadratic in momentum acts like a Gaussian measure on the remaining integral [1]. In this paper we will follow the same approach [11–12] and modified the Lagrangian as

$$L = e^{\sin\gamma t}\left(\frac{\dot{q}^2}{2} - \frac{\omega^2(t)\,q^2}{2}\right) + J(q)\,q, \tag{1}$$

where frequency $\omega(t)$ is assumed to be a function of time, $J(t)$ is the time dependent perturbed force, and γ is a damping factor. In obtaining equation (1) from the time-independent Lagrangian, the following assumptions can be made:

$$m(t) = e^{\sin\gamma t} \tag{2a}$$

$$k^2(t) = \omega e^{\sin\gamma t} \tag{2b}$$

$$J(t) = J(t)qe^{\sin\gamma t}. \tag{2c}$$

The equation of motion of the particle defined by equation (1) is given by

$$\ddot{q}(t) + \gamma\cos\gamma t\dot{q}(t) + \omega^2(t)\,q = J(t). \tag{3}$$

The Hamiltonian defined by

$$H = \frac{\partial L}{\partial q}\dot{q} - L(p,q,t) \tag{4}$$

is obtained from (1) as

$$H\left(p,q,t\right)=e^{-\sin\gamma t}\frac{P^{2}}{2}+e^{\sin\gamma t}\left(\frac{\omega^{2}\left(t\right)q^{2}}{2}+J\left(t\right)q\right). \tag{5}$$

Equation (5) reduces to time-independent Hamiltonian in the limits $\gamma\to 0$, $t\to 0$, and equation (5) satisfies in a compact form the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t}\left(t\right)=\hat{H}(t)\psi(t), \tag{6}$$

where $\hat{H}(t)$is the time dependent quantum Hamiltonian operator. In order to explain the dynamical behaviour of the system described by equation (5), it is therefore the primary objectives of this paper to evaluate an exact expression for the kernel (propagator) $K\left(q'',t'',q',t'\right)$ based on the Weyl-ordering prescription in the quantum Hamiltonian of the time-evolution operator which can be expressed as a sum over all possible paths connecting the point q' and $q''$with weight factor $\exp\left[\frac{i}{\hbar}S\left(q'',q',T\right)\right]$, where $S\left(q'',q',t'',t'\right)$ is the action that is

$$K\left(q'',t'',q',t'\right)=\int Dq(t)e^{{}^{i}/_{\hbar}S\left(q'',t'',q',t'\right)}\quad , \tag{7}$$

with $Dq(t)$ being the Feynman measure.

The propagator can be expressed semi-classically as $\exp\left(\frac{i}{\hbar}S_{\rm cl}\right)$, with $S_{\rm cl}$ being the classical action, times a prefactor; remarkably this leads to

$$K\left(q'',t'',q',t'\right)=\left(\frac{m}{2\pi i\hbar}\right)^{ND/2}\sqrt{\det\left(\frac{\partial^{2}S_{\rm cl}\left[q'',q'\right]}{\partial q'\partial q''}\right)}\exp\left(\frac{i}{\hbar}S_{\rm cl}\left[q',q''\right]\right), \tag{8}$$

where D is the dimension and the determinant

$$M=\det\left(\frac{\partial^{2}S_{\rm cl}\left[q'',q'\right]}{\partial q'\partial q''}\right) \tag{9}$$

is the known Pauli Van Vleck Morette determinant.

