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Exotic localized structures based on the symmetrical lucas function of the (2+1)-dimensional generalized Nizhnik-Novikov-Veselov system

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Abstract

Based on the Lucas Riccati method and a linear variable separation method, new variable separation solutions of the (2+1)-dimensional generalized Nizhnik-Novikov-Veselov system are derived. Then, we give a positive answer for the following question: Are there any localized excitations derived by the use of another functions? For this purpose, some attention will be paid to dromion, peakon, multi dromion-solitoff excitations, regular fractal dromions, stochastic fractal dromion structure and combined structures including bell-like compactons, peakon-like compactons and compacton-like semi-foldons based on the golden main. With the help of the modified Weierstrass function, we discuss the stochastic fractal dromion structure both analytically and graphically. Finally, we conclude the paper and give some features and comments.

Key Words: Lucas functions, localized excitations, variable separation solutions, generalized Nizhnik-Novikov-Veselov system

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1. Introduction

Nonlinear wave phenomena play an important role in the physical world and have been challenging topic of research for both mathematicians and physicists. Many phenomena in physics and other fields are described by nonlinear partial differential equations (NLPDEs). When we want to understand the physical mechanism of phenomena in nature, described by NLPDEs, exact solutions for NLPDEs have to be explored. There exists an extensive literature dealing with NLPDEs, in which exact solitary wave, kink wave and periodic wave solutions are discussed. Many powerful methods have been proposed by mathematicians and physicists to obtain special solutions of NLPDEs, including the inverse scattering method, Bäcklund and Darboux transformations [1–4], Hirota bi-linear method [5], homogeneous balance method [6], Jacobi elliptic function method [7, 8], tanh-function method [9, 10], extended tanh-function method [11–14], modified extended tanh-function method [15–17], F-expansion method, and so on [18–29].

There is well-known fact that two mathematical constants of Nature, the values π and e, play a great role in mathematics and physics. Their importance lie in that they "generate" the main classes of so-called "elementary functions": sine, cosine (the number π), exponential, logarithmic and hyperbolic functions (the number e). However, there is the one more mathematical constant playing a great role in modeling of processes in living nature termed the Golden Section, Golden Proportion, Golden Ratio, Golden Mean [30–36]. However, we should certify that a role of this mathematical constant is sometimes undeservedly humiliated in modern mathematics and mathematical education. There is the well-known fact that the basic symbols of esoteric (pentagram, pentagonal star, platonic solids, etc.) are connected to the Golden Section closely. Moreover, the "materialistic" science together with it's "materialistic" education had decided to "throw out" the Golden Section. However, in modern science, an attitude towards the Golden Section and connected to its Fibonacci and Lucas numbers is changing very quickly.

The outstanding discoveries of modern science based on the Golden Section have a revolutionary importance for development of modern science. These are enough convincing confirmation of the fact that human science approaches that of uncovering one of the most complicated scientific notions, namely, the notion of Harmony. Harmony is opposed to Chaos, hence meaning the organization of the Universe. In Euclid's The Elements we find a geometric problem called "the problem of division of a line segment in the extreme and middle ratio". This problem is often called the golden section problem [32–36]. Solution of the golden section problem reduces to the algebraic equation $x^2 = x + 1$, to which there are two roots. The positive root is $\alpha = \frac{1+\sqrt{5}}{2}$, known as the golden proportion, golden mean, or golden ratio.

El Naschie's works [32–36] develop the Golden Mean applications into modern physics. In [36], devoted to the role of the Golden Mean in quantum physics El Naschie concludes: "In our opinion it is very worthwhile enterprise to follow the idea of Cantorian space-time with all its mathematical and physical ramifications. The final version may well be a synthesis between the results of quantum topology, quantum geometry and may be also Rossler's endorphysics which like Nottale's latest work makes extensive use of the ideas of Nelson's stochastic mechanism". Thus, in the Shechtman's, Butusov's, Mauldin and Williams', El Naschie's, Vladimirov's works, the Golden Section occupied a firm place in modern physics and it is impossible to imagine the future progress in physical researches without the Golden Section.

In our present paper, we review symmetrical Lucas functions [25–27] and we find new solutions of the Riccati equation by using these functions. Also, we devise an algorithm called Lucas Riccati method to obtain new exact solutions of NLPDEs.

For a given NLPDE with independent variables $x = (x_0 = t, x_1, x_2, x_3, ..., x_s)$ and dependent variable u,

$$P(u,\ u_t,\ u_{x_i},\ u_{x_i x_j}, ...) = 0, \tag{1}$$

where P is in general a polynomial function of its argument and the subscripts denote the partial derivatives, in order to derive some new solutions with certain arbitrary functions, we assume that its solutions in the form

$$u(x) = \sum_{i=0}^{n} a_i(x)\, F^i(\varphi(x)), \tag{2}$$

with

$$F' = A + BF^2, \quad F' = \frac{d}{d\phi}F, \tag{3}$$

where $x = (x_0 = t, x_1, x_2, x_3, ..., x_n)$ and A, B are constants and the prime denotes differentiation with respect to $\varphi \equiv \varphi(x)$. To determine u explicitly, one may take the following steps: First, similar to the usual mapping approach, determine n by balancing the highest non-linear term(s) and the highest-order partial term(s) in the given NLPDE. Second, substituting equation (2) and equation (3) into the given NLPDE and collecting coefficients of polynomials of F, then eliminating each coefficient to derive a set of partial differential equations of a_i $(i = 0, 1, 2, \ldots, n)$ and φ. Third, solving the system of partial differential equations to obtain a_i and φ. Substituting these results into equation (1), then a general formula of solutions of equation (1) can be obtained. Choose properly A and B in ODE equation (3) such that the corresponding solution $F(\varphi)$ is one of the symmetrical Lucas function given bellow. Some definitions and properties of the symmetrical Lucas function are given in appendix A.

