Turk J Phys
35 (2011) , 209 – 240.

doi:10.3906/fiz-1003-16

Resonance Phenomena and time asymmetric quantum mechanics*

Arno Rudolf BOHM[1], **Fatih ERMAN**[2] **and Haydar UNCU**[3]
[1] *CCQS, Department of Physics, University of Texas at Austin, Austin, Texas 78712-1081, USA*
[2] *Department of Physics, Boğaziçi University, Bebek, 34342, İstanbul-TURKEY*
[3] *Department of Physics, Adnan Menderes University, Aytepe, 09100, Aydın-TURKEY*
e-mail: huncu@edu.edu.tr

Received: 12.03.2010

Abstract

This article is a review of time asymmetric quantum theory and its consequences applied to the resonance and decay phenomena. We first give some phenomenological results about resonances and decaying states to support the popular idea that resonances characterized by a width Γ and decaying states characterized by a lifetime τ are different appearances of the same physical entity. Based on Weisskopf-Wigner (WW) methods, one obtains approximately $\frac{\hbar}{\tau} \approx \Gamma$. However, using standard axioms of quantum physics it is not possible to establish a rigorous theory to which the various WW methods can be considered as approximations. In standard quantum theory, the set of states and the set of observables are mathematically identified and described by the same Hilbert space $\mathcal{H}$. Modifying this Hilbert space axiom to a Hardy space axiom one distinguishes the prepared (in) states and detected (out) observables. This leads to semi-group time evolution and to beginnings of time for individual microsystems. As a consequence of this time asymmetric theory one derives $\frac{\hbar}{\tau} = \Gamma$ as an exact relation, and this unifies resonances and decaying states. Finally, we show that this unification can also be extended to the relativistic regime.

Key Words: Time asymmetric quantum theory, resonances, decaying states

1. Phenomenological introduction

1.1. Resonances versus decaying states

Resonances and decaying states appear in all areas of quantum physics, both in the relativistic and the non-relativistic regime. They are defined by definite values of the discrete quantum numbers such as charge, hypercharge, isospin (particle species label), angular momentum and parity j^{π}. In addition, resonances are

*This article is dedicated to Prof. Erdal İnönü who is responsible that the authors become collaborators.

characterized by two real positive numbers, resonance energy E_R (or resonance mass M_R in the relativistic case) and resonance width Γ. Experimentally, they are identified by a sharp peak in the cross section as a function of energy. In contrast, decaying states are characterized also by an energy value E_D (or mass M_D in the relativistic case) and additionally by a lifetime τ, which is an entirely different physical quantity from the width and is measured by the exponential law for the decay rate*. We will compare their characteristics in Table 1.1 (see also chapters 18 and 21 in [1]).

Table 1. Resonances versus decaying states.

Resonances	Decaying states.
• Energy and Width (E_R, Γ) • Γ is measured by fitting the Breit-Wigner line shape to the cross section data	• Energy and (Average) Lifetime (E_D, τ) or Rate $(E_D, R = \frac{1}{\tau})$ • τ is measured by fitting the counting rate for any decay product (channel) η of the decaying state to the exponential law:
$\sigma_{j^\pi}^{BW}(E) \propto \frac{1}{(E-E_R)^2+(\Gamma/2)^2}$, (1a)	$\frac{1}{N}\frac{\Delta N_\eta(t_i)}{\Delta t_i} \approx R_\eta(t) = R_\eta(0)e^{-\frac{t}{\tau}}$, (1b)
as long as Γ/E_R is of the order 10^{-1}–10^{-4}. Here Γ is the full width of the peak of $\sigma_{j^\pi}^{BW}(E)$ and $0 \le E < \infty$ (see Figure 1).	where $\Delta N_\eta(t_i)$ is the number of decay products η registered in the detector during the time interval Δt_i around t_i. Here t_i is the lifetime of each *individual* decaying particle created at a time $t = 0$ $(= t_0)$ (see Figure 1).

Despite the difference in their experimental signatures, it is a common belief that resonances and decaying states are different appearances of the same physical entity, especially in non-relativistic quantum mechanics. Indeed, a resonance in a scattering process can be associated by some theoretical arguments to a large time delay. This suggests the interpretation that a decaying state has been formed (cf. page 458 [1]). Moreover, in the Weisskopf-Wigner (WW) approximation [2], the width Γ of the resonance and the lifetime τ of the decaying states are related by

$$\frac{\hbar}{\Gamma} \approx \tau \ . \tag{2}$$

Based on this approximate equality one often considers the calculated quantity $\tau^{\text{calc}} = \frac{\hbar}{\Gamma}$ as the lifetime of the resonance and the calculated quantity $\Gamma^{\text{calc}} = \frac{\hbar}{\tau}$ as the width of the decaying particle. However, as M. Levy stated in 1959, "*...There does not exist...a rigorous theory to which these various [Weisskopf-Wigner] methods can be considered as approximations*" [3].

1.2. Examples of non-relativistic resonances and decaying states

In non-relativistic as well as in relativistic physics one speaks of resonances and of decaying states. We will, in this and in the following subsections, give some examples of resonances and decaying states from atomic

*In the rest frame of the particle.

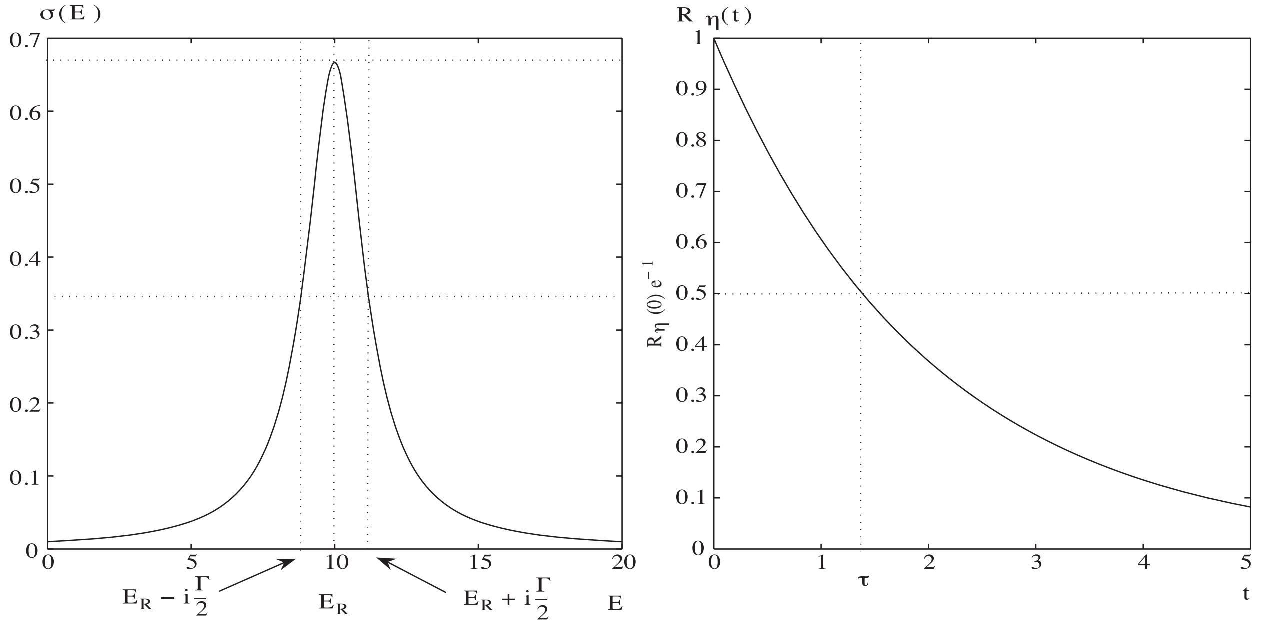

Figure 1. (a) Cross section vs. energy, and (b) counting rate vs. time.

and particle physics. The examples from atomic physics are in the non-relativistic regime, while the examples from particle physics are in the relativistic regime. Both resonances and decaying states in either regime can be obtained by formation or production experiments [1].

The electron-helium scattering process is a particularly suitable example to illustrate resonance phenomena, since resonances as well as decaying states are produced in one and the same energy-loss experiment (see Figure 2).

The formation experiment is the process

$$a + b \to R \to c + d,$$

where R denotes the resonance. An example of a resonance obtained in a formation experiment in atomic physics is the Schulz resonance (He^-) in the elastic scattering process:

$$e^- + \mathrm{He} \to \mathrm{He}^- \to \mathrm{He} + e^- \, . \tag{3}$$

If one measures the elastic scattering cross section as a function of the scattering energy (kinetic energy of the projectile e^- with respect to the target He), one observes that at the energy $E = E_{\mathrm{R}} = 19.31$ eV, the cross section changes violently (see Figure 3). Γ is measured to be approximately 20×10^{-3} eV, so that $\frac{\Gamma}{E_{\mathrm{R}}} \approx 1 \times 10^{-3}$. From $\Gamma = 20 \times 10^{-3}$ eV, one can calculate $\frac{\hbar}{\Gamma} \approx 10^{-12}\,\mathrm{s}$, which should be the lifetime according to (2). With such a value for Γ/E_{R}, we would call the He^- state a resonance state rather than a decaying state.

The production of a resonance is a process of the form:

$$a + b \to a + R$$
$$\hookrightarrow c + d\,.$$

A resonance R is produced and decays into two (or more) particles c and d.

An example of a production process is the inelastic scattering process in which a long-lived, singly excited intermediate state He^* is produced. (Only one of the two electrons in He is excited.) This He^* subsequently decays into the ground state of He and a photon:

$$e^- + \mathrm{He} \to e'^- + \mathrm{He}^* \qquad (4)$$
$$\hookrightarrow \mathrm{He} + \gamma\,.$$

The kinetic energy lost by the electron $E_e - E_{e'}$ excites the He into one of its excited states He^*. The electron intensity I (the electron current at the detector) as a function of the energy loss $E_e - E_{e'}$ is depicted in Figure 2. In this figure, the bumps in the intensity for the electrons as a function of the energy loss give the values of energy used to excite He into an excited state He^* or He^{**}. The experimental values of the energy loss for singly excited states He^* are of the order of 10 eV. This is the typical energy above the ground state for excited atomic levels. The lifetime τ of He^* is approximately 10^{-8} s, which is the typical lifetime of excited atomic levels.

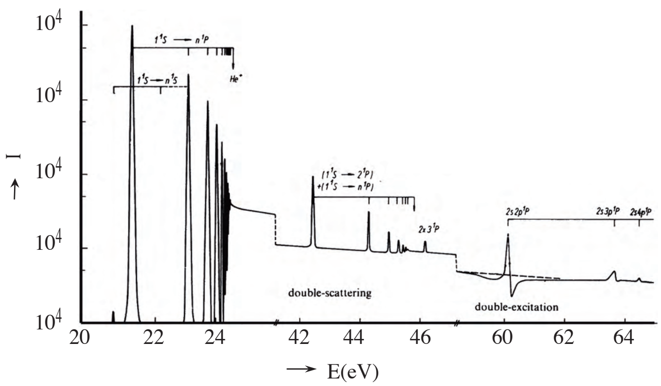

Figure 2. Energy loss spectrum of helium (from Figure 4.1, page 305 of [1]).

From $\tau \simeq 10^{-8}\,\mathrm{s}$, one calculates $\hbar/\tau \approx 10^{-7}\,\mathrm{eV}$, which should be the width Γ according to (2), so that $\Gamma/E_{\mathrm{R}} \approx \frac{10^{-7}\,\mathrm{eV}}{10\,\mathrm{eV}} = 10^{-8}$. This is much smaller than the energy resolution in the incoming beam. Thus, the width of the bumps in Figure 2 below 50 eV are due to the energy spread of the incoming beam, not due to a width of 10^{-7} eV for the energy levels of the He^*. The He^* states are called decaying states because Γ/E_{R} is of the order of 10^{-8}.

On the other hand, the bumps in the production experiment of Figure 2 above 50 eV have a width of the order 10^{-2} eV; they are the doubly excited states (also called Auger states) above the first ionization threshold, i.e. both electrons of the He are excited †. These Auger states He^{**} are of sufficiently high energy E_{R} that they can decay into $\mathrm{He}^+ + e^-$ as well as into $\mathrm{He} + \gamma$ (which is the only possibility for the He^* below the (first)

†The structure around 60.1 eV shows the typical profile of a resonance on top of a background, cf sect. XVIII.9 of [1] for details.

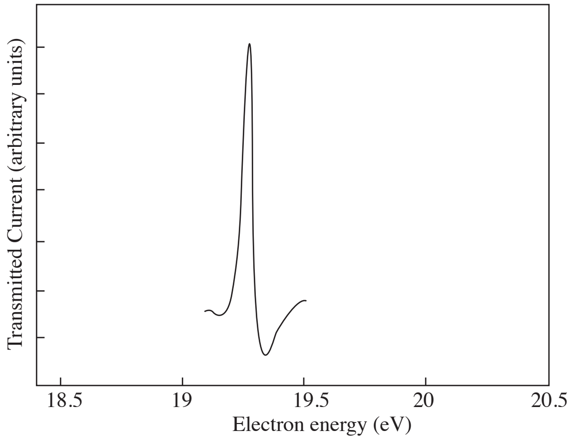

Figure 3. Variation with electron energy of the transmission of electrons through helium as observed in [4].

ionization threshold). Thus, for He^{**} two decay channels are open:

$$
\begin{array}{l}
e^- + \mathrm{He} \to e^- + \mathrm{He}^{**} \\
\qquad\qquad\qquad\quad \hookrightarrow \mathrm{He}^+ + e^- \\
\qquad\qquad\qquad\quad \hookrightarrow \mathrm{He} + \gamma .
\end{array}
\tag{5}
$$

The value of the width Γ of He^{**} is approximately measured as $\Gamma \simeq 4 \times 10^{-2}$ eV.