2. Particle Trajectory

The forced harmonic oscillators are of importance in quantum dynamics and in other quantum fields since these fields are represented as a set of forced harmonic oscillators. The solution of the time-dependent Schrödinger equation of equation (6) is given by

$$\psi\left(q,t\right)=\sum_{k}A_{k}e^{-iE_{k}t\phi(q,t)}\quad , \tag{10}$$

with A_k being the time-dependent expansion coefficient defined as

$$A_{k}=e^{+iE_{k}t}\left\langle\phi_{k}\mid\psi\right\rangle. \tag{11}$$

On substituting equation (11) into equation (1), Ituen [9] shows that

$$\psi\left(q'',t''\right)=\int K\left(q'',t'',q',t'\right)\psi\left(q,t'\right)dq',\quad t''>t' \tag{12}$$

which follows that the kernel (propagator) is obtained from equation (12) as

$$K\left(q'',t'',q',t'\right)=\int e^{-i\left(E_{k}t''+E_{k}t'\right)\psi\left(q'',t''\right)}. \tag{13}$$

The path traversed by the particle defined by the equation of motion in the absence of external field in equation (4) is

$$q(t) = Ae^{\frac{+\gamma\cos\gamma+\sqrt{\cos^2\gamma t-4\omega^2}}{}} + e^{\frac{+\gamma\cos\gamma t+\sqrt{\cos 2\gamma t-4\omega^2}}{}} = e^{\frac{+\gamma\cos\gamma t}{2}}\left(Ae^{\frac{\sqrt{\cos^2\gamma t-4\omega^2 t}}{2}} + Be^{\frac{-t\sqrt{\cos^2\gamma t-4\omega^2}}{2}}\right). \tag{14}$$

Equation (14) shows that the trajectory of the particle is a mixture of oscillatory and decaying terms; the oscillatory term arises from the time-independent Hamiltonian, and the decaying terms arises from the time-dependent Hamiltonian. For a small value of the damping factor (Figure 1 and Figure 2), we appropriate the quantities in the square root and obtain

$$\sqrt{\cos^2\gamma t - 4\omega^2} \cong \sqrt{(\gamma t)^2 - 4\omega^2} = 2i\omega\sqrt{1-\left(\frac{\gamma t}{2\omega}\right)^2}. \tag{15}$$

On substituting equation (15) into equation (14), the trajectory takes the form

$$q(t) = e^{\frac{+\gamma\cos\gamma t}{2}}\left(Ae^{\,i\omega\sqrt{1-(\gamma t/2\omega)^2 t}} + Be^{\,-i\omega\sqrt{1-(\gamma t/2\omega)^2 t}}\right), \tag{16}$$

which, as mention earlier, equation (18) reduces to the time-independent solution. When we take the limits $\gamma \to 0$, $t \to 0$ or simply $\gamma t \to \pi/4$, this shows that our choice of the Hamiltonian and Lagrangian given above are in appropriate, since it reduces to the known solution.

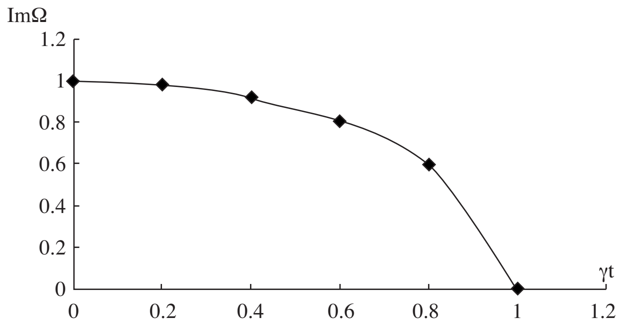


Figure 1. Variations of Ω with the damping factor. Note we obtain the time-independent solution as γt tends to zero.

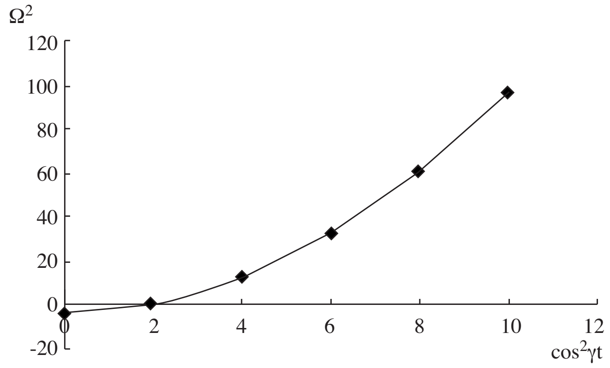


Figure 2. Variation of Ω^2 as a function of $\cos^2\gamma t$. Ω^2 has units of ω^2.

