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Exactly solvable potentials from decemvirate power potential

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Abstract

A transformation method is presented which consist of a coordinate transformation and a functional transformation that allow generation of exact analytic bound state solutions of the Schrodinger Green's function equation. The method proposed here, which involves the generation of an exact analytic solution from solvable decemvirate power potential within the framework of Green's functions technique, makes it possible to generate a number of solved quantum systems for original quantum systems with multi-term potentials. The generated quantum systems are in general energy dependent with a single normalized eigenstate. A method has been devised to convert a subset of the generated quantum systems with energy-dependent potentials to a single normal system with an energy-independent potential.

Key Words: Schrodinger Green's function equation, bound state solution, extended transformation method, decemvirate power potential

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1. Introduction

The exact solutions to the fundamental dynamical equation play an important role in physics. Some approximation methods are frequently used [1–9] to arrive at an exact analytic solution (EAS); and it is necessitated as the potential that governs a given quantum system (QS), more often than not, does not facilitate EAS of the Green's function (GF). For analytical accuracy within the framework of approximation, it is necessary that an exactly solvable potential (ESP) differ as little as possible from the given potential. Recently, the study of higher order anharmonic potentials has been much more desirable in different branches of physics and mathematics. In the context of the GF technique, Steiner [10, 11], in the course of his work on radial path integrals, obtained a connection between two 3-dimensional problems. His analysis involved a nonlinear space time transformation of the radial path integral, with a path-dependent change-of-time variable. S. S. Vasan, M. Seetharaman and K. Raghunathan [12] demonstrate a general connection between GFs for different potential, directly from the differential equation satisfied by the GF, and they connect certain 1-D problems with 3-D problems. However, success in lower dimensional cases (termed potentials) has been limited, as only a few states can be found analytically.

In this work we made use of a simple and compact mapping procedure called the Extended transformation (ET) method [13–20], to generate a new ESP from an already-known non-relativistic EAS of QS within the framework of the GF technique. It seems that, within the limit of anharmonic potentials, not much work has been carried out on the decemvirate potential, except for a simpler study [21] involving a lower dimensional scheme. In the present work we have attempted to generate the D-dimensional Schrodinger GF equation from decemvirate power potential. Our main objective is to generate an ESP and show their hierarchal connections, since ESPs facilitate physical explicabilities.

2. Formalism

The extended transformation method (ET) has been applied to generate new exactly solved potentials (ESPs) from an already known ESP. Let $V_A(r)$ be an exactly solved multi-term quantum mechanical central potential in D_A dimensional space

$$V_A(r) = a_1 r^2 + a_2 r^4 + \cdots = \sum_1^s a_s r^{2s}. \tag{1}$$

The radial part of Schrodinger GF equation [16] for D_A dimensional Euclidean space, henceforth called A-quantum system (A-QS) (with $\hbar = 1 = 2m$), is

$$\left(\partial_r^2 + \frac{D_A - 1}{r}\partial_r + E_n^A - V_A(r) - \frac{l_A(l_A + D_A - 2)}{r^2}\right) G_A\left(r, r_0, E_n^A, V_A(r), C_A\right) = \frac{\delta(r - r_0)}{r_0^{D_A - 1}}, \tag{2}$$

where r and r_0 are the dimensionless spatial coordinates, and C_A is the characteristic constant of exactly solved quantum potential in which transformation method is applied. The corresponding integral equation is $\psi_A(r) = \int G_A\left(r, r_0, E_n^A, V_A(r), C_A\right).\left(E_n^A - V_A(r)\right).\psi_A(r_0).r_0^{D_A-1} dr_0$, where the GF and energy eigenvalues E_n^A are known for the given $V_A(r)$. The completeness of the set of energy eigenfunctions allows us to have eigenfunction expansion of GF is

$$G_A\left(r, r_0, E_n^A, V_A(r), C_A\right) = \sum_{n=0}^{\infty} \frac{\psi_A^{(n)}(r)\,\psi_A^{*(n)}(r_0)}{E - E_n^A - i\in}, \tag{3}$$

from which we read off the analytic form of the wave function of the solved quantum system. Applying ET [16] to equation (2), which comprises the co-ordinate transformation $r \to g(r), r_0 \to g(r_0)$, followed by a functional transformation

$$G_B\left(r, r_0, E_N^B, V_B(r), C_B\right) = f_B^{-1}(r)\, G_A\left(g(r), g(r_0), g'^2 E_n^A, g'^2 V_A(g(r)), C_A\right) f_B^{-1}(r_0), \tag{4}$$

the resulting equation is found to be the same form as (2), but with new potential, energy eigenvalues and angular momentum quantum number. Consequently equation (2) of A-QS becomes

$$\Bigg[\partial_r^2 + \left(\frac{d}{dr}\ln\frac{f_B^2(r)\, g^{D_A-1}(r)}{g'(r)}\right)\partial_r + \left(\frac{d}{dr}\ln f_B(r)\right)\left(\frac{d}{dr}\ln\frac{f_B' g^{D_A-1}(r)}{g'(r)}\right) + g'^2\left(E_n^A - V_A(r)\right.$$

$$-\frac{l_A\left(l_A+D_A-2\right)}{r^2}\Bigg)\Bigg] G_B\left(r,r_0,E_N^B,V_B\left(r\right),C_B\right)=g'^2 f_B^{-1}\left(r\right)\frac{\delta\left(g\left(r\right)-g\left(r_0\right)\right)}{g_0^{D_A-1}\left(r_0\right)}f_B^{-1}\left(r_0\right). \quad (5)$$