From $\Gamma \simeq 4 \times 10^{-2}\,\mathrm{eV}$ one can calculate $\frac{\hbar}{\Gamma} \approx 10^{-13}\,\mathrm{s}$. With the observed value $E_{\mathrm{R}} \simeq 60$ eV, one has $\frac{\Gamma}{E_R} \approx \frac{4\times 10^{-2}}{60} \approx 10^{-3}$.

Since $\frac{\Gamma}{E_R} \approx 10^{-8}$ for He^{*} and its lifetime τ is measured, one calls the singly excited He^{*} levels decaying states . On the other hand, since $\frac{\Gamma}{E_R} \approx 10^{-3}$ for He^{**} and its width is measured, one calls the doubly excited He^{**} levels “resonance states”; both can be described by an energy E_{R} and a width Γ (in addition to discrete quantum numbers). Their widths vary widely but except for this quantitative difference there is no qualitative distinction between He^{*} and He^{**}. In other words, they are both excited states of He.

A conclusion from these experimental facts, and the relation (2) of the WW approximation, can be as follows:

- A resonance is the excited state for which the width Γ has been measured.
- A decaying state is the excited state for which the lifetime τ has been measured.

Relation (2), though based on the WW approximation, has been tested in the past [5] but not to a high degree of accuracy. Many physicists accepted (2) as an exact relation, although one did not have sufficiently accurate data to back this up for many years. But recently, Oates et al. [6] and Volz et al. [7] have performed non-relativistic experiments in which both lifetime τ and width Γ have been measured independently to a high degree of accuracy. These experimental values for the lifetime and for the width confirm the relation $\tau = \hbar/\Gamma$ to an accuracy that far exceeds the accuracy expected by the Weisskopf-Wigner approximation. The task is then to find a theory for which (2) holds *as an exact relation.* To find this theory for the non-relativistic case and to extend this new theory to a relativistic theory of resonances and decays is the subject of sections 2 and 3 of this paper.

1.3. Relativistic resonances and decaying states in particle physics

"Elementary particles" are listed in the Particle Data Table [8] and characterized by spin j and mass M (and some other quantum numbers such as "charges"). Most particles decay and this is expressed by their average lifetime τ. A typical excited state that decays by the weak or electromagnetic interactions has a lifetime of the order of 10^{-10} s or more and particles can travel a perceptible amount of distance during this lifetime. For other relativistic particles the table does not list a lifetime τ but a width Γ which is of the order of 100 MeV. Whether one measures the width Γ in an energy measurement or the lifetime τ in a time measurement is a question connected with the capabilities of the apparatuses, not a question related to the nature of quasistable particles. For some relativistic particles it is possible to measure the width and for the others the lifetime. There exists no relativistic particle for which both width and lifetime have been measured, whereas for non-relativistic quasistable states one has an example for which both lifetime and width have been measured [6, 7] with sufficiently high accuracy. Despite of this fact physicists use formula (2) also in the relativistic case to convert the values of Γ into a lifetime: $\tau = \frac{\hbar}{\Gamma}$. This lifetime of a typical relativistic resonance turns out to be on the order of 10^{-23} s. Such time intervals cannot be measured because even if the relativistic particle moved with the speed of light it could only travel a distance of about 1 fm $= 10^{-13}$cm which is a factor 10^{10} less than needed for observation. The width is measured as the full width at half maximum of the Breit-Wigner line shape for the cross section given by formulas similar to (1a). Although the equation (1a) is well accepted for the non-relativistic case, for a relativistic resonance one is even not sure what precise expression to use in place of (1a).

In non-relativistic physics one has a precise definition of lifetime by (1b) and of width by (1a). Then, one just needs to find a rigorous theory from which (2) can be derived. In the relativistic case, the Particle Data Group (PDG) sometimes listed several mass and width values such as, "Breit-Wigner mass" and "Breit-Wigner width." For some resonances, these mass values differ from each other by approximately 10 times the experimental error [9]. So, in contrast to the non-relativistic case, one also needs to find the correct *definition* of the mass and the width, and they need to be defined as relativistic invariant quantities. We shall return to this point later in more detail. By the lifetime for a relativistic decaying particle one means the lifetime in the rest frame of this particle which is a relativistic invariant. For the width of a resonance one is not sure what to take in the relativistic case [8, 9, 10].

The opinion that predominates in relativistic particle physics is that the quasistable particles are complicated phenomena that cannot be described as an exponentially decaying state or as a Breit-Wigner resonance characterized by two numbers like E_{R} and Γ. Relativistic resonances are believed to have a more complicated line shape than a Breit-Wigner, or at least an energy dependent width $\Gamma \equiv \Gamma(s)$, where $s = (p_A + p_B)^\mu (p_A + p_B)_\mu = (E_A^{\mathrm{cm}} + E_B^{\mathrm{cm}})^2$ (see page 324, equation (38.53) in [8]). The definition of the Z boson mass in standard renormalization theory can be considered as an example of this (see page 368 in [8]). The most popular mass and width values (M_z, $\Gamma_z \equiv \Gamma(M_z)$) quoted in [8] are fixed arbitrarily by the "*on-the-mass-shell renormalization scheme.*" This on-the-mass-shell definition is now known to be gauge dependent [10]. A better definition of mass and width, based on the S-matrix pole definition and relativistic invariance, is given in section 3.4.

We shall construct later a new theory for which the relation (2) between a resonance width Γ_R, defined by the position of the relativistic S-matrix pole or by a relativistic Breit-Wigner, and the lifetime τ, defined by exponential decay (1b) in the rest frame, holds. We shall show that this defines the width Γ_R of a relativistic resonance uniquely. But this width $\Gamma_R = \hbar/\tau$ does not agree with the standard definition of Γ_Z.

For now, we consider some concrete examples of relativistic resonances and relativistic decaying particles in formation and production experiments. We first consider the production processes

$$a + T \rightarrow b + R \qquad (6)$$
$$\quad\quad\quad\quad \hookrightarrow c + d\,,$$

whose diagram is given in Figure 4.

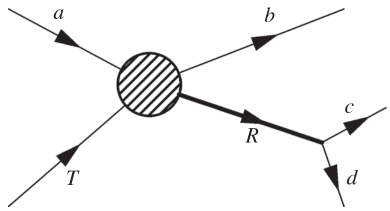

Figure 4. Schematic diagram for the production of the resonance R.

Examples of Resonance production:

1) Production of the ρ-resonance:

The ρ^0 resonance is observed as a bump in the invariant mass distribution, i.e. the maximum of the number $N(s_{\pi^+\pi^-})$ of $\pi^+\pi^-$, as a function of the invariant energy $\sqrt{s_{\pi_+\pi_-}} = \sqrt{(p_{\pi^+} + p_{\pi^-})_\mu (p_{\pi^+} + p_{\pi^-})^\mu}$; see Figure 5. Using (2) one interprets the bump near 0.8 GeV as the production of an ensemble of ρ^0 particles, each of which lives for a time whose average value is $\tau \approx 6 \times 10^{-24}$ s (too difficult to measure), then decays into $\pi^+\pi^-$.

2) Production of the Fermi resonance $\Delta(1232)$: Similar to ρ^0 resonance, an another observed reaction of a resonance production process is

in which the resonance particle Δ^{++} is observed as a bump in the invariant mass distribution $N(s_{\pi p})$, i.e. as the maximum of the number $N(s_{\pi p})$ of $\pi^+ p$ plotted as a function of the invariant mass square $s_{\pi p} = (p_\pi + p_p)^2$ of the πp system.

3) Associate production of the particles Λ and K_s^0:

This example is very similar to production processes (7) and (8), except that the K_s^0 has a lifetime of $\tau \approx 10^{-10}$ s—orders of magnitude larger than in (7) and (8) —which can be measured and has been measured numerous times (see Figure 6) using exponential law (1b).

The K_s^0 is produced in association with the Λ by the strong interaction in which the hypercharge Y is conserved. In the subsequent decay of the decaying particles K_s^0 into the particles π^+ and π^- the hypercharge is not conserved. The interaction that does not conserve Y is the weak interaction and consequently the lifetime of K_s^0 (and of Λ too) is orders of magnitude larger than that of the ρ and Δ resonance. The figure shows the logarithmic plot for $\Delta N(\pi^+\pi^-)/\Delta t$ as a function of the time interval $(t - t_0) = t_{\rm i}$ in the rest frame of the K_s^0 from the time of preparation to the time $t_{\rm i}$ of decay into $\pi^+\,\pi^-$.

$$\begin{array}{l} \pi^- \, p \to n \, \rho^0 \\ \qquad\qquad \hookrightarrow \pi^+ \, \pi^- \end{array} \tag{7}$$

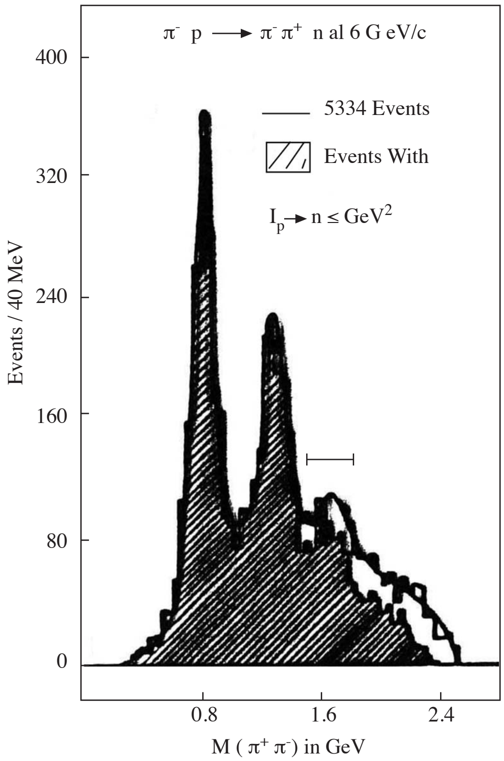

Figure 5. Invariant mass distribution of ρ-resonance [11].

The resonance formation processes of a relativistic resonance

$$a + b \to \mathrm{R} \to c + d$$

is depicted by Figure 7.

The following are examples of resonance formations.

a) The strong interaction scattering process with Δ-formation is

$$\Delta^{++} : \pi^+ \, p \to \Delta^{++} \to \pi^+ \, p \, . \tag{10}$$

In the scattering of $\pi^+ p$, the resonance Δ^{++} appears as the prominent bump in the cross section of Figure 8 [11].

b) The electroweak interaction process with Z-boson formation

$$Z^0 : e^+ \, e^- \to Z^0 \to e^+ \, e^- \tag{11}$$

is shown in Figure 9. It shows the cross section $\sigma(\sqrt{s})$ as a function of energy measured by two different detectors at LEP (CERN).

$$pp \to n\,\Delta^{++} \quad \text{with } \Delta^{++} \to \pi^{+}\,p \quad , \tag{8}$$

$$\pi^{-}\,p \to \Lambda\,K^{0}_{S} \quad \text{with } K^{0}_{S} \to \pi^{+}\,\pi^{-} \tag{9}$$

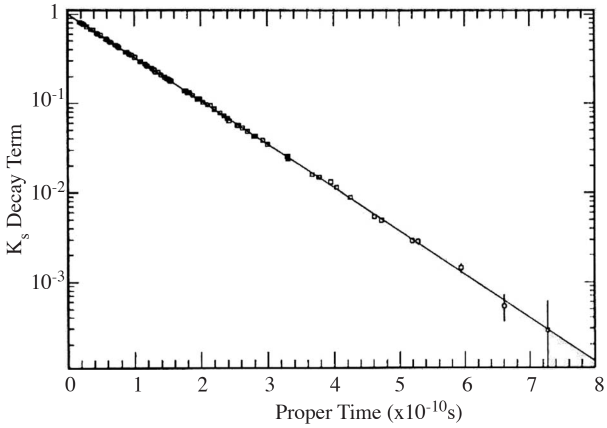

Figure 6. Exponential Decay of K^0 particles as a function of time.

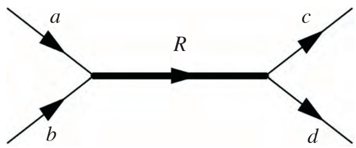

Figure 7. Schematic diagram of resonance formation process.

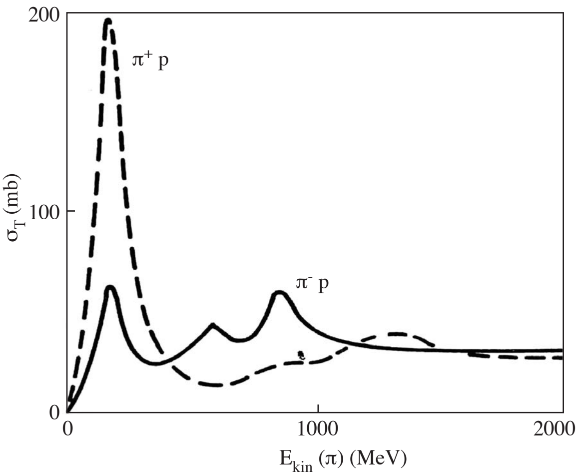

Figure 8. Total cross section as a function of pion kinetic energy for the scattering of positive and negative pions from protons $(1\,\text{mb} = 10^{-27}\,\text{cm}^2)$ [12].