3. Evaluation of the Time-Dependent Feynman Propagator

The Feynman propagator defined for the particle discussed above is expressed as

$$K(q'', t'', q', t') = \lim_{N\to\infty}\prod_{j=1}^{N-1} A_N \int_{-\infty}^{\infty} dq^j \left(\frac{i}{\hbar}\sum_{j=1}^{N}\frac{m}{2\varepsilon}\Delta^2 q^j - \varepsilon V\left(q^j\right) - \varepsilon J_k q_k\right), \tag{17}$$

where A_N is the normalization constant and the action in terms of equation (1) is approximated as

$$S = \int_{t'}^{t''} L dt = e^{\sin\gamma t}\left(\frac{1}{2\varepsilon}\left(q_k - q_{k-1}\right)^2 - \varepsilon\omega^2 q_k^2 - \varepsilon J_k q_k\right). \tag{18}$$

Substituting equation (18) into equation (19), we obtain

$$\begin{aligned} K\left(q'', t''; q', t'\right) = & \lim_{\substack{N\to\infty \\ \varepsilon\to\infty}} \left(\tfrac{e^{\sin\gamma t}}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}} \prod_{j-1}^{N-1} \int_{-\infty}^{\infty} dq^{(j)} \\ & \times \exp\left(\tfrac{i}{\hbar}\sum_{j-1}^{N} \tfrac{1}{2\varepsilon}\left(q_k - q_{k-1}\right)^2 - \tfrac{\varepsilon}{2}\omega_k^2 q_k^2 - \varepsilon J_k q_k\right), \end{aligned} \tag{19}$$

where we have made the assumption that $t'' - t' = \varepsilon$, and conditions of equation (2) have been applied in obtaining equation (19) from equation (17).

Equivalently, equation (19) can be written in the form

$$K\left(q'', t''; q', t'\right) = F(T)\exp\left(\frac{i}{\hbar}S_c\left(q'', t'', q', t'\right)\right) \tag{20}$$

with $F(T)$ evaluated to give

$$F(T) = \left(\frac{\omega e^{\sin\gamma t}}{2\pi i\hbar\sin\left(\omega T\right)}\right)\frac{N}{2}, \tag{21}$$

so that the propagator becomes

$$K\left(q'', t''; q', t'\right) = \left(\frac{\omega e^{\sin\gamma t}}{2\pi i\hbar\sin\left(\omega T\right)}\right)^{N/2} \exp\left[\frac{i}{\hbar}S_c\left(q'', q'\right)\right] \quad . \tag{22}$$

However, the Hamiltonian of the system describes a forced harmonic oscillator of mass $e^{\sin\gamma t}$for which we can calculate the classical action. In order to calculate the action explicitly, we recall the Euler-Lagrangian equations of equation (3) and its classical trajectory is the solution of the equation:

$$\left(\frac{d^2}{dt^2} + \gamma\cos\gamma t\frac{d}{dt} + \omega^2\right) q_{\rm cl}\left(t\right) = \frac{f(t)}{e^{\sin\gamma t}}. \tag{23}$$

