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Exact peaked wave solution of the osmosis $K(2,2)$ equation

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Abstract

By using the first-integral method, an exact peaked wave solution to the $K(2,2)$ equation with osmosis dispersion has been obtained directly. The obtained solution agrees well with the previously known solution in the literature. The first integral method is easier and quicker than other traditional techniques. It is shown that the first integral method is a standard and direct method, which may allow us to solve other more complicated solitary wave problems.

Key Words: Peaked wave solutions, the first-integral method, ring theory, $K(2,2)$ equation with osmosis dispersion.

1. Introduction

In 1993, Rosenau and Hyman [1] presented and studied a family of fully nonlinear KdV equations (also denoted by $K(m,n)$) as

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \; m > 0, \; 1 < n \leq 3. \tag{1.1}$$

They obtained a class of compactons of equation (1.1). Wazwaz [2] discussed two generalized forms of the $K(n,n)$ and KP equations that exhibit compactons. In [3], the nonlinear $K(m,n)$ equation was studied for all possible values of m and n. Lately, Wazwaz [4] gave explicit traveling wave solutions of variants of the $K(m,m)$ equation with compact and noncompact structures. By using the Adomian decomposition method, Wazwaz [5,6] studied the $K(m,n)$ equation

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \tag{1.2}$$

and obtained a compacton solution for $a=1$ and a peakon solution for $a=-1$. Recently, Xu and Tian [7] proposed and investigated the peaked wave solutions of the following $K(2,2)$ equation with osmosis dispersion(also called "osmosis $K(2,2)$ equation")

$$u_t+(u^2)_x-(u^2)_{xxx}=0, \tag{1.3}$$

which plays an important role in the research of motion laws of liquid drop and mixed flowing matter.

As is well known that searching for solitary solutions of nonlinear equations in mathematical physics has become more and more attractive in solitary theory. In order to obtain the exact solutions, a number of methods have been proposed, such as the homogeneous balance method [8], the hyperbolic tangent expansion method [9], the Jacobi elliptic function expansion method [10], F-expansion method [11], sine-cosine method [12], tanh function method [13] and so on. Recently, a new powerful method, the first integral method, which is based on the ring theory of commutative algebra, has been proposed by Feng [14] and developed to study the traveling wave solutions of various nonlinear evolution equations [15–19]. Different from other traditional methods, the first integral method has many advantages, which is mainly embodied in that it could avoid a great deal of complicated and tedious calculation and provide more exact and explicit traveling solitary solutions with high accuracy.

In the present work, we would like to extend the first integral method to solve the osmosis $K(2,2)$ equation.

The remainder of this paper is organized as follows. In section 2, using the first-integral method, we establish the exact peaked wave solution for equation (1.3), which is full agreement with the previously known result in the literature. However, the used first integral method is easier and quicker than other traditional techniques. Finally, some conclusions are given in section 3.

2. Exact Peaked Wave Solution for Equation (1.3)

In this section, we start out our study examining equation (1.3). Assume (1.3) has the traveling wave solution as

$$u(x,t)=\phi(\xi),\ \xi=x-ct, \tag{2.1}$$

where c is wave velocity. Substituting (2.1) into equation (1.3) yields

$$-c\phi'+2\phi\phi'-6\phi'\phi''-2\phi\phi'''=0, \tag{2.2}$$

where ϕ' and ϕ'' denote $\frac{d\phi}{d\xi}$ and $\frac{d^2\phi}{d\xi^2}$, respectively.

Integrating (2.2) once, it would become

$$-c\phi+\phi^2-2(\phi')^2-2\phi\phi''=g, \tag{2.3}$$

where g is an arbitrary integral constant.

If we let $\frac{d\phi}{d\xi}=y$, then equation (2.3) can be rewritten as the two-dimensional autonomous system

$$\begin{cases} \frac{d\phi}{d\xi}=y, \\ \frac{dy}{d\xi}=\frac{g+c\phi-\phi^2+2y^2}{-2\phi}. \end{cases} \tag{2.4}$$

Making the following transformation

$$d\xi = -2\phi d\tau, \tag{2.5}$$

then system (2.4) becomes

$$\begin{cases} \frac{d\phi}{d\tau} = -2\phi y, \\ \frac{dy}{d\tau} = g + c\phi - \phi^2 + 2y^2. \end{cases} \tag{2.6}$$

In order to find the traveling wave solutions of equation (1.3), we now applying the first-integral method, the key idea of which is to utilize the so-called Divisor theorem which is based on the ring theory of commutative algebra and to obtain first integrals to system (2.6) under various parameter conditions. Then using these first integrals, the above two-dimensional autonomous system (2.6) can be reduced to some different first-order integrable differential equations. Finally, through solving these first-order differential equations directly, traveling wave solutions for equation (1.3) can be established easily.

Next, let us recall the Divisor Theorem for two variables in the complex domain $\mathcal{C}$ [16].

2.1. Divisor Theorem

Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials of two variables ω and z in $\mathcal{C}[\omega, z]$ and $P(\omega, z)$ is irreducible in $\mathcal{C}[\omega, z]$. If $Q(\omega, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $\mathcal{C}(\omega, z)$ such that

$$Q(\omega, z) = P(\omega, z)G(\omega, z).$$

As Feng [19] pointed out, the Divisor Theorem follows immediately from Hilbert's Nullstellensatz theorem in commutative algebra [20].