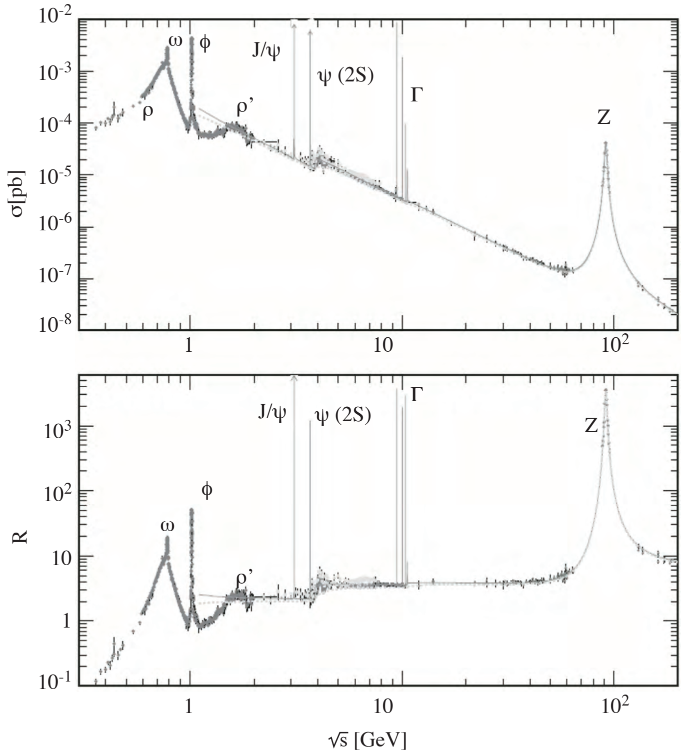

Figure 9. World data of the total cross-section e^+e^- going to hadrons. It shows the line-shape of the 2-boson at 91.188 GeV with a width of $\Lambda = 2.495$ GeV and the position of other relativistic resonances [8].

The mass and the width are obtained by the lineshape fit of these cross section data to a relativistic Breit-Wigner *and some background amplitude*. That is, the Δ resonance in Figure 8 as well as the Z-boson in Figure 9 are both seen as a bump in the cross section and the line shape is given by the Breit-Wigner amplitude plus a background $B(s)$:

$$\sigma(\sqrt{s}) \approx |a^{\mathrm{BW}}(s) + B(s)|^2; \quad a^{\mathrm{BW}}(s) = \frac{r}{s - s_R}, \quad s_R = \left(M - i\frac{\Gamma}{2}\right) . \tag{12}$$

The Δ is usually called a resonance (of strong interaction) and the Z is considered to be an electro-weakly interacting, fundamental particle. Thus, the property to be a relativistic resonance transcends the classification into fundamental (Z) and composite (Δ) particles.

We tabulate the mass, width and/or lifetime values of the particles discussed in the examples above (and also for some others) in Table 2 [8].

These experimental facts suggest the following conclusions.

Relativistic resonances and decaying particles are not different physical entities, they are just different appearances of the same physical entity, the quasistable particles. A particle decays if it can and a particle remains stable if selection rules for exactly conserved quantum numbers (e.g. charge, baryon number, lepton number) prevent it from decaying.

A quasistable particle decays weakly with a long lifetime if selection rules for approximately conserved quantities (e.g. hypercharge, charm) prevent it from decaying faster; in this case the lifetime is usually measured

Table 2. Examples of resonances and decaying states from particle physics.

Particle	Mass M in MeV	Width Γ in MeV	Lifetime τ in s	Γ/M
ρ	775.5 ± 0.4	146.4 ± 1.1	$\approx 4.4 \times 10^{-24}$	$\approx 2 \times 10^{-1}$
Δ^{++}	1231.88 ± 0.29	109.07 ± 0.48	$\approx 6.0 \times 10^{-24}$	$\approx 10^{-1}$
Z^0	$91,187.6 \pm 2.1$	$2,495.2 \pm 2.3$	$\approx 2.6 \times 10^{-25}$	$\approx 3 \times 10^{-2}$
π^0	134.9766 ± 0.0006	$\approx 7.8 \times 10^{-6}$	$(8.4 \pm 0.6) \times 10^{-17}$	$\approx 6 \times 10^{-8}$
K_S^0	497.648 ± 0.022	$\approx 7.4 \times 10^{-12}$	$(0.8958 \pm 0.0005) \times 10^{-10}$	$\approx 10^{-14}$
$\pi^{\pm}$	139.57018 ± 0.00035	$\approx 2.5 \times 10^{-14}$	$(2.6033 \pm 0.0005) \times 10^{-8}$	$\approx 10^{-16}$
$\mu^{\pm}$	$105.6583692 \pm 0.0000094$	$\approx 3 \times 10^{-16}$	$(2.19703 \pm 0.00004) \times 10^{-6}$	$\approx 3 \times 10^{-18}$

and the quasistable particle is called a decaying particle. The quasistable particle decays strongly if no such selection rules impede its decay, in this case the width is usually measured and the quasistable particle is called a resonance. Hence, stability or the value of the lifetime is not a criterion of elementarity. Both stable and quasistable states should be described on the same footing.

Moreover, what one usually calls resonances and what one usually calls decaying particles are not qualitatively different physical entities. They are just quantitatively different in the magnitude of Γ/M or of $\hbar/(\tau M)$. The resonance width Γ and the inverse lifetime (or decay rate) $\hbar/\tau$ are just different appearances of the same physical entity. In other words, resonances and decaying particles are not conceptually different objects and $\Gamma = \hbar/\tau$.

These observations suggest to look for a theory that unifies resonance and decay phenomena, from which the relation $\Gamma = \hbar/\tau$ follows as a prediction.

2. Time asymmetric quantum theory that unifies the theory of resonances and decays

2.1. The Hilbert space axiom and unitary group time evolution

One of the fundamental ideas in the foundations of quantum mechanics is the division of an experiment into the preparation of a state and a registration of the observable in this state. The experimental values are the probabilities for an observable in a state. In the scattering experiment, for example, the preparation consists of the acceleration and the collimation of the projectile, which interacts with the target—perhaps forming a resonance. The registration consists of the detection of scattered particles, e.g. the decay products of the resonance or decaying state, into different channels with different energies and scattering angles. To distinguish what is prepared in the preparation process from what is detected in the registration process, one uses the different words, state for what is prepared and observable for what is detected and registered (counted by a detector). Despite this experimental distinction between the prepared state and the detected observable, conventional axioms do usually not distinguish between the mathematical description for a state and for an observable.

A possible axiomatization of quantum mechanics has been given in the books [1, 13]. In this paper, we will follow the approach given in [1]. In this formulation, an observable is mathematically described by a self-adjoint operator A, or by a projection operator $\Lambda = \Lambda^2$, and in special cases by a vector ψ if Λ is a one-dimensional projector $\Lambda = |\psi\rangle\langle\psi|$. A state is described by a density operator, usually denoted by ρ or W

or also by a projection operator $P = P^2$ and a pure state $P = |\phi\rangle\langle\phi|$ is often described by a vector ϕ (though it is understood that this vector is defined only up to a phase factor). And every $|\phi\rangle\langle\phi|$ can represent a state but it could also represent an observable.

Experimentally, an observable is defined by a registration apparatus (e.g. detector) and a state is defined by a preparation apparatus (e.g. accelerator). Thus, observables and states are different physical concepts. Nevertheless, as a standard axiom of quantum mechanics, the set of states $\{\phi\}$ and the set of observables $\{\psi\}$ are mathematically identified and described by the same complete linear scalar product space (Hilbert Space) $\mathcal{H}$ [13]:

$$\{\phi\} = \{\psi\} = \mathcal{H} = \text{Hilbert space (norm complete)}. \tag{13}$$

However, there is no physical justification for this mathematical identification because states are physically described by the preparation apparatus and observables by the registration apparatus, so they may not belong to the same linear vector space. Therefore, we shall, in section 2.3, mathematically distinguish between the set of prepared states $\{\phi^+\}$ and the set of the detected observables $\{\psi^-\}$ by assigning them into different mathematical spaces [‡]:

$$\{\phi^+\} \Leftrightarrow \Phi_- \qquad\qquad \{\psi^-\} \Leftrightarrow \Phi_+ \ . \tag{14}$$

In experiments with quantum systems one measures ratios of large integers: N_i/N or $N(t)/N$, e.g. as ratios of detector counts from the i^{th} detector N_i and counts of all detectors N; or one measures ratios of detector counts $N(t)$ in the time interval between $t = 0$ and $t = t$, and the detector counts $N = N(\infty)$ at all times after a preparation time $t = 0$ [14]. The ratio of large numbers is interpreted as a probability

$$\frac{N_i}{N} \approx \mathcal{P}(\Lambda); \tag{15a}$$

or the probability to observe the property $\Lambda(t)$ in a state W at time t is

$$\frac{N(t)}{N} \approx \mathcal{P}_W(\Lambda(t)) \ . \tag{15b}$$

The symbol $\approx$ denotes the association of the experimentally measured quantity on the left hand side with the theoretically calculated quantity on the right hand side of equations (15a) and (15b). According to Born's interpretation of quantum theory, the probability, denoted by $\mathcal{P}_W(\Lambda(t))$, to measure an observable $\Lambda(t)$ in a state W at a time t is calculated as

$$\mathcal{P}_W(\Lambda(t)) \equiv \mathrm{Tr}(\Lambda(t)W_0) = \mathrm{Tr}(\Lambda_0 W(t)) \ , \tag{16a}$$

where $\Lambda_0 = \Lambda\ (t = 0)$ and $W_0 = W\ (t = 0)$. For the simplest case $\Lambda = |\psi^-\rangle\langle\psi^-|$ and $W = |\phi^+\rangle\langle\phi^+|$, this becomes

$$\mathcal{P}_W(\Lambda(t)) \equiv |\langle\psi^-(t)|\phi^+\rangle|^2 = |\langle\psi^-|\phi^+(t)\rangle|^2 \ . \tag{16b}$$

The parameter t in (16a) and (16b) is the continuous time parameter, and the observable Λ, or the state W, are continuous functions of time (see XII.A in [1]). Although the ratio $N(t)/N$ can only change in discrete steps, $\mathcal{P}_W(\Lambda(t))$ is a continuous function of t. Thus, we use the symbol $\approx$ in equations (15a) and (15b).

[‡] The discrepancy between the labels $\pm$ for the vectors and the labels $\mp$ for their spaces reflects the difference between the standard notation in physics for the vectors, and in mathematics for the spaces.

The dynamics of a system given by the Hamilton operator H can be described in different pictures. The first and second terms on the right hand side of equations (16a) and (16b) give the probability in the Heisenberg and Schrödinger picture, respectively. In the Heisenberg picture, the dynamical equation for the observables $\Lambda(t)$ is

$$i\hbar\frac{d\Lambda(t)}{dt} = [\Lambda(t), H] \,, \tag{17a}$$

or for the special observable $\Lambda(t) = |\psi^-\rangle\langle\psi^-|$, it becomes

$$i\hbar\frac{d\psi^-(t)}{dt} = -H\psi^-(t) \,. \tag{18a}$$

In the Schrödinger picture, the dynamical equation for the state $W(t)$ is

$$-i\hbar\frac{dW(t)}{dt} = [W(t), H] \,, \tag{17b}$$

or in the special case $W(t) = |\phi^+\rangle\langle\phi^+|$, for $\phi^+(t)$, this leads to

$$i\hbar\frac{d\phi^+(t)}{dt} = H\phi^+(t) \,. \tag{18b}$$

Using the assumption (13), the dynamical equations (18a)–(18b) have to be solved under the (boundary) condition that observable ψ^- and state ϕ^+ are elements of $\mathcal{H}$. As a consequence of the Stone-von Neumann theorem for the Hilbert space [13, 15], all solutions of the Heisenberg equations (18a) under the boundary conditions $\psi^- \in \mathcal{H}$, are given by

$$\psi^-(t) = e^{iHt}\psi_0^- , \qquad -\infty < t < \infty \,. \tag{19a}$$

Similarly, the solutions of the Schrödinger equation (18b) under the Hilbert space boundary condition are given by

$$\phi^+(t) = e^{-iHt}\phi_0^+ \,, \qquad -\infty < t < \infty \,. \tag{19b}$$

Here, $\hbar$ has been set to unity and we shall use the unit convention $\hbar = 1$ unless otherwise stated.

For a general observable $\Lambda(t)$, the solution of the Heisenberg equation (17a) is given by

$$\Lambda(t) = e^{iHt}\,\Lambda_0\,e^{-iHt}, \qquad -\infty < t < \infty \,, \tag{20a}$$

and for a general state $W(t)$, the solution of the Schrödinger equation (17b) is given by

$$W(t) = e^{-iHt}\,W_0\,e^{iHt}, \qquad -\infty < t < \infty \,. \tag{20b}$$

Evolutions (19a) and (19b), respectively, are special cases of (20a) and (20b).

In the equations (19a) and (20a) for the observables $\psi^-(t)$ and $\Lambda(t)$, the time evolution is given by a *one-parameter group of unitary operators* $U(t) = e^{iHt}$, $-\infty < t < \infty$; this means that every $U(t)$ has an inverse. The same holds for the operator $U^\dagger(t) = e^{-iH^\dagger t} = e^{-iHt} = U(-t)$ in (19b) and (20b), which describes the time evolution of the state $\phi^+(t)$ or $W(t)$, respectively. It means that the state ϕ^+ in the Schrödinger picture (or the observable ψ^- in the Heisenberg picture) evolves forward and backward in time.

The Born probabilities (16a) and (16b) can be predicted for all positive and negative values of time $-\infty < t < \infty$. Therefore, the conventional quantum mechanics is a time-symmetric theory. This means that if a state has been prepared at $t = t_0 = 0$ one can predict by (16b) the probability for the observable $\psi^-(t)$ at all times $t > 0$ after the state has been prepared. One can also predict by (16b) the probability for the observable $\psi^-(t = -|t|)$ in the state ϕ^+ *before* this state will be prepared at $t = 0$. However; such a prediction is nonphysical since it is in contradiction with the notion of causality [16]: A state needs to be prepared before an observable can be measured in it. Specifically, how can one detect the decay products of a decaying state before the decaying state has been prepared? Something is wrong with causality for the unitary group solutions (19). For reasons of causality, (16b) should not exist for $t \leq 0$.