The solution of equation (23) consists of a homogeneous and an inhomogeneous part and can be written as

$$q_c\left(t\right) = q_H\left(t\right) + q_I\left(t\right) \tag{24}$$

where $q_H(t)$ had been evaluated in equation (18) as

$$q_H\left(t\right) = e^{\frac{\gamma\cos\gamma t}{2}}\left(Ae^{i\omega\sqrt{1-\left(\frac{\gamma t}{2\omega}\right)^2 t}} + Be^{-i\omega\sqrt{1-\left(\frac{\gamma t}{2\omega}\right)t}}\right). \tag{25}$$

with A and B as arbitrary constants. The inhomogeneous solution is solved by the method of Green's functions. Here, the Green's function is defined by the equation

$$\left(\frac{d^2}{dt^2} + \gamma\cos\gamma t\frac{d}{dt} + \omega^2\right) G\left(t - t'\right) = -\delta\left(t - t'\right), \tag{26}$$

where the inhomogeneous solution can be written as

$$q_1\left(t\right) = \int_{t_j}^{t_f} dt' G\left(t - t'\right)\frac{f\left(t''\right)}{e^{\sin\gamma t}} \tag{27}$$

Defining the Green's function in the Fourier space as

$$G\left(t-t'\right)=\left(2\pi^{-4}\int d^{4}ke^{-ik(t-t_{1})}G\left(k\right)\right); \tag{28}$$

and substituting equation (28) into (26), we obtain

$$G\left(k\right)=\left(\left(2\pi\right)^{-4}\frac{1}{K^{2}-\omega^{2}-\gamma\cos\gamma tk}\right); \tag{29}$$

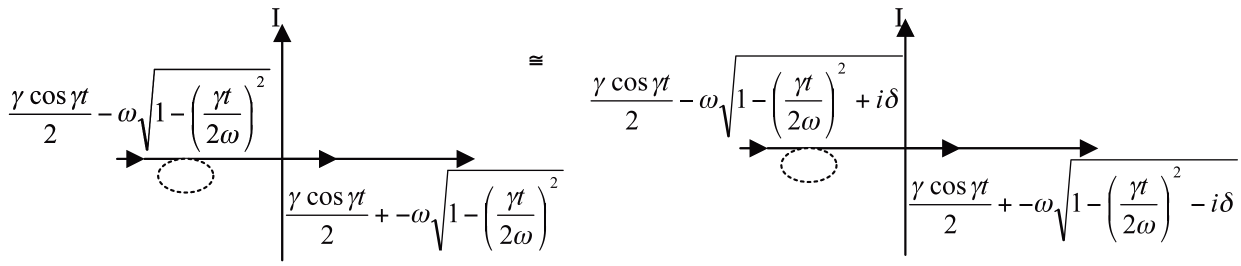
and subsequently, the Green's function takes the form

$$G\left(t-t'\right)=\left(\left(2\pi\right)^{-4}\frac{d^{4}ke^{-ik(t-t_{1})}}{K^{2}-\omega^{2}-\gamma\cos\gamma tk}\right). \tag{30}$$

A quick inspection of equation (3) shows that the integral has poles at

$$K=\frac{\gamma\cos\gamma t}{2}\pm\sqrt{1-\left(\frac{\gamma t}{2\omega}\right)^{2}}. \tag{31}$$

The fundamental Green's function needed in this analysis is that of Feynman which corresponds to choosing a contour as



Thus, the Feynman Green's function has the form

$$G_{F}\left(t-t'\right)=\frac{\theta\left(t-t'\right)e^{\left(\frac{\gamma\cos\gamma t}{2}-\omega\sqrt{1-(\gamma t/2\omega)^{2}}\right)\left(t-t'\right)}}{\left(\gamma\cos\gamma t+2i\omega\sqrt{1-\left(\gamma t/2\omega\right)^{2}}\right)}+\frac{\theta\left(t'-t\right)e^{\left(\frac{\gamma\cos\gamma t}{2}-\omega\sqrt{1-(\gamma t/2\omega)^{2}}\right)\left(t-t'\right)}}{\left(\gamma\cos\gamma t+2i\omega\sqrt{1-\left(\gamma t/2\omega\right)^{2}}\right)}. \tag{32}$$