Case 1: If $A = \ln\alpha$ and $B = -\ln\alpha$, then equation (3) possesses solutions

$$tLs(\phi), \quad cotLs(\phi).$$

Case 2: If $A = \frac{\ln\alpha}{2}$ and $B = -\frac{\ln\alpha}{2}$, then equation (3) possesses the solution

$$\frac{tLs(\varphi)}{1 \pm secLs(\varphi)}.$$

Case 3: If $A = \ln\alpha$ and $B = -4\ln\alpha$, then equation (3) possesses the solution

$$\frac{tLs(\varphi)}{1 \pm tLs^2(\varphi)}.$$

Case 4: If $A = 0$ then equation (3) possesses the solution $-\frac{1}{B\,F(\varphi)}$.

In section 2, we apply the Lucas Riccati method to obtain new localized excitations. Also in section 3, we pay attention to dromion, peakon, multi dromion-solitoff excitations, regular fractal dromions, stochastic fractal dromion structure and combined structures including bell-like compactons, peakon-like compactons and compacton-like semi-foldons based on the golden mean and the symmetrical hyperbolic and triangular Lucas functions. Finally, we give conclusions and comments.

2. New variable separation solutions of the (2+1)-dimensional generalized Nizhnik-Novikov-Vrselov system

We consider here the (2+1)-dimensional generalized Nizhnik-Novikov-Veselov (GNNV) system

$$\begin{aligned} &u_t + \rho\, u_{xxx} + \beta\, u_{yyy} + \gamma\, u_x + \delta\, u_y - 3\alpha(u\,v)_x - 3\beta(u\,w)_y = 0, \\ &v_y = u_x, \qquad w_x = u_y, \end{aligned} \tag{4}$$

where $\rho, \beta, \gamma, \alpha$ and δ are arbitrary constants. For $\gamma = \delta = 0$, the GNNV system degenerates to the usual (2+1)-dimensional NNV system, which is an isotropic Lax extension of the classical (1+1)-dimensional shallow water-wave KdV model. For $\rho = 1,\ \beta = \gamma = \delta = 0$ in system (4), we get the symmetric NNV equation, which may be considered as a model for an incompressible fluid. Some types of the soliton solutions of the GNNV equation have been studied by many authors. For instance, Boiti et al. [37] solved the GNNV equation via the inverse scattering transformation. Zhang obtained many exact solution of this system based on an extended homogeneous balance approach [38]. In [39], discussed new types of interactions between the multi-valued and the single-valued solitons of system (4). It is worth mentioning that this system has been widely applied in many branches of physics, such as plasma physics, fluid dynamics, nonlinear optics, etc. So, a good understanding of more solutions of the (2+1)-dimensional GNNV system (4) is very helpful.

Now we apply the Lucas Riccati method to system (4). Balancing the highest order derivative term with the nonlinear term in system (4), we have the following ansätz:

$$\begin{aligned}
u(x,y,t) &= a_0(x,y,t) + a_1(x,y,t)F(\varphi(x,y,t)) + a_2(x,y,t)F^2(\varphi(x,y,t)),\\
v(x,y,t) &= b_0(x,y,t) + b_1(x,y,t)F(\varphi(x,y,t)) + b_2(x,y,t)F^2(\varphi(x,y,t)),\\
w(x,y,t) &= c_0(x,y,t) + c_1(x,y,t)F(\varphi(x,y,t)) + c_2(x,y,t)F^2(\varphi(x,y,t)),
\end{aligned} \tag{5}$$

where $a_0(x,y,t) \equiv a_0,\ a_1(x,y,t) \equiv a_1,\ a_2(x,y,t) \equiv a_2,\ b_0(x,y,t) \equiv b_0,\ b_1(x,y,t) \equiv b_1,\ b_2(x,y,t) \equiv b_2, c_0(x,y,t) \equiv c_0,\ c_1(x,y,t) \equiv c_1,\ c_2(x,y,t) \equiv c_2$ and $\varphi(x,y,t) \equiv \varphi$ are functions to be determined.

Substituting equation (5) with equation (3) into equation (4), and equating each of the coefficients of $F(\varphi)$ to zero, we obtain a system of PDEs. Solving this system with the help of Maple, and selecting the variable separation ansätz $\varphi(x,y,t) = f(x,t) + g(y,t)$, we obtain the solution:

$$\begin{aligned}
&a_0 = AB\,f_x\,g_y, && a_1 = 0, && a_2 = B^2 f_x\,g_y,\\
&b_0 = \tfrac{\rho f_{xxx} + 2\rho ABf_x^3 + f_t + \gamma f_x}{3\rho f_x}, && b_1 = Bf_{xx}, && b_2 = B^2 f_x^2,\\
&c_0 = \tfrac{\beta g_{yyy} + 2\beta ABg_y^3 + g_t + \delta g_y}{3\beta g_y}, && c_1 = Bg_{yy}, && c_2 = B^2 g_y^2,
\end{aligned} \tag{6}$$

where f and g are two arbitrary functions of $\{x,\ t\}$ and $\{y,\ t\}$, respectively.

Now, based on the solutions of equation (3), one can obtain new types of localized excitations of the (2+1)-dimensional GNNV system. We obtain the general formulae of the solutions of the (2+1)-dimensional GNNV system

$$u = AB\,f_x\,g_y + B^2 f_x\,g_y\,F^2(f+g), \tag{7}$$

$$v = \frac{\rho f_{xxx} + 2\rho ABf_x^3 + f_t + \gamma f_x}{3\rho f_x} + Bf_{xx}F(f+g) + B^2 f_x^2 F^2(f+g), \tag{8}$$

$$w = \frac{\beta g_{yyy} + 2\beta ABg_y^3 + g_t + \delta g_y}{3\beta g_y} + Bg_{yy}F(f+g) + B^2 g_y^2 F^2(f+g), \tag{9}$$