2.2. The exponential decay law and Hardy spaces

If $\phi_k \equiv \phi_k\ (t = 0)$ is an eigenstate of a self-adjoint Hamilton operator H and $U^\dagger(t) = e^{-iH^\dagger t}$ is the unitary group, then

$$H\phi_k = E_k\phi_k \qquad (21a),$$

$$e^{-iH^\dagger t}\phi_k = e^{-iE_k t}\phi_k \qquad (21b)\ ,$$

for $-\infty < t < \infty$. These relations have always been assumed to hold, also, for Dirac kets:

$$H^\dagger|E, j, j_3, \eta\rangle = E|E, j, j_3, \eta\rangle, \quad (22a)$$

$$e^{-iH^\dagger t}|E, j, j_3, \eta\rangle = e^{-iEt}|E, j, j_3, \eta\rangle \quad (22b)$$

for $-\infty < t < \infty$. The basis vectors $|E, j, j_3, \eta\rangle$ are "eigenkets" of H (and a complete set of commuting observables (c.s.c.o) whose eigenvalues are called j (angular momentum), j_3 (its 3-component) and η denoting all the additional quantum numbers including particle species numbers. Whereas (21b) is a consequence of Stone-von Neumann theorem for Dirac kets, (22b) is not a consequence of it. (Here $j, j_3, \ldots$ are labels for degeneracy, e.g., the angular momentum quantum numbers.)

The probability to find the observable ψ_η of the decay product η in the state $\phi_k(t) = e^{-iHt}\phi_k$ at time t is given by the Born probability (16b). Therefore, as a result of (22), we have:

$$\mathcal{P}^\eta_{\phi_k(t)} = \mathcal{P}_{\phi_k(t)}(|\psi_\eta\rangle\langle\psi_\eta|) = |\langle\psi_\eta|\phi_k(t)\rangle|^2 = |\langle\psi_\eta|\phi_k\rangle e^{-iE_k t}|^2 = |\langle\psi_\eta|\phi_k\rangle|^2 = \text{const.} \qquad (23)$$

This means that for every eigenvector of H, the probability for its decay products η is constant in time i.e. it does not describe a decaying state. (However, the probability $\mathcal{P}_{\phi(t)}$ can change in time—albeit not exponentially—if $\phi(t)$ is not an eigenstate of H).

In contrast, experiments show that all spontaneously decaying quantum systems (radioactive decay) obey the exponential decay law: $\mathcal{P}^\eta(t) \sim e^{-t/\tau}$. *This exponential law can be considered as one of the best established laws of nature* [17]. Therefore we expect that there are plenty of state vectors $\phi(t)$ for which the Born probabilities (16b) obey the exponential law $\mathcal{P} = e^{-t/\tau}$. However, if one restricts oneself to states represented by vectors in the Hilbert space, then there is no state that has exponential decay. This is stated in the following theorem [18]

There exist no vector ϕ in the Hilbert space $\mathcal{H}$ such that the probability of any observable Λ_η fulfills the condition

$$\mathcal{P}_{\phi(t)}(\Lambda_\eta) \sim e^{-t/\tau} \ . \tag{24}$$

Thus, there is no vector $\phi \in \mathcal{H}$ with exponential time evolution, that could describe exponential decay.

Because of the overwhelming empirical evidence for the exponential decay law, we want to have exponentially decaying states included in our space of states. Therefore, we have to replace the Hilbert space by a space in which we can find a vector ψ^G, which is an eigenvector of the Hamilton operator H with complex eigenvalue. This means that we want to have something like

$$H^\times \psi^G = (E - i\Gamma/2)\, \psi^G \ , \tag{25}$$

where $H^\times$ is some kind of an extension or generalization of $H^\dagger$ in (22), and E and Γ are real, positive constants. And in analogy to (22b), the time evolution should be given by

$$\psi^G(t) = e^{-iHt}\psi^G = e^{-i(E-i\frac{\Gamma}{2})t}\psi^G \ . \tag{26}$$

For a vector ψ^G with the property (25) and (26) one would then obtain by a simple calculation a Born probability which fulfills the exponential law:

$$\begin{aligned}\mathcal{P}_{\psi^G(t)}(|\psi_\eta\rangle\langle\psi_\eta|) &= |\langle\psi_\eta|\psi^G(t)\rangle|^2 = |\langle\psi_\eta|e^{-iHt}\psi^G\rangle|^2 = |\langle\psi_\eta|\psi^G\rangle e^{-iEt}e^{-\frac{\Gamma}{2}t}|^2 \\ &= |\langle\psi_\eta|\psi^G\rangle|^2 e^{-\Gamma t} = \mathcal{P}_{\psi^G(0)}(|\psi_\eta\rangle\langle\psi_\eta|)e^{-\Gamma t} \ .\end{aligned} \tag{27}$$

In contrast to (21) and (22), the time t in (27) needs to have a lower bound $t \geq t_0$, since the probability (27) would otherwise grow exponentially (exponential catastrophe).

The vector ψ^G with property (26) is a good candidate to represent a decaying state. This idea was originally due to Gamow (1928) [19] and the vectors ψ^G will therefore be called Gamow vectors. Since the operator H is self-adjoint, the possibility (25) and (26) is not at all obvious, and looks wrong. Therefore Gamow vectors must be given a mathematical meaning which allows something like (25) and (26).

Standard formulation of quantum mechanics in Hilbert space $\mathcal{H}$ is not only insufficient for Gamow vectors but also for Dirac kets [20] because the eigenkets for a continuous set of eigenvalues are not in the Hilbert space. Dirac kets with continuous eigenvalues as in (22) had no mathematical meaning until they were defined as functionals by Schwartz [21, 22]. Similarly, the Gamow ket in (25) and (26) can also be defined as a functional or ket; then (26) with a self-adjoint Hamilton operator $H^\dagger$ will become an equality between functionals, as in the case of (22) for Dirac kets. This can be accomplished mathematically if one uses energy wavefunctions $\phi(E) = \langle E|\phi\rangle$ which are not only smooth and rapidly decreasing functions of E as is done for Dirac kets (22), but are also functions that can also be analytically continued to complex values of E in the complex energy semi-planes $\mathbb{C}_+$ or $\mathbb{C}_-$. In particular, we shall choose for the space of wave functions not just Schwartz space functions—like for Dirac kets—but *smooth Hardy functions*. Gamow vectors ψ^G with the property (25) and (26) can then be defined as functionals on Hardy spaces [23]. We will discuss all these in the following section.

2.3. The new axiom of quantum theory and semi-group time evolution

The experimental facts mentioned in the phenomenological introduction suggest looking for a theory in which the resonance and decay phenomena are unified and the relation (2) follows as a natural prediction of this theory. From the remarks in sections 2.1 and 2.2, we conclude that, using the standard quantum theory, this is not possible because the Hilbert space axiom (asymptotic completeness) leads to unitary time evolution and does not allow exponentially decaying states. In Hilbert space quantum mechanics, the eigenvalues of the (self-adjoint) Hamiltonian are real and Gamow vectors fulfilling (25) and (26)- which give a simple and natural description of decaying states- do not exist. Furthermore, in Hilbert space, due to the set of equations (18) and (19), the causality condition for Born probabilities $t \geq t_0$ cannot be fulfilled. All these problems disappear if we replace the Hilbert space axiom of standard quantum mechanics with a new axiom using Hardy spaces.

The word "time asymmetric" here is not to be misunderstood: Time asymmetry, irreversibility in macrosystems (entropy increase), time reversal non-invariance are (perhaps) different concepts. In this paper, we are only concerned with time asymmetry in quantum physics. This concept of time asymmetry comes from *time asymmetric* boundary conditions given by the Hardy spaces for *time symmetric* dynamical equations (18a) and (18b). Retarded and advanced Green's functions in electromagnetism are the examples of time asymmetry in classical physics. Time asymmetry in quantum physics is an expression of an arrow-of-time: First preparation then observation [14, 24]. By irreversibility in quantum physics one usually means the effect due to a measurement apparatus or the interaction with an external system (such as a reservoir), upon the quantum system (called extrinsic irreversibility). The time asymmetry which we mean is not this extrinsic irreversibility but it is the time asymmetry inherent in the dynamics of an isolated quantum system also called *intrinsic irreversibility*. Resonances and decaying states provide a particularly simple example of a quantum system with time asymmetry. This time asymmetry is based on the truism that a state must be prepared before an observable can be detected in it (also called *the preparation $\Rightarrow$ registration arrow of time*).

In the conventional mathematical theory for quantum physics:

- (A) The time t satisfies $-\infty < t < \infty$ due to unitary evolution.
- (B) The spectrum of the self-adjoint Hamilton operator H, the energy E, is bounded from below (because of stability of matter): $(0 =)\ E_0 \leq E < \infty$.

Then time symmetric evolution of the state vector $\phi^+(t)$ is given by the operator

$$U^\dagger(t) = e^{-iH^\dagger t}\ , \tag{28}$$

where Hamiltonian $H = H^\dagger$ is called the generator of the time evolution. The set of the operators $U^\dagger(t)$, $-\infty < t < \infty$, form a one-parameter group of unitary operators, i.e. the product of two operators exist:

$$U^\dagger(t_1)U^\dagger(t_2) = U^\dagger(t_1 + t_2); \tag{29-g}$$

and the inverse of every operator exists:

$$U^{\dagger\,-1}(t) = U^\dagger(-t). \tag{30-g}$$

Contrast to the properties (A) and (B) of Hilbert space quantum mechanics, for the solutions of practical problems of quantum physics—in particular when discussing the phenomena of resonance scattering and decay—one goes beyond the restrictions set by (B): The energy E of the j-th partial S-matrix $S_j(E)$ is continued into the complex plane $E \to z \in \mathbb{C}$. For the Lippmann Schwinger equation in scattering theory (and the propagator in field theory) one uses infinitesimal imaginary parts $E \to E \pm i\epsilon$ to express particular boundary conditions and for decaying states one even uses complex eigenvalues $E \to E - i\frac{\Gamma}{2}$ of the Hamiltonian, and all these violations of Hilbert space mathematics are pretty successful. Therefore, from the observation that (16a) and (16b) can have an experimental meaning only for $t \geq 0$ and that the complex energy values are needed to describe the scattering process and decay phenomena, one concludes that one needs a mathematical theory with the following features:

- (A') The time t has a preferred direction: $t_0 = 0 \leq t < \infty$.
- (B') The eigenvalue of the "essentially" self-adjoint § Hamilton operator H, the energy E, can take discrete and continuous values in the whole complex plane: $E \to z \in \mathbb{C}_{\pm}$.

We shall see that such a theory can be constructed and it will even turn out that the requirements $(A^{'})$ and $(B^{'})$ are not independent, and the implementation of $(B^{'})$ in a reasonable way will lead to $(A^{'})$ and to achieve $(A^{'})$ one has to restrict energy wave functions to the set of functions that can be continued into the complex plane. Thus, to implement $(A^{'})$ and $(B^{'})$ by a mathematically consistent theory requires a change of the Hilbert space boundary conditions. This is accomplished by restricting the set of allowable solutions for the Heisenberg equation (18a) to a subset Φ_+ of $\mathcal{H}$ and the set of allowable solutions for the Schrödinger equation (18b) to subset Φ_- of $\mathcal{H}$. These modifications can be explained in two steps.

Step 1. One implements the Dirac basis vector expansion:

i) Solutions of both the Heisenberg and the Schrödinger equations for observable ψ and state ϕ have a Dirac basis vector expansion with the basis vectors $|E, j, j_3, \eta\rangle$

$$\phi = \sum_{j,j_3,\eta} \int dE\, |E, j, j_3, \eta\rangle\langle E, j, j_3, \eta|\phi\rangle \qquad \psi = \sum_{j,j_3,\eta} \int dE\, |E, j, j_3, \eta\rangle\langle E, j, j_3, \eta|\psi\rangle\ , \tag{31}$$

where $\phi \in \Phi \subset \mathcal{H}$ and $|E, j, j_3, \eta\rangle \in \Phi^{\times}$ are functionals on the space Φ (see section I.7 of [1]). If one omits the $\sum_{j,j_3,\eta}$ over the discrete quantum numbers for the time being, one gets

$$\phi = \int dE\, |E\rangle\langle E|\phi\rangle\ \ . \tag{32}$$

The expansion (31) is the analog of the expansion of a vector $\vec{x}$ in the 3-dimensional Euclidean space $\vec{x} = \sum_{i=1}^{3} \vec{e}_i\, x^i$ with $\vec{e}_i$, $i = 1, 2, 3$ replaced by $|E\rangle$ and the summation over i replaced by integration over E, $0 \leq E < \infty$. Only the energy E is important here. The kets $|E\rangle$ fulfill the requirement

$$\langle H\phi|E\rangle \equiv \langle \phi|H^{\times}|E\rangle = E\langle \phi|E\rangle\ , \tag{33}$$

for all vectors ϕ and the kets $|E\rangle$. The main reason the space Φ was introduced is that some quantum mechanical observable in $\mathcal{H}$ are necessarily represented by discontinuous (unbounded) operators in $\mathcal{H}$ but one can find a dense subspace Φ in $\mathcal{H}$ with a stronger topology than the Hilbert space topology for which these operators are defined on the whole space Φ [26]. The superscript $\times$ denotes the conjugate operator and is defined by (33). Equation (33) is usually written as

$$H^{\times}|E\rangle = E|E\rangle\ . \tag{34}$$

§A symmetric operator is called "essentially" self-adjoint if its closure is self-adjoint[25].