Substituting equation (32) into equation (27), we obtain the inhomogeneous solution as

$$\begin{aligned}q_{1}(t)= &\frac{1}{e^{\sin\gamma t\left(\gamma\cos\gamma t+2i\omega\sqrt{1-(\gamma t/2\omega)^{2}}\right)}}\\&\times\left[\int_{t_I}^{t_f}dt'e^{\left(\frac{\gamma\cos\gamma t}{2}-i\omega\sqrt{1-(\gamma t/2\omega)^{2}}\right)\left(t-t'\right)}xf(t)+\int_{t_I}^{t_f}dt'e^{\left(\frac{\gamma\cos\gamma t}{2}-i\omega\sqrt{1-(\gamma t/2\omega)^{2}}\right)\left(t-t'\right)}f(t')\right].\end{aligned} \tag{33}$$

We can now write the classical trajectory as

$$\begin{aligned}&q_{c}\left(t\right)=q_{H}\left(t\right)+q_{I}(t)=Ae^{\frac{\gamma\cos\gamma t}{2}+i\omega\sqrt{1-(\gamma t/2\omega)^{2}}}+Be^{\frac{\gamma\cos\gamma t}{2}-i\omega\sqrt{1-(\gamma t/2\omega)^{2}}}\\&-\frac{1}{e^{\sin\gamma t\left(\gamma\cos\gamma t+2i\omega\sqrt{1-(\gamma t/2\omega)^{2}}\right)}}\\&\times\left[\int_{t_I}^{t_f}dt'e^{\left(\frac{\gamma\cos\gamma t}{2}-i\omega\sqrt{1-(\gamma t/2\omega)^{2}}\right)\left(t-t'\right)}f(t)+\int_{t_I}^{t_f}dt'e^{\left(\frac{\gamma\cos\gamma t}{2}-i\omega\sqrt{1-(\gamma t/2\omega)^{2}}\right)\left(t-t'\right)}f(t')\right].\end{aligned} \tag{34}$$

Imposing the boundary conditions; (34) is solved for A and B in terms of the initial and the final co-ordinates of trajectory, which is given by

$$q_{cI}(t) = \frac{1}{\sin\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)} T^{\left(qf\sin(\gamma\cos\gamma t + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)(t-t')} + q_I \sin\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)(t_f - t)$$

$$+\frac{1}{2e^{\sin\gamma t}\left(\frac{\gamma\cos\gamma t}{2}\omega\sqrt{1-(\gamma t 2\omega)^2}\right)}\int_{t_I}^{t_f} dt' f(t')\left[e^{-\left(\frac{\gamma\cos\gamma t}{2} i\omega\sqrt{1-(\gamma t/2\omega)^2}\right)\cos}\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)T\right.$$

$$\left.-\cos\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)(t_f + t_I - t - t')\right]$$

$$-\frac{1}{2ie^{\sin\gamma t}\left(\frac{\gamma\cos\gamma t}{2}\omega\sqrt{1-(\gamma t 2\omega)^2}\right)}\left[\begin{array}{l}\int_{t_I}^{t_f} dt' f(t') e^{-\left(\frac{\gamma\cos\gamma t}{2} i\omega\sqrt{1-(\gamma t/2\omega)^2}\right)\cos}\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)(t-t') \\ +\int_{t_I}^{t_f} dt' f(t') e^{-\left(\frac{\gamma\cos\gamma t}{2} i\omega\sqrt{1-(\gamma t/2\omega)^2}\right)\cos}\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)(t-t')\end{array}\right]. \tag{35}$$

We now evaluate the classical action from equation (1) and (18) to be

$$S_c(q_I, q_f) = \frac{\omega e^{\sin\gamma t}}{2\sin\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-\left(\sqrt{\gamma t/2\omega}\right)^2}\right)}\left[q_I^2 + q_f^2\cos\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)T - 2q_1 q_f\right]$$

$$+\frac{q_i}{\sin\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)}T\int_{t_f}^{t_f} dt f(t)\sin\left(\frac{\gamma\cos\gamma t}{2}\omega\sqrt{1-(\gamma t/2\omega)^2}\right)(t_f - t_i)$$