By selecting the special values of the A, B and the corresponding function F, we have the following solutions of (2+1)-dimensional GNNV system:

$$u_1 = f_x\,g_y[\ln\alpha]^2(-1 + tLs^2(f+g)), \tag{10}$$

$$v_1 = \frac{\rho f_{xxx} - 2\rho f_x^3[\ln\alpha]^2 + f_t + \gamma f_x}{3\rho f_x} - f_{xx}\ln\alpha\,tLs(f+g) + [\ln\alpha]^2 f_x^2 tLs^2(f+g), \tag{11}$$

$$w_1 = \frac{\beta g_{yyy} - 2\beta[\ln\alpha]^2 g_y^3 + g_t + \delta g_y}{3\beta g_y} - g_{yy}\ln\alpha\, tLs(f+g) + [\ln\alpha]^2 g_y^2 tLs^2(f+g), \tag{12}$$

$$u_2 = \; f_x\, g_y[\ln\alpha]^2(-1 + cotLs^2(f+g)), \tag{13}$$

$$v_2 = \frac{\rho f_{xxx} - 2\rho f_x^3[\ln\alpha]^2 + f_t + \gamma f_x}{3\rho f_x} - f_{xx}\ln\alpha\, cotLs(f+g) + [\ln\alpha]^2 f_x^2 cotLs^2(f+g), \tag{14}$$

$$w_2 = \frac{\beta g_{yyy} - 2\beta g_y^3[\ln\alpha]^2 + g_t + \delta g_y}{3\beta g_y} - g_{yy}\ln\alpha\, cotLs(f+g) + [\ln\alpha]^2 g_y^2 cotLs^2(f+g), \tag{15}$$

$$u_3 = \; \frac{1}{4} f_x\, g_y[\ln\alpha]^2\left(-1 + \left[\frac{tLs(f+g)}{1 \;\pm\; secLs(f+g)}\right]^2\right), \tag{16}$$

$$v_3 = \frac{\rho f_{xxx} - \frac{1}{2}\rho f_x^3[\ln\alpha]^2 + f_t + \gamma f_x}{3\rho f_x} - \frac{1}{2} f_{xx}\ln\alpha\left[\frac{tLs(f+g)}{1 \;\pm\; secLs(f+g)}\right] + \frac{1}{4} f_x^2[\ln\alpha]^2\left[\frac{tLs(f+g)}{1 \;\pm\; secLs(f+g)}\right]^2, \tag{17}$$

$$w_3 = \frac{\beta g_{yyy} - \frac{1}{2}\beta g_y^3[\ln\alpha]^2 + g_t + \delta g_y}{3\beta g_y} - \frac{1}{2} g_{yy}\ln\alpha\left[\frac{tLs(f+g)}{1 \;\pm\; secLs(f+g)}\right] + \frac{1}{4} g_y^2[\ln\alpha]^2\left[\frac{tLs(\xi)}{1 \;\pm\; secLs(\xi)}\right]^2, \tag{18}$$

$$u_4 = \; 4\, f_x\, g_y[\ln\alpha]^2\left(-1 + 4\left[\frac{tLs(f+g)}{1 \;\pm\; tLs^2(f+g)}\right]^2\right), \tag{19}$$

$$v_4 = \frac{\rho f_{xxx} - 8\rho f_x^3[\ln\alpha]^2 + f_t + \gamma f_x}{3\rho f_x} - 4 f_{xx}\ln\alpha\left[\frac{tLs(f+g)}{1 \;\pm\; tLs^2(f+g)}\right] + 16 f_x^2[\ln\alpha]^2\left[\frac{tLs(f+g)}{1 \;\pm\; tLs^2(f+g)}\right]^2, \tag{20}$$

$$w_4 = \frac{\beta g_{yyy} - 8\beta g_y^3[\ln\alpha]^2 + g_t + \delta g_y}{3\beta g_y} - 4 g_{yy}\ln\alpha\left[\frac{tLs(f+g)}{1 \;\pm\; tLs^2(f+g)}\right] + 16 g_y^2[\ln\alpha]^2\left[\frac{tLs(f+g)}{1 \;\pm\; tLs^2(f+g)}\right]^2, \tag{21}$$

$$u_5 = \; \frac{f_x\, g_y}{(f+g)^2}, \tag{22}$$

$$v_5 = \frac{\rho\, f_{xxx} + f_t + \gamma f_x}{3\rho f_x} - \frac{f_{xx}}{(f+g)} + \frac{f_x^2}{(f+g)^2}, \tag{23}$$

$$w_5 = \frac{\beta g_{yyy} + g_t + \delta g_y}{3\beta g_y} - \frac{g_{yy}}{(f+g)} + \frac{g_y^2}{(f+g)^2}, \tag{24}$$

where $f \equiv f(x,t)$ and $g \equiv g(y,t)$ are two arbitrary variable separation functions. The potential $U = \; u_5$ has the form

$$U = \; \frac{f_x\, g_y}{(f+g)^2}. \tag{25}$$

3. Novel localized structures of the (2+1)-dimensional GNNV system

All rich localized coherent structures, such as non-propagating solitons, dromions, peakons, compactons, foldons, instantons, ghostons, ring solitons, and the interactions between these solitons [40–52], can be derived by the quantity U expressed by equation (25) with the help of the hyperbolic and triangular functions. It is known that, for the (2+1)-dimensional integrable models, there are many more abundant localized structures than in (1+1)-dimensional case because some types of arbitrary functions can be included in the explicit solution expression [40–52]. Moreover, the periodic waves also have been studied by some authors. In this paper, we try to give an answer for the following question: Are there any localized excitations derived by the use of another functions? Fortunately, the answer is still positive due to some arbitrariness of the functions $f(x,t)$ and $g(y,t)$ in the potential U given by the equation (25). In order to answer this question, some attention will be paid to dromion, peakon, dromion lattice, multi dromion-solitoff excitations, regular fractal dromions, stochastic fractal dromion structure, combined structures, including bell-like compactons, peakon-like compactons and compacton-like semi-foldons based on the golden main and the symmetrical hyperbolic and triangular Lucas functions for the potential field U in (2+1) dimensions.