Now, we apply the above Divisor Theorem to look for the first integral of system (2.6). Suppose that $\phi = \phi(\tau)$ and $z = z(\tau)$ are the nontrivial solutions to (2.6), and $\Omega(\phi, y) = \sum\limits_{i=0}^{m} a_i(\phi) y^i$ is an irreducible polynomial in $\mathcal{C}(\phi, y)$ such that

$$\Omega(\phi(\xi), y(\xi)) = \sum_{i=0}^{m} a_i(\phi) y^i = 0, \tag{2.7}$$

where $a_i(\phi)(i = 1, 2, \ldots, m)$ are polynomials of ϕ and $a_m(\phi) \neq 0$. We start our study with $m = 1$. Note that $\frac{d\Omega}{d\tau}$ is a polynomial of ϕ and y, and $\Omega(\phi(\tau), y(\tau)) = 0$ implies that $\frac{d\Omega}{d\tau}|_{(2.6)} = 0$. According to the Divisor Theorem, there exists a polynomial $H(\phi, y) = p(\phi) + q(\phi)y$ in $\mathcal{C}(\phi, y)$ such that

$$\frac{d\Omega}{d\tau}|_{(2.6)} = H(\phi, y)\Omega(\phi, y), \tag{2.8}$$

that is

$$\sum_{i=0}^{1} a_i'(\phi) y^i \cdot (-2\phi y) + \sum_{i=0}^{1} a_i(\phi) i y^{i-1} [g + c\phi - \phi^2 + 2y^2] = [a_0(\phi) + a_1(\phi)y][p(\phi) + q(\phi)y]. \tag{2.9}$$

Equating the coefficients of $y^i (i = 2, 1, 0)$ on both sides of (2.9), we obtain that

$$\mathbf{e}'(\phi) = \mathbf{B}(\phi) \cdot \mathbf{e}(\phi), \tag{2.10}$$

and

$$\left[g + c\phi - \phi^2, -p(\phi)\right] \cdot \mathbf{e}(\phi) = 0, \tag{2.11}$$

where $\mathbf{e}(\phi) = (a_1(\phi), a_0(\phi))^T$ and

$$\mathbf{B}(\phi) = \begin{pmatrix} \frac{2-q(\phi)}{2\phi} & 0 \\ \frac{p(\phi)}{-2\phi} & \frac{q(\phi)}{-2\phi} \end{pmatrix}.$$

Since $a_i(\phi)(i = 0, 1)$ are polynomials of ϕ, from (2.10), we can deduce that $q(\phi)$ must be a constant, and $a_1(\phi) = c_0 e^{\int \frac{2-q(\phi)}{2\phi} d\phi}$, where c_0 is integral constant. Obviously, the simplest case for $a_0(\phi)$ and $a_1(\phi)$ is that $q(\phi) = 2$. That is, $a_1(\phi)$ is a constant. For computational convenience, we assume that $a_1(\phi) = 1$. Meanwhile, from (2.10), we have

$$p(\phi) = -2a_0'(\phi)\phi - 2a_0(\phi), \tag{2.12}$$

which implies that $\deg p(\phi) \leq \deg a_0(\phi)$.

From (2.11), we have also

$$p(\phi) \cdot a_0(\phi) = g + c\phi - \phi^2.$$

It means that when both $p(\phi)$ and $a_0(\phi)$ are all polynomials of ϕ, so we can deduce that $\deg p(\phi) = \deg a_0(\phi) = 1$.

Without loss of generality, we assume that $a_0(\phi) = A_0 + A_1\phi, p(\phi) = B_0 + B_1\phi$, where $A_i, B_i(i = 0, 1)$ are all constants to be determined. Substituting them into Eqs.(2.10)–(2.11) and equating the corresponding coefficients of ϕ^2, ϕ and constant terms, we have

$$\begin{cases} A_0 B_0 = g, \\ A_1 B_0 + A_0 B_1 = c, \\ A_1 B_1 = -1, \\ B_0 = -2A_0, \\ B_1 = -4A_1. \end{cases} \tag{2.13}$$

Solving equation (2.13), we can obtain that

$$\begin{cases} A_0 = \pm\frac{c}{3}, \\ B_0 = \mp\frac{2c}{3}, \\ A_1 = \mp\frac{1}{2}, \\ B_1 = \pm 2, \\ g = -\frac{2}{9}c^2. \end{cases} \tag{2.14}$$

In this case, (2.7) becomes

$$y = \pm(\frac{c}{3} - \frac{1}{2}\phi). \tag{2.15}$$

Using this first integral, the second-order ordinary differential equation (2.4) reduces to

$$\frac{d\phi}{d\xi} = \pm(\frac{c}{3} - \frac{1}{2}\phi). \tag{2.16}$$

On integrating equation (2.16) and setting $\phi = 0$ at $\xi = 0$, we obtain the solution

$$\phi(\xi) = -\frac{2}{3}ce^{-\frac{1}{2}|\xi|} + \frac{2}{3}c, \tag{2.17}$$

which implies that equation (1.3) has peaked wave solution

$$u(x,t) = -\frac{2}{3}ce^{-\frac{1}{2}|x-ct|} + \frac{2}{3}c, \tag{2.18}$$

It is easy to see that peaked wave solution (2.18) agrees well with the solution described in [7]. However, it should be noted here that the solution obtained by the bifurcation method of planar dynamical system requires drawing the bifurcation of phase portraits which is not so with the solution obtained by the first integral method, though the two solutions are in complete agreement.

3. Conclusions

In this paper, the first-integral method is directly applied to obtain the exact peaked wave solution for the $K(2,2)$ equation with osmosis dispersion. It can be easily seen that the first integral method is easier and quicker than other traditional techniques. So it indicates that the validity and great potential of this method in solving complicated solitary wave problems.

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