$$+\frac{q_f}{\sin\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)}T\int_{t_f}^{t_f} dt f(t)\sin\left(\frac{\gamma\cos\gamma t}{2}\omega\sqrt{1-(\gamma t/2\omega)^2}\right)(t - t_i)$$

$$-\frac{1}{e^{\sin\gamma t}\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)\sin\left(\frac{\gamma\cos\gamma t}{2} + \omega\sqrt{1-(\gamma t/2\omega)^2}\right)}T\left[\int_{t_f}^{t_f} dt x(t)\right.$$

$$\left. x\int_{t_i}^{t} dt' f(t')\sin\left(\frac{\gamma\cos\gamma t}{2}\omega\sqrt{1-(\gamma t/2\omega)^2}\right)(t_f - t)\, x\sin\left(\frac{\gamma\cos\gamma t}{2}\omega\sqrt{1-(\gamma t/2\omega)^2}\right)(t' - t_i f(t'))\right]. \tag{36}$$

Combining equations (21) and (32) give a complete transition amplitude for the time harmonic oscillator interacting with a time dependent external source.

4. Uncertainty Relation

We consider again a time-dependent oscillator with variable mass $m(t) = e^{\sin\gamma t}$ and frequency in the absence of an external field, our equation (5) reduces to Caldirola-kanai oscillator [13–15]

$$H_{ck} = \frac{e^{-\sin\gamma t}P^2}{2} + \frac{\omega^2(t)e^{\sin\gamma t}}{2}q^2 \tag{37}$$

On the other hand, Lewis and Riesenfeld [16–17] introduces the invariant method to find the Fock space of exact states of time-dependent oscillators which has been currently reviewed in Reference [13, 18–20]. The quantum theory of the damped oscillator is prescribed by the time-dependent Schrödinger equation [13]

$$i\hbar\frac{\partial}{\partial t}\Psi(q,t) = H_{ck}(P,q,t)\Psi(q,t) \tag{38}$$

However, the eigenfunction associated with Equation (38) are found by different methods [21–22].

Two linear invariant operators are introduced [21, 23, 24] as

$$\begin{aligned} \hat{a}(t) &= i[U^*(t)\hat{P}_q - \dot{U}^*(t)\hat{q} \\ \hat{a}(t) &= -i[U(t)\hat{P}_q - \dot{U}(t)\hat{q} \end{aligned} \tag{39}$$

where the pair of oscillators are first order in position and momentum operators and $U(t)$ is a complex solution to the classical equation of motion of Equation (3) when $J(t)$ is set to zero as

$$\ddot{U}(t) = \frac{\dot{m}(t)}{m(t)}\dot{U}(t) + \omega(t)U(t) = 0$$

or

$$\ddot{U}(t) + \gamma\cos\gamma t\dot{U}(t)\omega^2(t)U(t) = 0 \tag{40}$$

The set of operators in Equation (39) are required to satisfy the quantum Liouville-Von Neumann equations:

$$\begin{aligned} i\hbar\tfrac{\partial}{\partial t}\hat{a}(t) + [\hat{a}(t), \hat{H}_{ck}(p,q,t)] = 0 \\ i\hbar\tfrac{\partial}{\partial t}\hat{a}^+(t) + [\hat{a}^+(t), \hat{H}_{ck}(p,q,t)] = 0 \end{aligned} \tag{41}$$