3.1. Dromion, peakon and dromion lattice excitations

According to solution U, we first discuss its dromion excitation which is one of the significant localized excitations localized exponentially in all directions are driven by multiple straight-line ghost solitons with some suitable dispersion relation. Also, multiple dromion solutions are driven by curved line and straight line solitons. When the simple selections of the functions $f(x,t)$ and $g(y,t)=g(y)$ are given to be

$$f(x,t)=1+\sum_{i=1}^{M} a_i\, tLs[k_i(x+c_i\, t)+x_{0i}], \quad g(y)=1+\sum_{j=1}^{N} b_j\, tLs(K_j y+y_{0j}), \tag{26}$$

where a_i, k_i, c_i, b_j, K_j, x_{0i}, y_{0j} are arbitrary constants, and M and N are integers. We can obtain a two-dromion excitation for the physical quantity U, as shown in Figure 1, and the parameter selections as

$$\begin{aligned} &M=2, \;\; N=k_1=k_2=-\tfrac{c_2}{2}=c_1=K_1=1, \;\; x_{01}=y_{01}=x_{02}=0, \\ &2a_1=a_2=2b_1=0.2 \end{aligned} \tag{27}$$

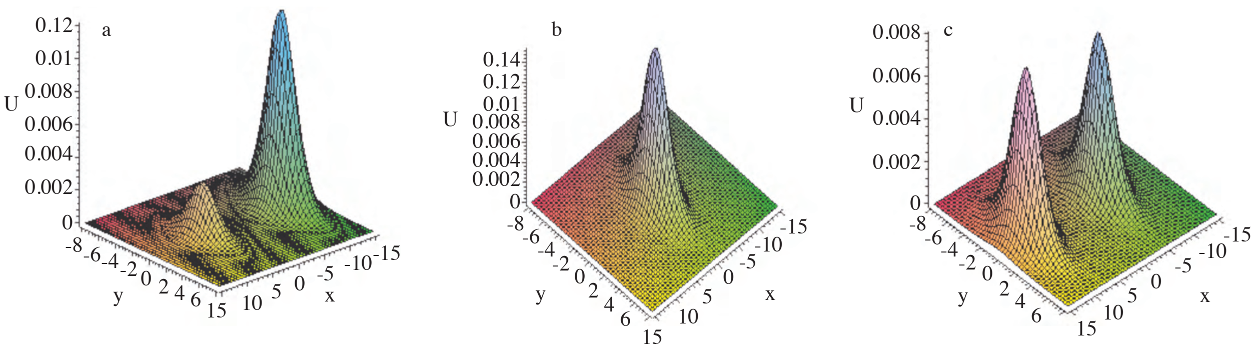

Figure 1. Time evolutional plots of an interaction between two travilling dromions for the potential U with functional forms (26) and (27); when (a) $t=-5$, (b) $t=0$, and (c) $t=5$.

It is well-known that the interactions of solitons in (1+1)-dimensional nonlinear models are usually considered elastic. That means there is no exchange of any physical quantities like the energy and the momentum among interacting solitons. That is, the amplitude, velocity and wave shape of a soliton do not undergo any change after the nonlinear interaction. But in higher dimensional non-linear models the interactions between solitary waves may be completely elastic or non-completely elastic. From Figure 1, we show that the interaction between a dromion-dromion is non-completely elastic since their shapes and amplitudes are not completely preserved after interaction. Similarly, based on the field U we can obtain some important weak localized excitations such as peakons (weak continuous solution) which is discontinuous at its crest $[u(x,t) = -k + c\,e^{(-|x-ct|)},\ \ k \to 0]$, was first found in the celebrated (1+1)-dimensional Camassa-Holm equation [53]

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

We find many (2+1)-dimensional nonlinear models also possess these soliton excitations [15–33]. We select $f(x,t)$ and $g(y)$ to be piecewise smooth functions

$$f(x,t) = 1 + \sum_{i=1}^{M} \begin{cases} f_i(x + c_i t) & x + c_i t \le 0, \\ -f_i(-x - c_i t) & x + c_i t > 0, \end{cases} \qquad g(y) = 1 + \sum_{i=1}^{N} \begin{cases} g_i(y) & y \le 0, \\ -g_i(-y) & y > 0, \end{cases} \tag{28}$$

where the functions $f_i(x + c_i t)$ and $g_i(y)$ are differentiable functions of the indicated arguments and possess boundary condition $f_i(\pm\infty) = C_{\pm i}, i=1,2,\ldots,M$ and $g_i(\pm\infty) = D_{\pm i}, i=1,2,\ldots,N$ with $C_{\pm i}$ and $D_{\pm i}$ being constants, even infinity. For instance, when choosing

$$f_1 = 0.1\alpha^{x+t}, \quad f_2 = 0.05\alpha^{x-2t} \quad g(y) = 0.1\alpha^{y}, \quad M = 2N = 2, \tag{29}$$

we can derive a propagating two-peakon excitation for the potential field U. The corresponding two-peakon excitation profile is depicted in Figure 2. Also, a simple example of peaked solitary waves with periodic behavior is depicted in Figure 3, in which the selection function is

$$f_1 = 0.1\alpha^{x+t}, \quad f_2 = 0.05\alpha^{x-2t} \quad g(y) = \alpha^{sTLs(y)}, \tag{30}$$

where $sTLs(y) = \frac{1}{i}(\alpha^{iy} - \alpha^{-iy}),\ (i = \sqrt{-1})$ is the symmetrical triangular Lucas sine function [33].

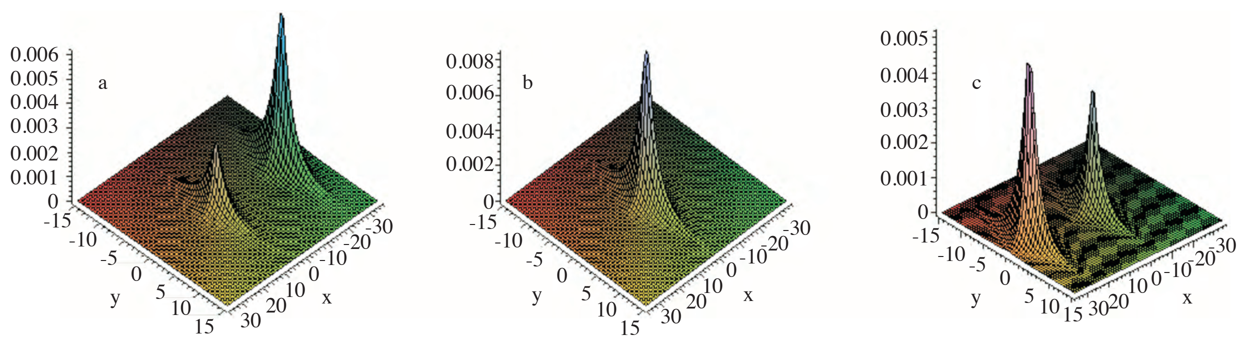

Figure 2. Time evolutional plots of the two peaked solitary waves for the potential U with functional forms (28) and (29); when (a) $t = -5$, (b) $t = 0$, and (c) $t = 5$.