Here, $g(r)$ and $g(r_0)$ are the transformation functions, which are continuous and at least three times differentiable function and $f_B\left(r\right)$ and $f_B\left(r_0\right)$ are the r dependent modulated amplitude to be determined. C_B is the characteristic constant of the daughter QS. To mould equation (5) to the form of a Schrodinger GF equation form in a chosen D_B dimensional Euclidean space, we consider the unspecified modulation function $f_B\left(r\right)$ and set

$$\frac{d}{dr}\ln\frac{f_B^2\left(r\right)g^{D_A-1}\left(r\right)}{g'\left(r\right)}=\frac{d}{dr}\ln r^{D_B-1}. \quad (6)$$

Integrating, we get

$$\ln\frac{f_B^2\left(r\right)g^{D_A-1}\left(r\right)}{g'\left(r\right)}=\ln r^{D_B-1}-2\ln N, \quad (7)$$

where N is the normalization constant. This gives

$$f_B\left(r\right)=Ng'^{1/2}g^{-\left(\frac{D_A-1}{2}\right)}\left(r\right)\cdot r^{\frac{D_B-1}{2}}. \quad (8)$$

The corresponding D_B dimensional standard Schrodinger GF equation for B-QS found to be

$$\Bigg[\partial_r^2+\frac{D_B-1}{r}\partial_r+\frac{1}{2}\{g,r\}+g'^2\left(E_n^A-V_A\left(g(r)\right)\right)-\left(l_A+\frac{D_A}{2}-1\right)^2\left(\frac{g'}{g}\right)^2$$

$$-\frac{\left(D_A-2\right)^2}{4}\left(\frac{g'}{g}\right)^2-\frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g'}{g}\right)^2+\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2}\Bigg]$$

$$\times G_B\left(r,r_0,E_N^B,V_B\left(r\right),C_B\right)=\frac{\delta\left(r-r_0\right)}{r_0^{D_B-1}}, \quad (9)$$

where $\{g,r\}=\frac{g'''}{g'}-\frac{3}{2}\left(\frac{g''}{g'}\right)^2$ denotes the Schwarz derivative symbol.

To implement ET on A-QS potential, we have to select a term of $V_A\left(g(r)\right)$ as a working potential $V_A^W\left(g(r)\right)$ and make the following ansatz:

$$g'^2V_A^W\left(g(r)\right)=E_N^B \quad (10)$$

$$g'^2\left(E_n^A-V_A\left(g(r)\right)+V_A^W\left(g(r)\right)\right)=-V_B\left(r\right) \quad (11)$$

and

$$\frac{g'^2\left(l_A+\frac{D_A}{2}-1\right)^2}{g^2}=\frac{\left(l_B+\frac{D_B}{2}-1\right)^2}{r^2} \quad (12)$$

Invoking the ansatz (10) to (12), equation (9) becomes

$$\Bigg[\partial_r^2+\frac{D_B-1}{r}\partial_r+\frac{1}{2}\{g,r\}+E_N^B-V_B\left(r\right)-\left(l_B+\frac{D_B}{2}-1\right)^2\frac{1}{r^2}-\frac{\left(D_A-2\right)^2}{4}\left(\frac{g'}{g}\right)^2$$

$$-\frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g'}{g}\right)^2+\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2}\Bigg]G_B\left(r,r_0,E_N^B,V_B\left(r\right),C_B\right)=\frac{\delta\left(r-r_0\right)}{r_0^{D_B-1}}. \tag{13}$$

In the case of a power law type potential, $V_A\left(r\right)=a_Ar^{b_A}$, the dimension of the Euclidean space to which the transformed system gets transported, can be arbitrarily pre-assigned only when ET is performed and the expression $\frac{1}{2}\left\{g,r\right\}-\left(l_B+\frac{D_B}{2}-1\right)^2\frac{1}{r^2}-\frac{(D_A-2)^2}{4}\left(\frac{g'}{g}\right)^2-\frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g'}{g}\right)^2+\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2}$, in equation (13) is reduce to the correct form of the centrifugal barrier term, $-\frac{l_B\left(l_B+D_B-2\right)}{r^2}$ [22], in D_B dimensional space. The transformed B-QS Green's function equation is established as

$$\left[\partial_r^2+\frac{D_B-1}{r}\partial_r+E_N^B-V_B\left(r\right)-\frac{l_B\left(l_B+D_B-2\right)}{r^2}\right]G_B\left(r,r_0,E_N^B,V_B\left(r\right),C_B\right)=\frac{\delta\left(r-r_0\right)}{r_0^{D_B-1}} \tag{14}$$