Imposing the Wronskian condition

$$\hbar e^{\sin\gamma t}[\dot{U}^*(t)U(t) - \dot{U}(t)U^*(t)] = i \tag{42}$$

allows the operators in Equation (2\39) to satisfy the usual commutation relation at equal times

$$[\hat{a}(t), \hat{a}^+(t)] = 1 \tag{43}$$

and its play the roles of time-dependent annihilation and creation operators. Using Equation 40 we obtain the solution for the undamped motion $\left(\gamma \le \frac{\omega}{2}\right)$ as

$$U(t) = \frac{1}{\sqrt{2\hbar - \Omega}}e^{-\gamma\cos\omega t}e^{-i\Omega t} \tag{44}$$

where

$$\Omega = 2\omega\sqrt{1 - \left(\frac{\gamma t}{2\omega}\right)^2} \tag{45}$$

The number operators defined by

$$\hat{N}(t) = \hat{a}^+(t)\hat{a}(t) \tag{46}$$

satisfies Equation 41 and each state

$$\hat{N}(t)|n,t\rangle = n|(n,t) \tag{47}$$

is an exact equation state of equation (38) up to a time-dependent phase factor [24]. Thus, the eigenfunction of the number state satisfying Equation 37 is given as [25]

$$\Psi_n(q,t) = \left(\frac{\Omega e^{\sin\gamma t}}{\pi\hbar}\right)^{\frac{1}{4}} \frac{e^{-i(n+\frac{1}{2})\Omega t}}{\sqrt{2^n n!}} \times \exp\left[-e^{\sin\gamma t}\left(\frac{\Omega}{2\hbar} + i\frac{\gamma}{4\hbar}\right)q^2\right] \tag{48}$$

where H_n is the Hermite Polynomial and Equation 36 reduces to [24] when $\sin \simeq \gamma t$ is appropriate to first order. The eigenfunction in Equation 48 has the dispersion relations

$$\langle x^2 \rangle = \hbar^2 U^*(t)U(t) = \frac{\hbar^2}{2\hbar\Omega} e^{\frac{-\gamma \cos \gamma t}{2}} e^{-i\Omega t} e^{\frac{-\gamma \cos \gamma t}{2}} e^{-i\Omega t} = \frac{\hbar^2}{2\Omega} e^{-\gamma \cos \gamma t} \tag{49}$$

$$\langle P_q^2 = \hbar^2 (m'(t))^2 \dot{U}(t)\dot{U}(t) \tag{50}$$

where $m'(t)$ is the reduced variable mass of $m(t)$ and is defined as

$$m'(t) = \exp\left[\frac{d}{dt}\sin \gamma t\right] = e^{\gamma \cos \gamma t} \tag{51}$$

Using equation (43) and its time derivative and equation (51) in Equation 50, we obtain

$$\langle P_q^2 \rangle = \frac{\hbar}{2\Omega}\left(\left(\frac{\gamma^2}{2}\right)^2 + \Omega^2\right) e^{\gamma \cos \gamma y} \tag{52}$$

Thus, we obtain the uncertainty relation as

$$(\Delta q)(\Delta P_q) = \frac{\hbar}{2}\left[1 + \left(\frac{\gamma^2}{2\Omega}\right)^2\right]^{\frac{1}{2}} \tag{53}$$

These results indicate that the uncertainty relation is satisfied during the time evolution of the wave function.

5. Conclusion

We introduced the Lagrangian of equation (1) and analyzed its dynamical behaviour by evaluating the transition amplitude of the particle. In Section 2, we saw that the solution describing the motion of the particle reduces to one of a free particle as $\gamma \to 0$ and $t \to 0$, or when $\gamma t = \pi/4$. In Section 3, we evaluated the exact path integral to yield the result of the form

$$K\left(q'', t''; q', t'\right) = F(t'', t') \exp\left[\frac{i}{\hbar} S_c(q'', t''; q', t')\right]. \tag{36}$$

Here, $F(t'', t')$ is independent of the space coordinate q'', t'' and $S_c(q'', t''; q', , t')$is the classical action function connecting the initial and final space-time points $(q', t')$and (q'', t'') as given in equation (21) and equation (36), respectively.

Finally, the propagator evaluated in equation (22) combining equation (21) and (36) reduced to the time-independent by taking the limits $\gamma \to 0$, $t \to 0$ or $\gamma t = \pi/4$, which agrees with the known solution of the harmonic oscillator in the field of an external source.

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