From figure 2, we show that the interaction between a peakon-peakon is non-completely elastic since their shapes and amplitudes are not completely preserved after interaction. Also with the help of figure 3, one can easily say that the multi-peakon excitation possess periodic behavior.

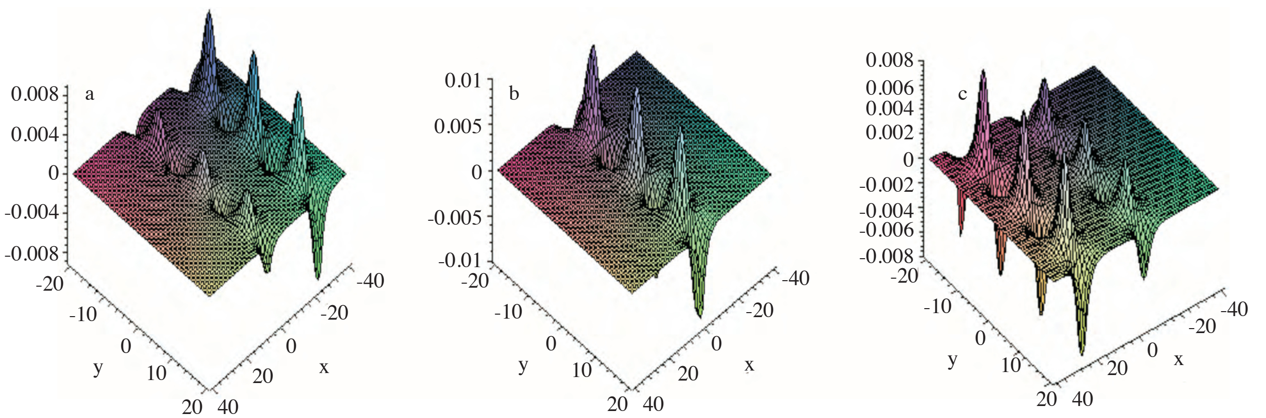

Figure 3. Time evolutional plots of the two peaked solitary waves with periodic behavior for the potential U with the functional forms (28) and (30); when (a) $t = -5$, (b) $t = 0$, and (c) $t = 5$.

3.2. Multi dromion-solitoff excitations

According to the potential U in the form of its multi dromion-solitoff excitations. That can be expressed by means of Lucas functions in the form

$$f(x,t) = \sum_{n=-N}^{N} 0.2\, secLs(x + 5n + t), \quad g(y) = \sum_{m=-M}^{M} 0.2\, secLs(y + 5m). \tag{31}$$

If $m = n = 2$, we can obtain a multi dromion-solitoff excitation for the physical quantity U depicted in Figure 4 with $t = 0$.

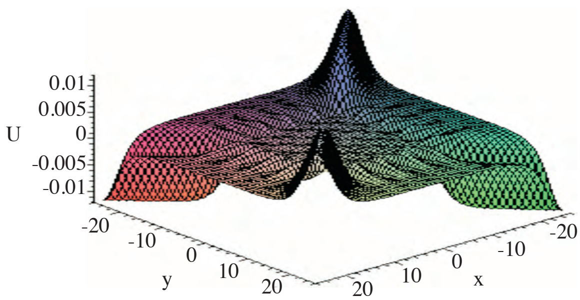

Figure 4. A plot of a special type of multi dromion-solitoff structure for the physical quantity U with the choice of relation (31) and $t = 0$.

For instance, when $f(x,t)$ and $g(y)$ are

$$\begin{aligned} f(x,t) &= \sum_{n=-N}^{N} [0.2\, secLs(x+5n+t) + 0.2\, secLs(x+5n-t)], \\ g(y) &= \sum_{m=-M}^{M} 0.2\, secLs(y+5m), \end{aligned} \tag{32}$$

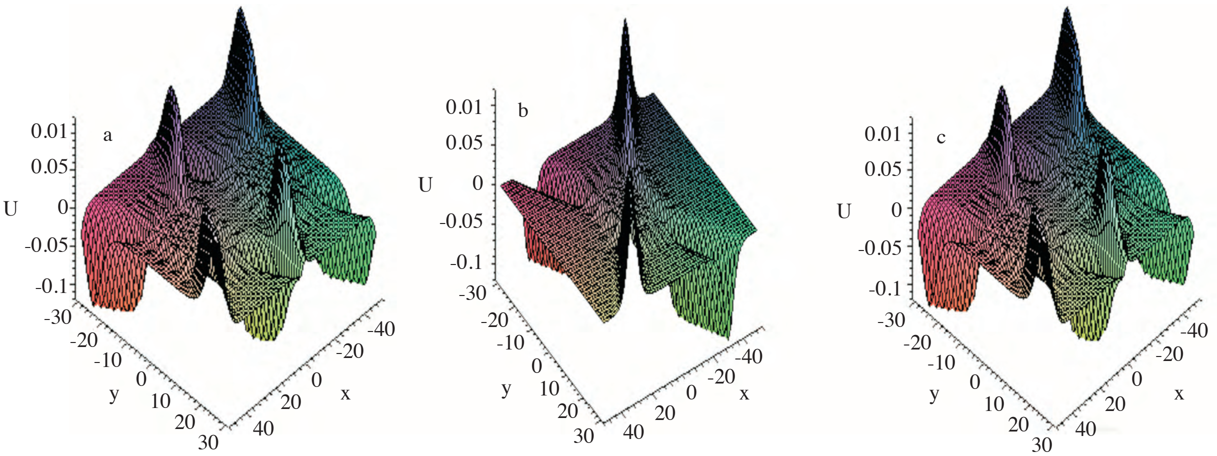

Figure 5. Time-evolution plots of an interaction between two multi dromion-solitoffs for the potential U with functional form (32); when (a) t = -25; (b) t = 0, and (c) t = 25.

We can obtain the interactions between two multi dromion-solitoffs. Figure 5 shows an evolutional profile corresponding to the physical quantity U. From Figure 5 and through detailed analysis, we find that the shapes, amplitudes, and velocities of the two multi dromion-solitoffs are completely conserved after their interactions. Consequently, the interactions between two multi dromion-solitoffs are completely elastic.