In the standard Dirac notation [20], the superscript $\times$ on H was omitted and (34) was written as

$$H|E\rangle = E|E\rangle \ . \tag{35}$$

ii) The meaning of the eigenkets in the equation (33) depends on the choice of the space Φ of vectors $\{\phi\}$. The equation (31) cannot hold for all vectors ϕ from the Hilbert space. In the textbooks of quantum mechanics one usually does not worry about these mathematical questions when one uses (31). If one does, one chooses for Φ the Schwartz space (the space of smooth rapidly decreasing functions) as done for example in [27]. This means that the co-ordinates or "scalar product", or the bra-ket $\langle E|\phi\rangle$ are Schwartz space functions:

$$\phi(E) \equiv \langle E|\phi\rangle = \overline{\langle\phi|E\rangle} \in \mathcal{S}_{\mathbb{R}_+} = \text{Schwartz function space} \tag{36}$$

Precisely, there is a one-to-one correspondence (bicontinuous mapping) between the vectors $\phi \in \Phi$ and the set of smooth rapidly decreasing functions $\phi(E) \in \mathcal{S}_{\mathbb{R}_+}$

$$\Phi \ni \phi \Leftrightarrow \phi(E) \in \mathcal{S}_{\mathbb{R}_+} \ . \tag{37}$$

The function $\phi(E)$ corresponds to the vector ϕ in the same way as the coordinates $x^i = (\vec{e}_i, \vec{x})$ corresponds to the vector $\vec{x}$ in $\vec{x} = \sum_{i=1}^{3} \vec{e}_i\, x^i$. One calls this correspondence (37) of the abstract linear topological space Φ by the function space $\mathcal{S}_{\mathbb{R}_+}$ a "realization" of Φ by $\mathcal{S}_{\mathbb{R}_+}$.

The triplet of the spaces $\Phi \subset \mathcal{H} = \mathcal{H}^\times \subset \Phi^\times$, where the spaces $\Phi^\times$ and $\mathcal{H}^\times$ denote the space of antilinear continuous functionals on Φ and $\mathcal{H}$, respectively, form a Rigged Hilbert Space (RHS), which is also called a Gel'fand Triplet. The RHS is briefly described in section I.7 of [1]. If we choose for Φ, the space realized by the Schwartz function space, we obtain the Schwartz RHS

$$\{\phi\} = \{\psi\} = \Phi \subset \mathcal{H} \subset \Phi^\times \ . \tag{38}$$

It is equivalent (algebraic and topological) to the triplet of the Schwartz function spaces, with $\mathcal{H}$ realized by the space of Lebesgue square integrable functions $L^2_{\mathbb{R}_+}$ and the space $\Phi^\times$ realized by the space of tempered distributions $\mathcal{S}^\times_{\mathbb{R}_+}$:

$$\{\phi(E)\} = \{\psi(E)\} = \mathcal{S}_{\mathbb{R}_+} \subset L^2_{\mathbb{R}_+} \subset \mathcal{S}^\times_{\mathbb{R}_+} \ . \tag{39}$$

The Schwartz space triplet is sufficient to give a mathematical meaning to the Dirac kets and to prove the Dirac basis ket expansion: The nuclear spectral theorem (31) [22].

However, the Schwartz space triplet (38) or (39) for the Dirac kets will not yet accommodate Gamow vectors and provide time asymmetry. The reason is that the solutions of the dynamical (Schrödinger (18b) and Heisenberg (18a)) equations under the Schwartz space boundary conditions $\phi \in \Phi$, $\psi \in \Phi$ are also given by a unitary group, as in the case of Hilbert space boundary conditions [28]:

$$\phi(t) = e^{-iHt}\phi(0), \qquad \psi(t) = e^{iHt}\psi(0), \qquad \text{for } -\infty < t < \infty \ . \tag{40}$$

Thus, for the Schwartz-Dirac kets (i.e. the Dirac kets defined as functionals on the Schwartz space Φ) one obtains again,

$$e^{-iH^{\times}t}|E\rangle = e^{-iEt}|E\rangle \qquad \text{for } -\infty < t < \infty\,. \tag{41}$$

This shows that in standard quantum mechanics, even when amended with the Dirac formalism in a Schwartz-Rigged Hilbert Space (38), there is *no* time asymmetry and *no* beginning of time $t_0 > -\infty$.

Most physicists are quite happy with a time-symmetric theory. However, in a time-symmetric quantum theory one cannot obtain a consistent description of resonances and decay phenomena. One just gets approximate methods to which "there does not exist a rigorous theory" [3]. There have been many arguments in favor of quantum mechanical time asymmetry, e.g., [14, 24, 29] and purely outgoing boundary conditions [30]. Therefore, we suggest a second modification of standard quantum mechanics, which accommodates quantum mechanical time asymmetry.

Step 2. We go to two riggings of the same Hilbert space:

Because of the physical distinction between the prepared states ("in-states") ϕ^+, defined by the preparation process, and detected observable ("out-observables") ψ^-, defined by the registration process, we associate with the set of states and with the set of observables different dense subspaces of the same Hilbert space $\mathcal{H}$. Thus, we take as a *new axiom*:

$$\text{set of prepared states } : \{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^{\times} \tag{42$-$}$$

$$\text{set of detected observables } : \{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^{\times}\,. \tag{42+}$$

The spaces $\Phi_{\mp}$ are mathematically constructed such that the two triplets (42$-$, 42+) form a Gel'fand triplet or a rigged Hilbert space (RHS) [31]. Then, using the mathematical properties of the RHS, one knows that there exist a complete set of generalized eigenvectors $|E^{\pm}\rangle \in \Phi_{\mp}^{\times}$ of the self-adjoint Hamiltonian H (nuclear spectral theorem [22, 31]). This means that there exist $|E^-\rangle \in \Phi_+^{\times}$ such that the Dirac basis vector expansion for every out-observable $\psi^- \in \Phi_+$ is given by

$$\Phi_+ \ni \psi^- = \int_0^\infty dE|E^-\rangle\langle^-E|\psi^-\rangle = \sum_{j,j_3,\eta}\int_0^\infty dE\,|E,j,j_3,\eta^-\rangle\langle^-E,j,j_3,\eta|\psi^-\rangle\,, \tag{43}$$

and there exist $|E^+\rangle \in \Phi_-^{\times}$ such that the basis vector expansion for every prepared in-states $\phi^+ \in \Phi_-$ is given by

$$\Phi_- \ni \phi^+ = \int_0^\infty dE|E^+\rangle\langle^+E|\phi^+\rangle = \sum_{j,j_3,\eta}\int_0^\infty dE\,|E,j,j_3,\eta^+\rangle\langle^+E,j,j_3,\eta|\phi^+\rangle\,. \tag{44}$$

In the summation on the far right, all quantum numbers have been included and we often suppress the discrete quantum numbers j, j_3, η because they are handled in the usual way. The kets $|E,j,j_3,\eta^{\pm}\rangle$ are generalized eigenvectors of the exact Hamilton operator $H = H_0 + V$,

$$H|E^{\mp}\rangle = E|E^{\mp}\rangle \tag{45}$$

or in detail

$$H|E,j,j_3,\eta^{\pm}\rangle = E|E,j,j_3,\eta^{\pm}\rangle\,, \tag{46}$$

with $0 \le E < \infty$. The expansions (43) and (44) is the same as in conventional scattering theory, where one uses as basis vectors the solution of the Lippmann-Schwinger equation [32]:

$$|E^{\pm}\rangle = |E \pm i\epsilon\rangle = |E\rangle + \frac{1}{E - H \pm i\epsilon} V|E\rangle = \Omega^{\pm}|E\rangle \qquad \lim \epsilon \to 0^{+} \ . \tag{47}$$

In (43) and (44) the basis kets $|E^{\pm}\rangle$ are mathematically defined as functionals on the Hardy spaces $\Phi_{\mp}$

$$|E, j, j_3, \eta^{+}\rangle \in \Phi_{-}^{\times} \ , \quad |E, j, j_3, \eta^{-}\rangle \in \Phi_{+}^{\times} \ . \tag{48}$$

Whether these basis vectors fulfill also the Lippmann-Schwinger equations must be checked for every particular problem with a particular Hamiltonian $H = H_0 + V$. Because the term $+i\epsilon$ in the Lippmann-Schwinger equation suggests analytic continuation to complex energy, the energy wave functions

$$\phi^{+}(E) = \langle^{+}E, j, j_3, \eta|\phi^{+}\rangle = \overline{\langle\phi^{+}|E, j, j_3, \eta^{+}\rangle} \tag{49}$$

have to be the boundary values of analytic functions in the lower complex energy semi-plane. For complex energy $z = \overline{E + i\epsilon} = E - i\epsilon$, this value is immediately below the real axis of the 2nd sheet of the S-matrix $S_j(z)$[¶]. The Hardy space axiom (42−) and (42+), i.e. the requirement that the energy wave functions are Hardy functions, is stronger than the requirement of analyticity.

As a consequence of our new Hardy space axiom (42), the energy wave functions $\phi^{+}(E)$ and $\psi^{-}(E)$ are not only smooth functions on the positive real energy axis as would be the case under the Schwartz space hypothesis (39), but they are analytic on the lower and upper complex energy semi-planes in the 2nd sheet, respectively. In other words, the $\phi^{+}(z) = \langle^{+}z|\phi^{+}\rangle$ are analytic on the lower complex plane $\mathbb{C}_{-}$ and they are smooth Hardy functions on $\mathbb{C}_{-}$; whereas the $\psi^{-}(E) = \langle^{-}E|\psi^{-}\rangle \to \langle^{-}z|\psi^{-}\rangle$ are analytic on the upper complex plane and they are smooth Hardy functions on $\mathbb{C}_{+}$. That is, the energy wave functions $\langle^{+}E|\phi^{+}\rangle$ of the prepared states ϕ^{+} must be functions that can be extended from $\mathbb{R}_{+}$ to analytic functions on $\mathbb{C}_{-}$, and the energy wave functions $\langle^{-}E|\psi^{-}\rangle$ of the detected observables ψ^{-} must be functions that can be extended from the real line $\mathbb{R}_{+}$ to analytic functions on $\mathbb{C}_{+}$. Therefore, we propose an axiom for the wave functions:

The set of prepared in-state wave functions on the positive real semi-axis $E \in \mathbb{R}_{+}$

$$\{\langle^{+}E|\phi^{+}\rangle\} = \mathcal{S} \cap \mathcal{H}_{-}^{2}|_{\mathbb{R}_{+}} \tag{50}$$

are smooth Hardy functions of the lower complex energy plane. The set of observed out-state wave functions

$$\{\langle^{-}E|\psi^{-}\rangle\} = \mathcal{S} \cap \mathcal{H}_{+}^{2}|_{\mathbb{R}_{+}} \tag{51}$$

are smooth Hardy functions of the upper complex energy plane.

$\mathcal{S} \cap \mathcal{H}_{\pm}^{2}|_{\mathbb{R}_{+}}$ are the realization of the abstract spaces $\Phi_{\pm}$ in the same way as $L^2(\mathbb{R}_{+})$ is the realization of the abstract Hilbert space $\mathcal{H}$ by the space of Lebesgue square integrable functions. One can show that—as a consequence of the Paley-Wiener theorem for Hardy functions [33]—the Schrödinger equation (18b) and the Heisenberg equation (18a) have asymmetric time evolutions in the Hardy spaces Φ_{-} and Φ_{+}:

$$\phi^{+}(t) = e^{-iHt}\phi^{+}(0) \qquad 0 \le t < \infty \ , \tag{52a}$$

[¶]The reason we choose the 2nd sheet of the S-matrix $S_j(z)$ is that the resonance poles are located on the 2nd sheet, which one reaches when one passes through the cut along the positive real axis.

$$\psi^-(t) = e^{iHt}\psi^-(0) \qquad 0 \le t < \infty \ . \tag{52b}$$

In summary, if we replace the Hilbert space axiom (13) of standard quantum mechanics by the Hardy space *axiom*:

- The set of prepared (in-) states defined by the preparation apparatus (e.g. accelerator) are mathematically described by (42−) and the set of (out-) observables defined by the registration apparatus (e.g. detector) are mathematically described by (42+).
- All solutions of the Schrödinger equation (18b) with $H = H|_{\Phi_-}$ and the boundary condition $\phi^+ \in \Phi_-$ are given by the semigroup (52a), not by the unitary group (19b). Similarly, all solutions of the Heisenberg equation (18a) and the boundary condition $\psi^- \in \Phi_+$ are given by the other semigroup (52b), not by the unitary group (19a).

As far as the description of the physical apparatuses is concerned, the modification of the Hilbert space axiom to the Hardy space axiom has no observable consequence because it is probably imperceptible whether the resolutions of the accelerator or detector are described by smooth rapidly decreasing functions of energy or only by those smooth rapidly decreasing functions which can be analytically continued into the complex energy plane. But mathematically the replacement of the Hilbert space boundary condition (13) by the Hardy space boundary condition (42−) and (42+) is a severe modification and it has serious consequences, like the semigroup time evolution (52a) and (52b). We shall discuss the other consequences in the next section.

2.4. Resonances and decaying states in time asymmetric quantum mechanics

The new axiom (42−) and (42+) of Time Asymmetric Quantum Theory (TAQT) leads to a consistent theory of resonances and decaying states. For instance, it allows us (A) to define an exponentially decaying ("Gamow") state vector, (B) to associate it with a Breit-Wigner resonance amplitude and (C) to derive it from a resonance pole of the S-matrix.

In standard quantum mechanics a Gamow vector is not possible; a resonance can be defined by a Breit-Wigner energy distribution (1a) but the decay rate and/or the survival probability for this state with Breit-Wigner distribution contains a term that is not exponential in time ("deviation from exponential decay law" [18]). Gamow vector and Breit-Wigner amplitude are at best "approximately" related (Weisskopf-Wigner "approximation" [2]). There is also no direct connection between the S-matrix pole and a Breit-Wigner amplitude or a Gamow vector; a Gamow vector is an out-cast of a mathematical theory of scattering.