Now we focus our attention on the intriguing evolution of a dromion in a background wave for the potential field U. For instance, if we choose $f(x,t)$ and $g(y)$ as

$$\begin{aligned} f(x,t) &= 3 + 0.12\ tLs(0.5x - t) + 0.02\ sn(0.4x, 0.3), \\ g(y) &= 3 + 0.12\ tLs(0.5y) + 0.02\ sn(0.4y, 0.3), \end{aligned} \tag{33}$$

where, sn denotes the Jacobi sine function, we can obtain an evolutional profile of single-dromion in the background wave for the physical quantity U presented in Figure 6 at different times when (a) $t = -10$, (b) $t = 0$, and (c) $t = 10$. From Figure 6 and through detailed analysis, this figure shows the corresponding profile of the complex wave excitation presenting the propagation of a dromion moving on the determined double periodic wave background. However, its wave shape and wave velocity do not suffer any change, which is very close to many actual physical processes in the natural world.

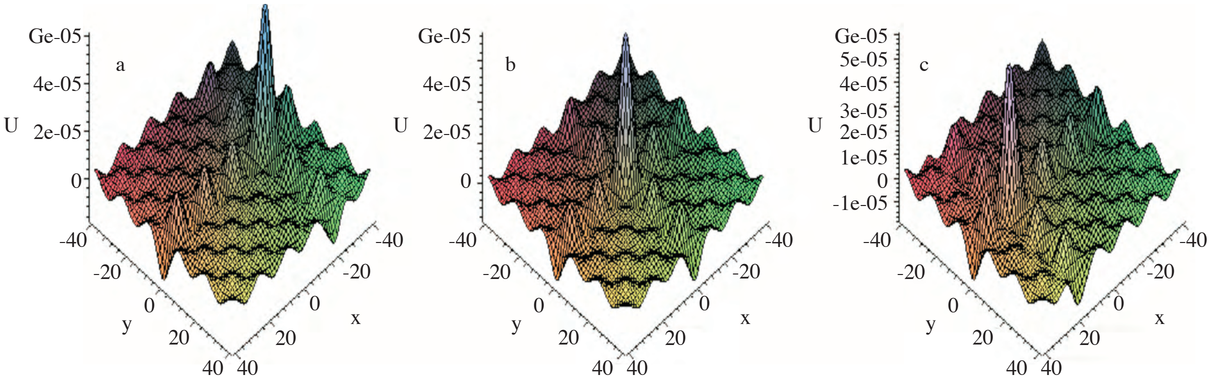

Figure 6. Time evolutional plots of the single dromion in the background wave for the potential U with selection (33), when (a) t = -10, (b) t = 0, and (c) t = 10.

3.3. Regular fractal dromions and stochastic fractal dromion structure

Recently, it has been found that many lower-dimensional piecewise smooth functions with fractal structure can be used to construct exact localized solutions of the higher-dimensional soliton system which also possesses fractal structures. If we appropriately select the arbitrary functions $f(x,t)$ and $g(y)$, we find that some special types of fractal dromions for the potential U can be revealed. For example, if we take

$$\begin{aligned} f(x,t) &= 1+\alpha^{|x-t|[cTLs(\ln(x-t)^2)+sTLs(\ln(x-t)^2)]^2}, \\ g(y) &= 1+\alpha^{|y|[cTLs(\ln(y)^2)+sTLs(\ln(y)^2)]^2}, \end{aligned} \tag{34}$$

where $cTLs(y)=\alpha^{iy}+\alpha^{-iy}$, $(i=\sqrt{-1})$ is the symmetrical triangular Lucas cosine function [33], then we can obtain a simple fractal dromion.

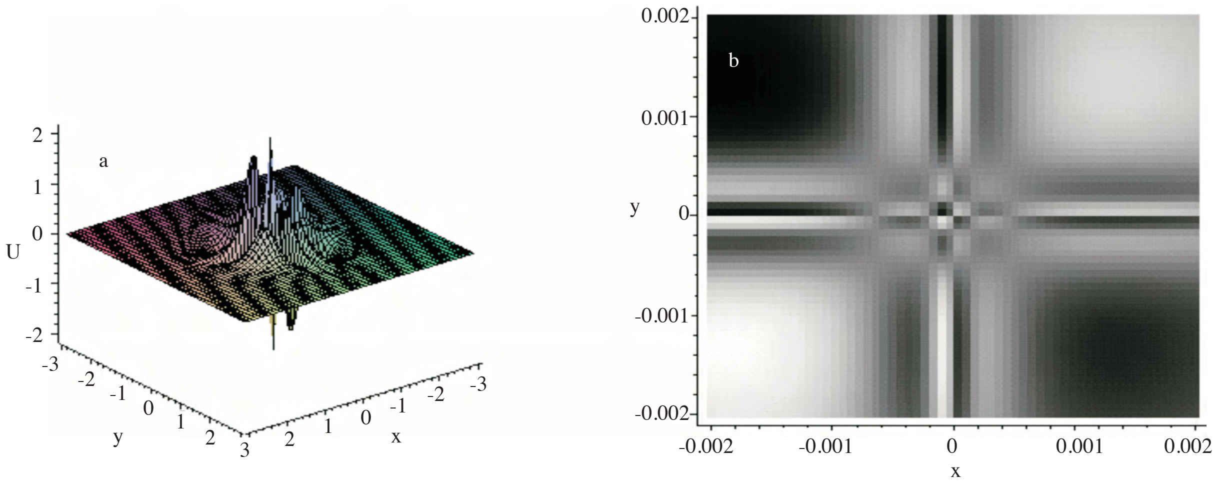

Figure 7. (a) Fractal dromion structure for the potential U with the choice if functional form (34), at t = 0. (b) Density of the fractal structure of the dromion in the region $\{x=$ [-0.002, 0.002], $y=$ [-0.002, 0.002]$\}$.

From Figure 7 and through detailed analysis, Figure 7(a) shows a plot of this special type of fractal dromion structure for the potential U given by equation (25) with the selection functions (34) for t = 0. Figure

7(b) shows the density of the fractal structure of the dromion in the region $\{x=$ [-0.002, 0.002]$, y=$ [-0.002, 0.002]$\}$. To observe the self-similar structure of the fractal dromion clearly, one may enlarge a small region near the centre of Figure 7(b). For instance, if we reduce the region of figure 7(b) to $\{x=$ [-0.0002, 0.0002], $y=$ [-0.0002, 0.0002]$\}$, $\{$ $x=$ [-0.00001, 0.00001], $y=$ [-0.00001, 0.00001]$\}$, and so on, we find a totally similar structure to that presented in Figure 7(b).

In addition to the regular fractal dromion, the lower-dimensional stochastic fractal functions may also be used to construct high-dimensional irregular stochastic or chaotic fractal localized excitations. By the help of the symmetrical triangular Lucas sine function [33], we define the modified Weierstrass function $\wp$ as

$$\wp \equiv \wp(\xi) = \sum_{j=0}^{M} \lambda^{(q-2)j}\, sTLs(\lambda^j \xi), \quad M \to \infty, \tag{35}$$

where $\lambda > 1,\ q < 2$ and the independent variable ξ may be a suitable function of $\{x, t\}$ or $\{y\}$, say $\xi = x + v\,t$ and $\xi = y$ in the functions $f(x,t)$ and $g(y)$, respectively, for selections

$$f(x,t) = \wp(x + v\,t) + (x + v\,t)^2 + 1000, \qquad g(y) = \wp(y) + y^2 + 1000, \tag{36}$$

and

$$\begin{aligned} f(x,t) &= 0.02\,\wp(x + v\,t)\ \ tTLs(3[x + v\,t] - 20) + 1, \\ g(y) &= 0.04\,tTLs(y) + 0.12\,tTLs(y+8) + 0.08\,tTLs(2\,y - 16) + 1, \end{aligned} \tag{37}$$

and

$$\begin{aligned} f(x,t) &= 0.02\,\wp(x + v\,t)\ \ tTLs(3[x + v\,t] - 20) + 1, \\ g(y) &= 0.02\,\wp(y)\ \ tTLs(3\,y - 20) + 1. \end{aligned} \tag{38}$$

If the modified Weierstrass function is included in the field U, then we can derive some stochastic fractal patterns. Figure 8 shows three plots of typical stochastic fractal patterns for the potential function U given by equation (25) under the function selections equations (35)–(38) with the parameters $\lambda = 1.5,\ q = 1$ at $t = 0$. From Figure 8, one can find that the amplitude of the localized or non-localized patterns is randomly changed.

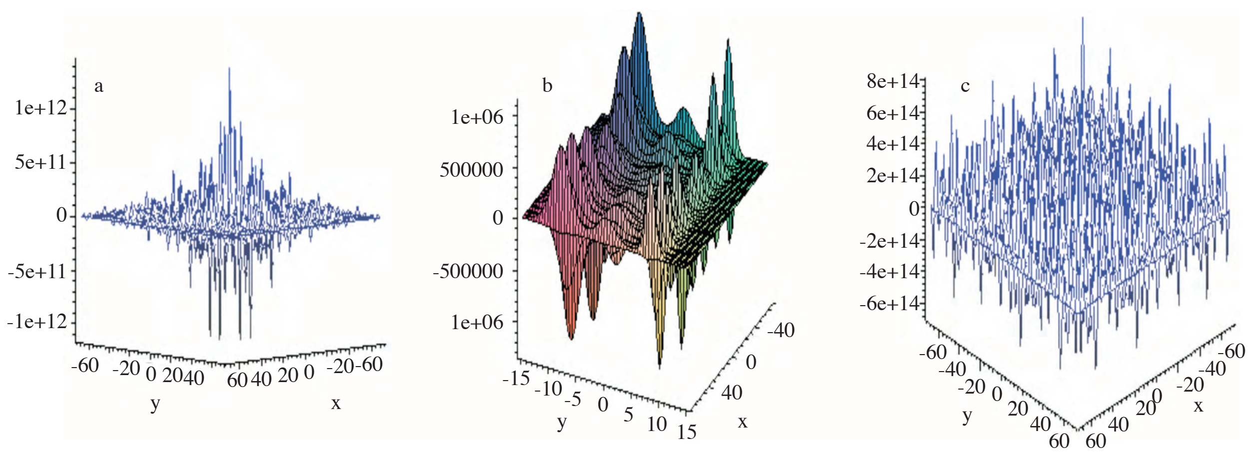

Figure 8. Stochastic fractal dromion structure for the potential U : (a) with selections (35) and (36), (b) with selections (35) and (37), (c) with selections (35) and (38) with $\lambda = 1.5,\ q = 1$ at $t = 0$

3.4. Special localized structures

Based on the physical quantity (25), multi-valued localized structures can be discussed, when the function $g(y)$ is a single-valued function and $f(x,t)$ is selected via the relations

$$f_x = \sum_{j=1}^{M} F_j(\xi + \omega_j t), \qquad x = \xi + \sum_{j=1}^{M} X_j(\xi + \omega_j t), \tag{39}$$

where ω_j are all arbitrary constants and F_j, X_j are all localized excitations with properties $F_j\ (\pm\infty) = 0$, $X_j\ (\pm\ \infty)$=0.

From the second equation of equation (39), one knows that ξ may be a multi-valued function in some suitable regions of x by selecting the functions X_j appropriately. Therefore, the function f_x, which is obviously an interaction solution of M localized excitations since the property $\xi|_{x\to\infty} \to \infty$ may be a multi-valued function of x in these area though it is a single valued functions of ξ. Actually, most of the known multi-loop solutions are the special situations of equation (39). Now, some attention will be paid to combined structures, including bell-like compactons, peakon-like compactons and compacton-like semi-foldons based on the golden main and the symmetrical hyperbolic Lucas functions:

$$f_x = 0.5\, secLs^2(\xi), \qquad x = \xi - \frac{A}{\ln\alpha} tLs(\xi), \tag{40}$$

$$g(y) = 8 + \begin{cases} 0, & y \le -\frac{\pi}{2}, \\ \sin y + 1, & -\frac{\pi}{2} < y < \frac{\pi}{2} \\ 2 & y > \frac{\pi}{2}, \end{cases} , \tag{41}$$

where A is a characteristic parameter, determining the localized structure. Figure 9 describe three localized structures, i.e. bell-like compacton, peakon-like compacton and compacton-like semi-foldon with A =0.05, 0.95, 1.5, respectively. They localize as a compacton in the y-direction and a bell-like soliton, peakon and loop soliton in the x-direction, respectively.