However, just one new axiom (42−) and (42+) changes all this. Using this axiom one can derive that

1) To every (first order) S-matrix pole there corresponds a Breit-Wigner energy distribution, albeit with energy extending over $-\infty < E < \infty$.

2) Every Breit-Wigner energy distribution leads to a (mathematically well defined as elements of $\Phi_-^\times$) Gamow vector with exponential time dependence, albeit with a semigroup evolution $0 \le t < \infty$.

We start with the Born probability amplitude $(\psi^-|\phi^+)$ to register the observable $\Lambda^- = |\psi^-\rangle\langle\psi^-|$, $\psi^- \in \Phi_+$ in the state $W^+ = |\phi^+\rangle\langle\phi^+|$, $\phi^+ \in \Phi_-$ for a scattering experiment. Using the standard notions of scattering theory, we express it as the S-matrix element

$$\mathrm{Tr}(\Lambda^- W^+) = (\psi^-|\phi^+) = (\psi^{\mathrm{out}}, S\phi^{\mathrm{in}}) \ . \tag{53}$$

This statement is essentially the same as in standard scattering theory except that in the traditional language of scattering theory one speaks of in-state ϕ^+, and of out-state ψ^- but then restricts oneself mostly to $t \geq 0$. The important difference here is that for in-states ϕ^+, the time evolution is given by (52a) and for the out-observables ψ^-, the time evolution is given by (52b). Since Born probabilities correlate observables with states, not states with states, the definition of the S-matrix element by the Born probability amplitude is given for $t \geq 0$.

Using (43) and (44) with energy and angular momentum conservation laws, the matrix element $(\psi^-|\phi^+)$ can be expressed as [34]

$$(\psi^-|\phi^+) = \sum_{\eta,\eta',j,j_3} \int_0^\infty dE \langle \psi^-|E,j,j_3,\eta^-\rangle S_j^{\eta\eta'}(E) \langle {}^+E,j,j_3,\eta'|\phi^+\rangle \ . \tag{54}$$

Here, $S_j^{\eta\eta'}(E) = \langle {}^-E,j,j_3,\eta|E,j,j_3,\eta'^+\rangle$ is the S matrix element and it is related to the scattering amplitude $a_j(E)$ ($\sigma_j(E) \sim |a_j(E)|^2$) by

$$S_j^{\eta_0}(E) = 2ia_j^{\eta_0}(E) + 1 \qquad \text{(elastic channel from } \eta_0 \to \eta_0) \ , \tag{55}$$

$$S_j^{\eta}(E) = 2ia_j^{\eta}(E) \qquad \text{(reaction channel from } \eta_0 \to \eta) \ . \tag{56}$$

Under the new hypothesis (42$-$) and (42+), the energy wave functions in (54) are not only smooth rapidly decreasing or square integrable functions, but they are also Hardy functions, which means that they are analytic in such a way that the integral along the positive real axis on the physical sheet (see Figure 10) can be continued through the cut along the positive real axis into the lower half complex energy plane of the second sheet. From the integral along the real axis on the first sheet (spectrum of H) one goes to an integral along the infinite semi-circle in the lower half-plane $\mathcal{C}_-$ plus integral around the pole z_R and an integral from $-\infty$ to 0 on the second sheet. The integral along the infinite semicircle is zero (due to the Hardy property of $\langle\psi^-|z^-\rangle\langle{}^+z|\phi^+\rangle$).

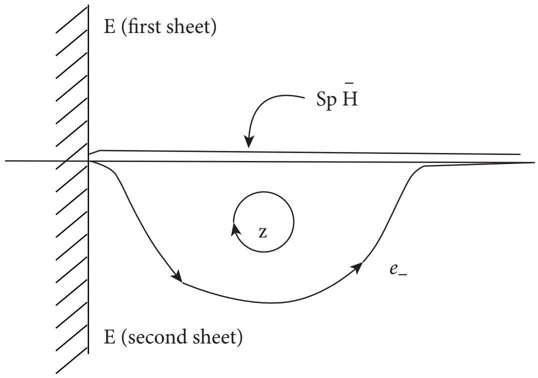

Figure 10. Deformation of the path of integration into the second sheet of the lower half complex energy plane for the decaying state.

Then one remains with the two integrals

$$(\psi^-|\phi^+) = \int_0^{-\infty_{\rm II}} dz \langle\psi^-|z^-\rangle S(z) \langle{}^+z|\phi^+\rangle + \oint_C dz \langle\psi^-|z^-\rangle \frac{R}{z - z_R} \langle{}^+z|\phi^+\rangle \ . \tag{57}$$

Here, C is the circle around the position of the pole z_R and by $\infty_{\rm II}$ we mean it is infinity on the second sheet. One also has to insert the S-matrix expansion

$$S^{\eta\eta'} = \frac{R}{z - z_R} + R_0 + R_1(z - z_R) + \cdots \tag{58}$$

into the second integral around the pole position z_R and use the analyticity property of Hardy functions. Since the first integral on the right hand side of equation (57) is not related to the resonance per se, we shall not consider it here for now. The second integral on the right hand side of (57) can be evaluated separately by using the well-known Cauchy theorem

$$f(z_R) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_R}\, dz \ . \tag{59}$$

So, we find that

$$(\psi^-|\phi^+)_{\rm pole} = -2\pi i R \langle \psi^-|z_R^-\rangle\langle^+ z_R|\phi^+\rangle \ . \tag{60}$$

On the other hand, if one uses the Hardy property of the wave functions $\langle\psi^-|z^-\rangle$, $\langle^+z|\phi^+\rangle$ and the Titschmarsh theorem [23] for the function $G_-(E) \equiv \langle\psi^-|E^-\rangle\langle^+E|\phi^+\rangle$, then for the second integral on the right hand side of (57), the contour of integration around the pole can be deformed into the contour of integration along the real axis from $-\infty_{\rm II}$ to $+\infty$ (second sheet lower rim of the real axis, which is the same as the first sheet upper rim on the positive real axis):

$$(\psi^-|\phi^+)_{\rm pole} = \int_{-\infty_{\rm II}}^{\infty} dE \langle\psi^-|E^-\rangle\langle^+E|\phi^+\rangle \frac{R}{E - z_R} \ . \tag{61}$$

Here the subscript II in $-\infty_{\rm II}$ indicates that the analytic continuation has been done in the second sheet of the analytic S-matrix where the position $z_R = E_R - i\Gamma_R/2$ of the resonance pole is located. Comparing (60) with (61) leads to the following definition of the Gamow vectors $|z_R^-\rangle$ as functionals on the space $\{\psi^-\} \equiv \Phi_+$:

$$\langle\psi^-|z_R^-\rangle = \frac{i}{2\pi} \oint_C \frac{\langle\psi^-|z^-\rangle}{z - z_R}\, dz = \frac{i}{2\pi} \int_{-\infty_{\rm II}}^{\infty} \frac{\langle\psi^-|E^-\rangle}{E - z_R}\, dE \ , \tag{62a}$$

or in ket notation for all $\psi^- \in \Phi_+$

$$|z_R^-\rangle = \frac{i}{2\pi} \int_{-\infty_{\rm II}}^{\infty} |E^-\rangle \frac{1}{E - z_R} \ . \tag{62b}$$

The integral extends over the negative real axis from $-\infty_{\rm II}$ to 0 in the second sheet and then along the positive real axis $0 \le E < \infty$, which are the physical scattering energies. Since the Hardy function has the remarkable property that the values of $\langle\psi^-|E^-\rangle$ for negative E are already determined by their values for $0 \le E < \infty$ [35], the values of $\langle\psi^-|z^-\rangle$ are completely determined from the values $\langle\psi^-|E^-\rangle$ for physical scattering energies. Because of this property of the Hardy function, the phenomenological Breit-Wigner formula defined for $0 \le E < \infty$ can be extended uniquely to $-\infty_{\rm II} < E < \infty$. This definition of the Breit-Wigner is one of the important points where the Hardy axiom is needed.

If we start from the definition of a resonance as a simple pole of the partial S-matrix $S_j(E)$ of angular momentum j at the complex value $z = z_R$, then by (54)–(62a) we obtain from this pole the Gamow vector: the values of the functional $|z_R^-\rangle$ for every vector $\psi^- \in \Phi_+$; that means the element $\psi_j^G = |z_R, j, j_3, \eta^-\rangle\sqrt{2\pi\Gamma}$ is defined as a functional on the Hardy space Φ_+. If one omits the arbitrary ψ^- in the equality (62a) one obtains the relation between generalized vectors

$$a_j^{\rm BW} = \frac{R}{E - z_R} \ , -\infty < E < \infty \iff \psi_j^G \equiv \sqrt{2\pi\Gamma}|z_R, j, j_3, \eta^-\rangle \ . \tag{63}$$

This means that to a first order pole of the S-matrix there correspond a pole term given by a Breit-Wigner energy distribution and to every Breit-Wigner there corresponds a generalized vector ψ_j^G with the integral representation given by the right hand side of (62a). This can be done for every first order poles at the values $z_R = E_R - i\Gamma/2$. Thus, one can conclude that every Gamow vector is given by the continuous linear superposition of the Lippmann-Schwinger-Dirac kets $|E, j, j_3, \eta^-\rangle$ with Breit-Wigner energy wave functions $\langle^- E|\psi_j^G\rangle = \langle^- E, j, j_3, \eta|\psi_j^G\rangle \sim a_j^{\rm BW}(E)$ albeit with E extending over $-\infty < E < \infty$ [‖].

Using the property of Hardy functions, one can further derive that [23]:

$$\langle H\psi_\eta^-|\psi^G\rangle = \langle\psi_\eta^-|H^\times|\psi^G\rangle = (E_{\rm R} - i\Gamma/2)\langle\psi_\eta^-|\psi^G\rangle \tag{64}$$

for all $\psi_\eta^- \in \Phi_+$ if the Hamiltonian $H = H_0 + V$ is a self-adjoint operator and has semi-bounded spectrum $0 \leq E < \infty$. Hence, the Gamow vector ψ_j^G is not an ordinary vector with a Breit-Wigner energy wave function, but it is a generalized vector and does not have a smooth energy wave function on $\mathbb{R}_+$ (the physical values of scattering energy E) [36]. The equation (64) states that ψ_G is an eigenket of $H^\times$ (on the Hardy space) with a discrete complex eigenvalue z_R (as Gamow desired).

In Dirac's notation, (64) can be written as

$$H^\times|E_R - i\Gamma/2^-\rangle = (E_R - i\Gamma/2)|E_R - i\Gamma/2^-\rangle \ , \tag{65}$$

where $H^\times$ is the extension of the Hamilton operator $\bar{H} = H^\dagger$ in the completion of the Hilbert space $\mathcal{H}$ to the space $\Phi_+^\times$ of (42+).

If there are N poles in the $j^{\rm th}$ partial wave amplitude $a_j(E)$, then one obtains a sum over the Breit-Wigner-resonance amplitudes. In addition one obtains always a background function $B(E)$, which has its origins in the first term on the right side of (57) [23]

$$a_j(E) = \sum_{i=1}^N \frac{R_i}{E - z_{R_i}} + B(E) \ . \tag{66}$$

In addition, there is a vector ϕ^{bg} which is connected to the background amplitude $B(E)$ of (66)

$$\phi_j^+ = \sum_i |z_{R_i}, j, j_3, \eta^-\rangle c_i + \phi^{bg} \ . \tag{67}$$

‖It cannot be expressed in terms of a Lorentzian function $\frac{1}{E - z_R}$ or any other smooth function on the positive real axis $0 \leq E < \infty$ only[36].

Each background vector can be represented by a continuous superposition of $|E, j, j_3, \eta^-\rangle$ with $0 \leq E < \infty$ [37]. Then, one has the following correspondences

$$
\begin{aligned}
a_j(E) = \sum_i \frac{R_i}{E - z_{R_i}} + B(E) \Leftrightarrow \phi_j^+ &= \sum_i |z_{R_i}, j, j_3, \eta^-\rangle c_i + \phi^{bg} , \\
\frac{R_i}{E - z_{R_i}} &\Leftrightarrow |z_{R_i}, j, j_3, \eta^-\rangle , \\
B(E) &\Leftrightarrow \phi^{bg} .
\end{aligned}
\tag{68}
$$

Now, one can calculate the probability amplitude to find an observable $\psi_\eta^-(t)$ (decay products) in Gamow state $|E_{\rm R} - i\Gamma/2^-\rangle$ at the time t and one obtains [23]:

$$
\begin{aligned}
\langle\psi_\eta^-(t)|E_{\rm R} - i\Gamma_R/2^-\rangle &= \langle e^{iHt}\psi_\eta^-|E_{\rm R} - i\Gamma/2^-\rangle = \langle\psi_\eta^-|e^{-iH^\times t}|E_{\rm R} - i\Gamma/2^-\rangle \\
&= e^{-iE_{\rm R}t}e^{-(\Gamma/2)t}\langle\psi_\eta^-|E_{\rm R} - i\Gamma/2^-\rangle \quad \text{for} \quad t \geq 0 \, \text{only}.
\end{aligned}
\tag{69}
$$

This holds only for $t \geq 0$ because the operator $U^\times(t) = e^{-iH^\times t}$ is defined on $\Phi_+^\times$ only for $t \geq 0$. Here, $t = 0$ is the time at which the Gamow state $|E_{\rm R} - i\Gamma/2^-\rangle$ has been prepared. A discussion of the physical meaning of this time t=0, which is mathematically represented by semi-group time $t = 0$ of the semi-group time evolution operator $U^\times = e^{-iH^\times t}$, $0 \leq t < \infty$, can be found in [38].

From (69) it follows that

$$
|\langle\psi_\eta^-|\psi_j^G(t)\rangle|^2 = e^{-\Gamma t}|\langle\psi_\eta^-|\psi_j^G(0)\rangle|^2 \quad \text{for}\, t \geq 0 \, \text{only}. \tag{70}
$$

The derivations of these results rely heavily on the Hardy Space Axiom. Thus, the Gamow state ψ_j^G (62a) associated to the pole of the $j^{\rm th}$ partial S-matrix has a Breit-Wigner energy distribution which extends $-\infty < E < \infty$ and cannot be represented by a smooth function on $\mathbb{R}_+$. From (69) and (70), it follows that one has an exact exponential time evolution with lifetime given by

$$
\tau = 1/\Gamma (= \hbar/\Gamma) . \tag{71}
$$

The exponential decay law is the prediction of time asymmetric quantum theory. The Gamow vector in (63) represents the resonance per se, (without background). The deviation from the exponential decay is due to the background vector ϕ^{bg} in (67) which comes from the always present background amplitude $B(E)$ in (66). This is nothing to do with the resonance per se, which is given by the Breit-Wigner amplitude $\frac{R_i}{E - z_{R_i}}$ and by the Gamow ket ψ_j^G.

Summary: Time asymmetric quantum theory connects the S-matrix pole, the Breit-Wigner resonance amplitude and the Gamow vectors with each other and identifies these as the signatures of the quasistable exponentially decaying-resonance state. But it also explains the deviations from the exponential decay law as the effect of the ever present background given by the amplitude $B(E)$ which is connected to a vector ϕ^{bg}. The Weisskopf-Wigner approximation is obtained if one omits these (continuous) background amplitude in (66) and the background vector ϕ^{bg} in (67).

3. Brief remarks about relativistic resonances

In the previous sections, we explained that an exact theory of resonances and decaying phenomena does not exist within the framework of the standard axioms of quantum mechanics and we have presented a unified, consistent and exact theory for resonances and decaying states by changing one axiom of standard quantum mechanics. That is, we have replaced the Hilbert space axiom with a *Hardy space axiom.*

Resonances and decaying phenomena are also observed in relativistic particle physics, as we have mentioned in the phenomenological part of this paper. For relativistic systems, the situation is more complicated since there is no agreement on how to characterize relativistic resonance in terms of the two numbers, like mass and width. For example, there are several parameterizations of the Z-boson resonance amplitude. The most common parametrization is the relativistic Breit-Wigner amplitude with energy dependent width $\Gamma(s)$ for which one applies "the on-shell renormalization scheme" (which turned out to be gauge dependent) [10]:

$$a_j^{om}(s) = -\frac{\sqrt{s}\Gamma_e(s)\Gamma_f(s)}{s - M_Z^2 + i\sqrt{s}\Gamma_Z(s)} \approx \frac{R_Z}{s - M_Z^2 + i\frac{s}{M_Z}\Gamma_Z}. \tag{72}$$

Another possible parametrization of the relativistic Breit-Wigner amplitude is the S-matrix pole definition with constant width:

$$a_j^{\mathrm{BW}} = \frac{R_Z}{s - s_R} , \tag{73}$$

where different parameterizations of a complex pole position s_R are used. Some of them are

$$s_R = \left(M_R - i\frac{\Gamma_R}{2}\right)^2 , \tag{74}$$

$$s_R = \bar{M}_Z^2 - i\bar{M}_Z\bar{\Gamma}_Z . \tag{75}$$

By fitting these to the experimental data, one obtains several mass and width values for the Z-boson [8]. PDG listed mass and width values for the Z-boson (M_Z, Γ_Z) as

$$M_Z = 91.1876 \mp 0.0021\,\mathrm{GeV} \qquad \Gamma_Z = 2.4952 \mp 0.0023\,\mathrm{GeV} \tag{76}$$

and for the other parameterizations (74) and (75), they differ from each other:

$$M_Z - M_R = 0.0026 \pm 0.004\,\mathrm{GeV}, \quad M_Z - \bar{M}_Z = 0.035 \pm 0.004\,\mathrm{GeV}, \tag{77}$$

and

$$\Gamma_Z - \bar{\Gamma}_R \approx 0.0012\,\mathrm{GeV}, \quad \Gamma_Z - \bar{\Gamma}_Z \approx 0.0009\,\mathrm{GeV}. \tag{78}$$

Notice that this difference for the masses is significant since it amounts to 10 times the experimental error. So, the question is: How do we distinguish between these different mass definitions? Is there a right definition of mass for the Z-boson or is the mass value of the Z-boson and other fundamental decaying particles just a convention?

The same problem also appears for ρ resonance. PDG [8] gives different mass and width values:

$$M_\rho = 775.5 \mp 0.4\,\mathrm{MeV} \qquad \Gamma_\rho = 146.4 \mp 1.1\,\mathrm{MeV} , \tag{79}$$

$$\bar{M}_\rho = 757.5 \mp 1.5\,\text{MeV} \qquad \bar{\Gamma}_\rho = 142.5 \mp 3.5\,\text{MeV}\ , \tag{80}$$

which again differ significantly from each other.

For the Δ-resonances, the PDG quotes two different values for mass and width; the one called pole position, which is obtained from a fit to the relativistic Breit-Wigner amplitude (73) with the parametrization (75), is given in (82) below. The other, called "Breit-Wigner mass," comes from a fit (of essentially the same data) to the amplitude $a_j^{om}(s)$ in (72), and gives the values in (81)

$$M_\Delta = 1231.88 \mp 0.29\,\text{MeV} \qquad \Gamma_\Delta = 109.07 \mp 0.48\,\text{MeV}\ , \tag{81}$$

$$\bar{M}_\Delta = 1212.50 \mp 0.24\,\text{MeV} \qquad \bar{\Gamma}_\Delta = 97.37 \mp 0.42\,\text{MeV}\ . \tag{82}$$

Since two different values of mass and width are in circulation for the three best measured relativistic resonances, one can ask which of these (M, Γ) is the right mass and width? We shall discuss this problem in the following subsections and suggest a solution. The solution will be in favor of the relativistic Breit-Wigner amplitude (73) which originates from the pole of the relativistic S-matrix.

3.1. How to define mass and width uniquely?

In order to obtain a unique definition of a mass and width and unify relativistic resonances with decaying states, in analogy with the non-relativistic case, one has to define relativistic Gamow vector and relativistic Lippmann-Schwinger kets rigorously and find how the relativistic Gamow vectors transform under causal Poincaré transformations [9, 34].

The problems with gauge invariance led to the choice of (73) for the relativistic Breit-Wigner amplitudes [9]. This means that a relativistic resonance is defined by a pole term of the j^{th} partial S matrix given by the relativistic Breit-Wigner amplitude (73) and associated to a pole of the S matrix at $s = s_R$. A relativistic Gamow vector is then defined in analogy to the non-relativistic case: 1. We replace the non-relativistic energy in (63) by s: $E \to s = p_\mu p^\mu$ and $z_R \to s_R$. The resulting relativistic Gamow kets span a representation space of the Poincaré semigroup transformations in the froward light cone. 2. From the transformation properties of the Gamow kets under Poincaré transformations, one can define a Γ uniquely by the condition that $\hbar/\Gamma$ be the lifetime τ of the relativistic resonance. This condition will lead to Γ_R and therewith M_{R} in (74) as the right values for the width and mass of relativistic resonances. This will be briefly explained in the following subsections.

3.2. Relativistic hardy space axiom

In the relativistic case, all we have to do is to use in place of the non-relativistic angular momentum eigenvectors, the relativistic basis vectors which span the representation space $[s, j]$ of Poincaré transformations [34]:

$$|E, j, j_3, \eta\rangle \to |[s, j]\hat{\mathbf{p}}, j_3, n\rangle\ . \tag{83}$$

We use, in place of the standard Wigner momentum eigenkets $|[s, j]\mathbf{p}, j_3, \eta\rangle$ for the basis vectors of the Poincaré group, the eigenkets of the space component of the 4-velocity $\hat{\mathbf{p}} = \gamma\mathbf{v} = \mathbf{p}/\sqrt{s}$. Here, $\gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}} = \hat{p}^0$, $\sqrt{s} = m = E^{\text{cm}}$ (center of mass energy).

We have, in analogy with the non-relativistic basis vector expansion (43) and (44), the expansions familiar from the representations of the Poincaré transformations:

$$\phi_n^+ = \int_{(m_1+m_2)^2}^{\infty} ds \sum_{jj_3} \int_{-\infty}^{\infty} \frac{d^3\hat{p}}{2\hat{p}^0} |[s,j]\hat{\mathbf{p}}, j_3, n^+\rangle\langle^+ n, j_3, \hat{\mathbf{p}}[s,j]|\phi^+\rangle \tag{84}$$

$$\equiv \int_{s_0}^{\infty} ds |s^+\rangle\langle^+ s|\phi^+\rangle \tag{85}$$

$$\psi_n^- = \int_{(m_3+m_4)^2}^{\infty} ds \sum_{jj_3} \int_{-\infty}^{\infty} \frac{d^3\hat{p}}{2\hat{p}^0} |[s,j]\hat{\mathbf{p}}, j_3, n^-\rangle\langle^- n, j_3, \hat{\mathbf{p}}[s,j]|\psi^-\rangle \tag{86}$$

$$\equiv \int_{s_0}^{\infty} ds |s^-\rangle\langle^- s|\psi^-\rangle \tag{87}$$

The expansions (85) and (87) are the abbreviated versions of the expansions (84) and (86) in which only the invariant mass $s = p_\mu p^\mu$ has been retained, because that is the quantity which replaces E in (43) and (44). The Hardy space axiom for the relativistic case then states that

$$\langle^+ s|\phi^+\rangle \equiv \langle^+ n, j_3, \hat{\mathbf{p}}[s,j]|\phi^+\rangle = \overline{\langle^+\phi|s^+\rangle} \in \tilde{\mathcal{S}} \cap \mathcal{H}_-^2|_{\mathbb{R}_{s_0}} \otimes \mathcal{S}(\mathbb{R}^3) \tag{88}$$

is analytic in the lower complex s-plane (second Riemann sheet of the S-matrix) and that

$$\langle^- s|\psi^-\rangle \equiv \langle^- n, j_3, \hat{\mathbf{p}}[s,j]|\psi^-\rangle = \overline{\langle^-\psi|s^-\rangle} \in \tilde{\mathcal{S}} \cap \mathcal{H}_+^2|_{\mathbb{R}_{s_0}} \otimes \mathcal{S}(\mathbb{R}^3) \tag{89}$$

is analytic in the upper complex s-plane and $\mathbb{R}_{s_0} = \{s|(m_1+m_2)^2 \leq s < \infty\}$. This is the relativistic analogue of (50) and (51) in the non-relativistic case. In the relativistic case, the Schwartz space is slightly modified into the $\tilde{\mathcal{S}}$ [34]. The non-relativistic theory of resonances and decaying states was the motivation for the axiom (88) and (89). Using the Hardy spaces (88) and (89) in s, one is able unify the relativistic resonance and decay phenomena.

3.3. Causal space-time transformations and relativistic resonance states

For the relativistic space-time evolution the transformations of the detected observables relative to the prepared state form a semigroup representation in the forward light cone $U_+(\Lambda, x)$, with

$$\mathcal{P}_+ \equiv \{(\Lambda, x)|\Lambda \in \overline{SO(3,1)}, \Lambda_0^0 \geq 1, \det\Lambda = +1, x \in \mathbb{R}_{1,3}, x^2 = t^2 - \mathbf{x}^2 \geq 0, t \geq 0\}\ . \tag{90}$$

This semigroup consists of all proper orthochronous Lorentz transformations and of space-time translations in the forward light cone. The restriction of the transformations (of observables relative to state) to semigroup transformations of the light cone $U_+(\Lambda, x), (\Lambda, x) \in \mathcal{P}_+$ has the following physical meaning [9]:

1) $t \geq 0$: A state must be prepared first (at $t = 0$) before one can speak of probabilities for observables (causality).

2) $x^2 = t^2 - \mathbf{x}^2 \equiv t^2 - \frac{r^2}{c^2} \geq 0$ or $t^2 \geq \frac{r^2}{c^2}$: Born probabilities ("the signal") can only propagate with a velocity r/t which is smaller than the speed of light, $r/t \leq c$ (Einstein causality).

In order to solve the problem of determining the mass and width values of relativistic resonances and decaying states uniquely, we need a general formula for the transformation of the Gamow kets under the Poincaré transformations $(\Lambda, x) \in \mathcal{P}_+$.

Starting with the S-matrix element

$$(\psi^{\mathrm{out}}, \phi^{\mathrm{out}}) = (\psi^{\mathrm{out}}, S\phi^{\mathrm{in}}) = (\psi^-, \phi^+) = \int_{s_0}^{\infty} ds \langle^- \psi | s^- \rangle S_j(s) \langle^+ s | \phi^+ \rangle \tag{91}$$

and following the similar arguments as in non-relativistic case, we associate relativistic Breit-Wigner distribution with relativistic Gamow ket:

$$a_j^{\mathrm{BW}} = \frac{r}{s - s_R} \Longleftrightarrow |[s_R, j]\hat{\mathbf{p}}, j_3{}^- \rangle = \frac{i}{2\pi} \int_{-\infty_{\mathrm{II}}}^{\infty} ds \, |[s, j]\hat{\mathbf{p}}, j_3{}^- \rangle \, \frac{1}{s - s_R} \, . \tag{92}$$

The right hand side of this equation is in fact the analog of the representation for the non-relativistic Gamow vectors (63).

Line shape alone cannot discriminate between (M_R, Γ_R), $(\bar{M}_Z, \bar{\Gamma}_Z)$, etc, because they are just different parameterizations of $a_j^{\mathrm{BW}}(s)$. The selection of the "right" (M, Γ) requires a theory that relates M or Γ to the particle aspects of a relativistic decaying state.