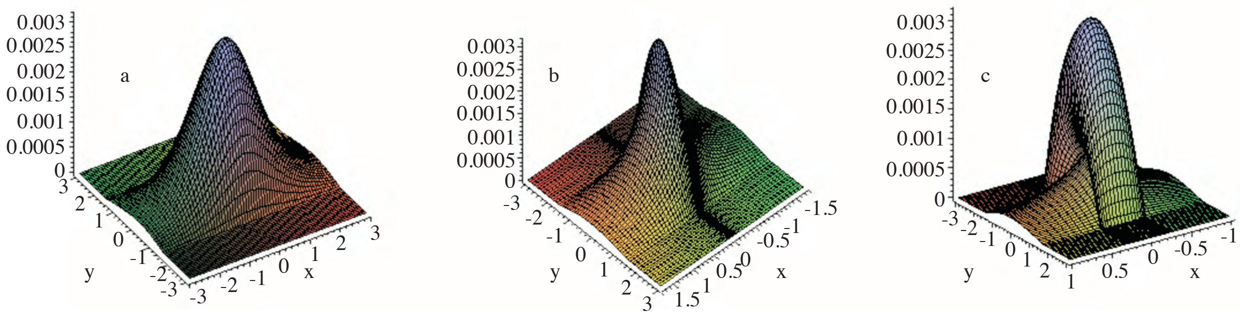

Figure 9. (a) Bell-like compacton structure for the potential U with selections (40) and (41) when $A = 0.05$, (b) Peakon-like compacton structure for the potential U with selections (40) and (41) when $A = 0.95$, (c) Compacton-like semi-foldon structure for the potential U with selections (40) and (41) when $A = 1.5$.

4. Summary and discussion

In conclusion, the Lucas Riccati method is applied to obtain variable separation solutions of the (2+1)-dimensional GNNV system. With the help of quantity (25), some localized excitations, such as dromion, peakon, multi dromion-solitoff excitations, regular fractal dromions, combined structures, including bell-like compactons, peakon-like compactons and compacton-like semi-foldons based on the golden main and the symmetrical hyperbolic and triangular Lucas functions. With the help of the modified Weierstrass function, we discuss the stochastic fractal dromion structure both analytically and graphically. We think that all localized solutions of the (2+1)-dimensional GNNV system can not be constructed in general for all localized solutions of the system. We hope that in future experimental studies these localized excitations obtained here can be realized in some fields. Actually, our present short paper is merely a beginning work; more application to other nonlinear physical systems should be conducted and deserve further investigation. In our future work, on the one hand, we devote to generalizing this method to other (2+1)-dimensional nonlinear systems such as the ANNV system, BKK system, KdV system, Boiti-Leon-Pempinelle system etc. On the other hand, we will look for more interesting localized excitations.

5. Appendex

Stakhov and Rozin in [30] introduced a new class of hyperbolic functions that unite the characteristics of the classical hyperbolic functions and the recurring Fibonacci and Lucas series. The hyperbolic Fibonacci and Lucas functions, which are an extension of Binet's formulas for the Fibonacci and Lucas numbers in continuous domain, transform the Fibonacci numbers theory into "continuous" theory because every identity for the hyperbolic Fibonacci and Lucas functions has its discrete analogy in the framework of the Fibonacci and Lucas numbers. Taking into consideration a great role played by the hyperbolic functions in geometry and physics, ("Lobatchevski's hyperbolic geometry", "Four-dimensional Minkowski's world", etc.), it is possible to expect that the new theory of the hyperbolic functions will bring to new results and interpretations on mathematics, biology, physics, and cosmology. In particular, the result is vital for understanding the relation between transfinitness *i.e.* fractal geometry and the hyperbolic symmetrical character of the disintegration of the neural vacuum, as pointed out by El Naschie.

The definition and properties of the symmetrical Lucas functions The symmetrical Lucas sine function (sLs), the symmetrical Lucas cosine function (cLs) and the symmetrical Lucas tangent function (tLs) are defined [30, 31] as

$$sLs(x) = \alpha^x - \alpha^{-x}, \quad cLs(x) = \alpha^x + \alpha^{-x}, \quad tLs(x) = \frac{\alpha^x - \alpha^{-x}}{\alpha^x + \alpha^{-x}}. \tag{I}$$

They are introduced to consider so-called symmetrical representation of the hyperbolic Lucas functions and they may present a certain interest for modern theoretical physics taking into consideration a great role played by the Golden Section, Golden Proportion, Golden ratio, Golden Mean in modern physical researches [30, 31]. The symmetrical Lucas cotangent function (cotLs) is $cotLs(x) = \frac{1}{tLs(x)}$, the symmetrical Lucas secant function (secLs) is $secLs(x) = \frac{1}{cLs(x)}$, the symmetrical Lucas cosecant function (cscLs) is $cscLs(x) = \frac{1}{sLs(x)}$. These functions satisfy the following relations [30–31]

$$cLs^2(x) - sLs^2(x) = 4, \qquad 1 - tLs^2(x) = 4secLs^2(x)$$
$$cotLs^2(x) - 1 = 4cscLs^2(x) \tag{II}$$

Also, from the above definition, we give the derivative formulas of the symmetrical Lucas functions as follows:

$$\frac{d\,sLs(x)}{dx} = cLs(x)\ln\alpha, \quad \frac{d\,cLs(x)}{dx} = sLs(x)\ln\alpha\,, \quad \frac{d\,tLs(x)}{dx} = 4secLs^2(x)\ln\alpha. \tag{III}$$

The above symmetrical hyperbolic Lucas functions are connected with the classical hyperbolic functions by the following simple correlations:

$$sLs(x) = 2\sinh(x\ln\alpha), \quad cLs(x) = 2\cosh(x\ln\alpha), \quad tLs(x) = \tanh(x\ln\alpha). \tag{IV}$$

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