Wigner's unitary representations $[m^2, j]$ of the Poincaré group $\mathcal{P}$ for the stable relativistic particle of mass m and spin j transform the Dirac basis kets in the following way:

$$U^\dagger(\Lambda, x) |[m, j]\hat{\mathbf{p}}, j_3\rangle = e^{-im\gamma(t - xv)} \sum_{j_3'} D^j_{j_3 j_3'}(W(\Lambda^{-1}, \hat{\mathbf{p}})) |[m, j]\Lambda^{-1}\hat{\mathbf{p}}, j_3'\rangle \, . \tag{93}$$

Here, (Λ, x) are elements of Poincaré group, $-\infty < x_\mu < \infty$, $W(\Lambda^{-1}, \hat{\mathbf{p}})$ is the Wigner rotation, and $D^j_{j_3 j_3'}$ is the rotation matrix corresponding to j^{th} angular momentum. The basis kets in (93) are Schwartz-space kets $|[m, j]\hat{\mathbf{p}}, j_3\rangle \in \Phi^\times$ and $U^\dagger(\Lambda, x)$ is the group of unitary operators in the Hilbert space. In contrast to this, the relativistic Gamow kets have the following transformation properties:

$$U^\times(\Lambda, x) |[s_R, j]\hat{\mathbf{p}}, j_3{}^-\rangle = e^{-i\sqrt{s_R}\gamma(t - xv)} \sum_{j_3'} D^j_{j_3 j_3'}(W(\Lambda^{-1}, \hat{\mathbf{p}})) |[s_R, j]\Lambda^{-1}\hat{\mathbf{p}}, j_3'{}^-\rangle. \tag{94}$$

$U^\times(\Lambda, x)$ are operators in the dual space $\Phi_+^\times$ and $|[s_R, j]\Lambda^{-1}\hat{\mathbf{p}}, j_3'{}^-\rangle \in \Phi_+^\times$, and $U^\times(\Lambda, x)$ are defined for the *semigroup* transformations $(\Lambda, x) \in \mathcal{P}_+$ in the forward light cone, i.e., they are defined *only* for $x^2 = t^2 - \mathbf{x}^2 \geq 0$ and $t \geq 0$. The difference between Wigner's formulation of relativistic stable particles and relativistic time asymmetric quantum theory for the relativistic quasistable particles is that the former is described by the irreducible unitary representations of Poincaré group and characterized by a real mass, and the latter is described by the irreducible "minimally complex" semigroup representations of the Poincaré transformations into the forward light cone [9, 34]. The representations of these semigroup transformations are characterized in addition to spin j by a complex mass

$$\sqrt{s_R} = (M_{\mathrm{R}} - i\Gamma_R/2) = \sqrt{\bar{M}_Z^2 - i\bar{M}_Z\bar{\Gamma}_Z} \, . \tag{95}$$

3.4. Exponential decay of relativistic breit-wigner resonances

Gamow vectors are generalized eigenvectors of the mass operator $M = (P_\mu P^\mu)^{1/2}$ and momentum operators P^μ with complex eigenvalues

$$\begin{aligned} \mathbf{P}^\times |[s_R, j]\hat{\mathbf{p}}, j_3{}^-\rangle &= \sqrt{s_R}\ \hat{\mathbf{p}}|[s_R, j]\hat{\mathbf{p}}, j_3{}^-\rangle & (96)\\ P_0^\times |[s_R, j]\hat{\mathbf{p}}, j_3{}^-\rangle &= \gamma\sqrt{s_R}\ |[s_R, j]\hat{\mathbf{p}}, j_3{}^-\rangle & (97)\\ M^\times |[s_R, j]\hat{\mathbf{p}}, j_3{}^-\rangle &= \sqrt{s_R}\ |[s_R, j]\hat{\mathbf{p}}, j_3{}^-\rangle. & (98) \end{aligned}$$

If we go to the rest frame $\hat{\mathbf{p}} = (1, 0, 0, 0)$, $\mathbf{v} = 0$, and $P_0 = H$, we easily find

$$H^\times |[s_R, j]\hat{\mathbf{p}} = 0, j_3{}^-\rangle = \sqrt{s_R}\ |[s_R, j]\hat{\mathbf{p}} = 0, j_3{}^-\rangle. \tag{99}$$

The time translation of the Gamow vector in the rest frame is therefore given by

$$\begin{aligned} \Psi^G_{s_R}(t) &= e^{-iH^\times t}|[s_R, j]\hat{\mathbf{p}} = 0, j_3{}^-\rangle = e^{-i\sqrt{s_R}t}|[s_R, j]\hat{\mathbf{p}} = 0, j_3{}^-\rangle \\ &= e^{-iM_{\mathrm{R}}t}e^{-(\Gamma_R/2)t}|[s_R, j]\hat{\mathbf{p}} = 0, j_3{}^-\rangle \qquad \text{for } t \geq 0 \text{ only} . \end{aligned} \tag{100}$$

From this, we see that the Born probability density for detecting the decay products given by the observable $|\psi^-\rangle\langle\psi^-|$ ** in the quasistable state $\Psi^G_{s_R,j}(t) \equiv |Z^-\rangle$ is proportional to:

$$|\langle\psi^-|\Psi^G_{s_R}(t)|^2 = e^{-\Gamma_R t}|\langle\psi^-|\Psi^G_{s_R}(0)|^2 , \qquad t \geq 0 . \tag{101}$$

If the lifetime τ and the width Γ are to fulfill the standard relation $\tau = \hbar/\Gamma$, then, of the many possible parametrizations of s_R, only the parametrization $(\Gamma_R, M_{\mathrm{R}})$ given by (74) does the job. And from (72), a relativistic Gamow vector (92) cannot be obtained. Thus, of the many possible definition of (M, Γ) for a relativistic resonance, and the many possible parametrizations of the resonance pole position s_R, only (74) will give the lifetime $\hbar/\Gamma_R$. Therefore we would call

$$M_{\mathrm{R}} = \mathrm{Re}\sqrt{s_R} = 91.1611 \pm 0.0023\,\mathrm{GeV} = M_Z - 0.026\,\mathrm{GeV} \tag{102}$$

and

$$\Gamma_R = -2\mathrm{Im}\sqrt{s_R} = 2.4943 \pm 0.0024\,\mathrm{GeV} \tag{103}$$

the "right" values of mass and width for the Z-boson. It differs significantly from the value M_Z quoted by the PDG [8] for the mass of the Z-boson.

To summarize: The transformation property under causal Poincaré transformations of the Gamow state vector chooses $(M_{\mathrm{R}}, \Gamma_R)$ of (74) as the mass and width values for relativistic resonances. According to (101), the Gamow vector (92) with the relativistic Breit-Wigner line shape $\frac{1}{s - s_R}$ and the parameterization of the S-matrix pole position s_R given by (74) has the lifetime $\tau = \hbar/\Gamma_R$ where $\Gamma_R = -2\mathrm{Im}\sqrt{s_R}$.

**e.g. $|\psi^-\rangle\langle\psi^-| = |e^+e^-\rangle\langle e^+e^-|$

Acknowledgment

The authors gratefully acknowledge Prof. Erdal İnönü, with whose help Haydar Uncu and Fatih Erman came to collaborate with Prof. Arno Bohm. This work would have been impossible without his efforts. The authors are also indebted to Prof. Ali Mostafazadeh who had invited Prof. Bohm to İstanbul for a series of lectures which are the basis for this article. This work is part of a collaboration with the University of Texas at Austin, sponsored by the US NSF Award No. OISE-0421938, and was completed while the authors visited the Faculty of Sciences, Helwan University, Cairo. The authors wish to express their appreciation to Professor S. Komy for the hospitality extended to them at Helwan. A. B. also acknowledges support of the Center for Complex Quantum Systems.

References

[1] A. Bohm, *Quantum Mechanics: Foundations and Applications*, 3^{rd} Ed. (Springer Verlag, New York 2001), Ch.1 and Ch.2.

[2] V. Weisskopf and E. P. Wigner, *Z.* Phys., **63**, (1930), 54; **65**, (1930), 18.

[3] M. Levy, *Nuovo Cimento*, **13**, (1959), 115.

[4] D. E. Golden, H. Nakano, *Phys. Rev.*, **144**, (1966), 71.

[5] H. Frauenfelder, *The Mössbauer Effect* (W. A. Benjamin, New York, 1962), Sec.4.2.

[6] C. W. Oates, K. R. Vogel, and J. L. Hall, *Phys. Rev. Lett.*, **76**, (1996), 2866.

[7] U. Volz, M. Majerus, H. Liebel, A. Schmitt, and H. Schmoranzer, *P*hys. Rev. Lett., **76**, (1996), 2862.

[8] Particle Data Group, W-M Yao et al, *J. Phys. G: Nucl. Part. Phys.*, **33**, (2006), $1-1232$.

[9] A. Bohm, Y. Sato, *Phys. Rev. D.*, **71**, (2005), 085018.

[10] A. Sirlin, *Phys. Rev. Lett.*, **67**, (1991), 2127; *Phys. Lett. B*, **267**, (1991), 240;
S. Willenbrock and G. Valencia, *Phys. Lett. B*, **259**, (1991), 373;
R. G. Stuart, *Phys. Lett. B*, **262**, (1991), 113; **272**, (1994), 353, *Phys. Rev. D*, **50**, (1997), 4454;
A. Leike, T.Riemann, and J. Rose, *Phys. Lett. B*, **273**, (1991), 513;
T. Bhattacharya and S. Willenbrock, *Phys. Rev. D*, **47**, (1993), 4022 ;
H. G. J. Veltman, *Z. Phys. C*, **62**, (1994), 35.

[11] E. M. Henley and H. Garcia *Subatomic Physics*, 3rd Edition, (World Scientific Publishing, New Jersey, 2007), p.428.

[12] L. K. Gibbons et al. *Phys. Rev. Lett.,* **70**, (1993), 1199

[13] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, (Princeton University Press, Princeton, 1955).

[14] A. Bohm, *Phys. Rev. A*, **60**, (1999), 81.

[15] M. H. Stone, *Ann. Math.*, **33**, (1932), 643.

[16] A. Bohm and H. Kaldass, *Rigged Hilbert Space, Resonances and Time Asymmetric Quantum Mechanics*, arXiv:quant-ph/9909081; ed. S. Albeverio et al., *Mathematical Physics and Stochastic Analysis*,(World Scientific Publication, New Jersey, 2000), p.99 .

[17] E. B. Norman, S. B. Gazes, S. G. Crane, and D. A. Bennett, *Phys. Rev. Lett.* **60**, (1988), 2246.

[18] A. L. A. Khalfin, *Soviet Phys. Dokl.*, **2**, (1958), 340.

[19] G. Gamow, *Z. Phys.*, **51**, (1928), 204,; see also J. J. Thompson, *Proc. London Math. Soc.* **15**, (1884), 197.

[20] P. A. M. Dirac, *The Principles of Quantum Mechanics*, (Clarendon, Oxford, 1958), Ch.2.

[21] L. Schwartz, *Thèorie des Distributions*, (Hermann, Paris, 1950).

[22] I. M. Gelfand and N. Ya. Vilenkin, *Generalized Functions Vol 4* (Academic Press, New York, 1964) p.108.

[23] A. Bohm, *J. Math. Phys.*, **22**, (1981), 2813;
A. Bohm, S. Maxson, M. Loewe and M. Gadella, *Physica A*, **236**, (1997), 485.

[24] A. Bohm, H. Kaldass, and P. Patuleau, *International Journal of Theoretical Physics*, **38**, (1999), 115.

[25] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, vol.1, (Academic Press, Inc., London 1980) p256.

[26] A. Bohm, *Rigged Hilbert Spaces*, (Lecture Notes in ICTP, Trieste, 1964).

[27] S. Wightman, *PCT, Spin and Statistics, and All That*, (Princeton University Press 1980).

[28] S. Wickramasekara, A. Bohm, *J. Phys. A*, **35**, (2002), 807.

[29] M. Gell-Mann and J. B. Hartle, *in Physical Origins of Time Asymmetry*, ed. J. J. Halliwell et.al. (Cambridge University, Cambridge, 1994); M. Gell-Mann and J. B. Hartle, University of California at Santa Barbara, Report No. UCSBTH-95-28 (1995), *arXiv:gr-qc/9509054* and references thereof.

[30] R. Peierls, *Proc. R. Soc. London Ser. A* **166**, 277 (1938); R. Peierls, ed. E. M. Bellamy et.al. *in Proceedings of the 1954 Glasgow conference on Nuclear Meson Physics*, (Pergamon Press, New York 1995).

[31] A. Bohm and M. Gadella, *Dirac Kets, Gamow vectors, and Gel'fand Triplets, Lecture Notes in Physics 348*, (Springer-Verlag, Berlin, 1989).

[32] R. G. Newton, *Scattering theory of Waves and Particles*, 2. ed., (Springer Verlag, Berlin, 1982), Ch.7.

[33] P. Koosis, *Introduction to H_p spaces*, (Cambridge University of Press, 1998), p. 130.

[34] A. Bohm, H. Kaldass, S. Wickramasekara, Fortschr. Phys., **51**, No 6, 569 (2003); *Fortschr. Phys.*, **51**, (2003), 604.

[35] C. van Winter, *J. Math. Annal. Appl.*, **47**, (1974), 633.

[36] I. Antoniou, Z. Suchanecki, S. Tasaki, *Generalized Functions, Operator Theory and Dynamical Systems*, ed. I. Antoniou and G. Lumer, vol. 399, (Chapman and Hall Research Notes in Mathematics, 1999), p. 130.

[37] M. Gadella, *Int. J. Theor. Phys.*, **36**, (1997), 2271.

[38] A. Bohm, H. Kaldass, S. Komy, and W. P. Schleich, *Time Asymmetry in Quantum Physics II*, arXiv: quant-ph/071031163.