In equation (10), $V_A\left(g(r)\right)$ is termed as the working potential (WP). WP can be chosen in principle in 2^n-1 different ways. In fact we can pick any number of terms of the multi-term potential, the least being a single term and designate it as the WP. Let for simplicity i^{th} term as WP, in which $V_A^W\left(r\right)=a_ig^{b_i}$. Ansatz (10) is now $g'^2a_ig^{b_i}=-E_N^B$, and with a simple integration yields

$$g\left(r\right)=\left[\pm\frac{b_i+2}{2}\sqrt{-\frac{E_N^B}{a_i}r+C}\right]^{\frac{2}{b_i+2}}, \tag{15}$$

where C is the integration constant. For power law $g\left(r\right)$, $g'\left(r\right)$ is also a power law function of r; hence by equation (11), one gets a power law $V_B\left(r\right)$. The energy eigenvalues E_N^B of the B-QS is simply obtained by putting the coefficient of the r-independent part of $V_B\left(r\right)$—which would be a product of a function $F\left(E_N^B\right)$ and E_n^A—equal to the characteristic constant C_B^2 of the B system. E_N^B is specified in terms of the known E_n^A of the A system. However the quantum number n and N are different as l_A and l_B are in general different. The relation between the angular momentum quantum numbers are obtained from equation (12) and is

$$4l_A=\left(b_A+2\right)\left(2l_B+D_B-2\right)\left(D_A-2\right). \tag{16}$$

From equations (5) and (6) the eigenfunction expansion of B-QS Green's function is

$$G_B\left(r,r_0,E_N^B,V_B\left(r\right),C_B\right)=\sum_{n=0}^{\infty}\frac{f_B^{-1}\left(r\right)\psi_A^{(n)}\left(g\left(r\right)\right)\psi_A^{*(n)}\left(g\left(r_0\right)\right)f_B^{-1}\left(r_0\right)}{E-E_N^B-i\in}=\sum_{N=0}^{\infty}\frac{\psi_B^{(N)}\left(r\right)\psi_B^{*(N)}\left(r_0\right)}{E-E_N^B-i\in}. \tag{17}$$

The B-QS energy eigenfunctions $\psi_B^{(N)}\left(r\right)$ can be read off from equation (17) in conjunction with equation (8) as

$$\psi_B^{(N)}\left(r\right)=g'^{-1/2}g^{\left(\frac{D_A-1}{2}\right)}\left(r\right).r^{-\left(\frac{D_B-1}{2}\right)}\psi_A^{(n)}\left(g\left(r\right)\right). \tag{18}$$

Equation (19) and (20) holds good for any parent and daughter QS.

3. Generation of new ESP from decemvirate power potential

We consider the decemvirate power potential as A-QS,

$$V_A\left(r\right)=ar^2-br^4+cr^6-dr^8+er^{10}, \tag{19}$$

to generate the new exactly solved QS. Coefficients a, b, c, d and e are the parameters of the potential. She-Hai Dong and Zhong Qi Ma [21] had given the energy eigenfunctions as

$$\psi_A(r) = N_0 r^{l_A} \exp\left[\frac{1}{2}\alpha.r^2 - \frac{1}{4}\beta.r^4 + \frac{1}{6}\tau.r^6\right], \tag{20}$$

where the parameters of the potential, energy eigenfunctions and the angular momentum quantum number l_A are interrelated as $\alpha^2 - 2\beta \cdot l_A - 3\beta = a, \quad 5\tau - 2\tau \cdot l_A - 2\alpha\beta = -b, \quad \beta^2 + 2\alpha\tau = c$, and $\tau^2 = e$.

The constraints on the parameters of the decemvirate power potential are

$$a = \frac{d^4 - 8ced^2 + 16c^2e^+ 64de^2\sqrt{e}\left(l_A + {}^3\!/_2\right)}{64e^2} \tag{21}$$

and

$$b = \frac{8e^2\sqrt{e}\,(5 + 2l_A) - d\left(d^2 - 4ce\right)}{8e^2}. \tag{22}$$

The energy eigenvalues are

$$E_A = -\alpha\,(1 + 2l_A) = -\frac{(1 + 2l_A)\left(d^2 - 4ce\right)}{8e\sqrt{e}}. \tag{23}$$

Corresponding to $D_A = 1$, the dimensional differential equation is

$$\left[\partial_r^2 + E_A - \left(ar^2 - br^4 + cr^6 - dr^8 + er^{10}\right) - \frac{l_A\,(l_A - 1)}{r^2}\right] G_A\left(r, r_0, E_A, V_A(r), C_A\right) = \delta\,(r - r_0)\,. \tag{24}$$

Here, $V_A(r)$ is a five term potential, as given in equation (19). From this multi-term potential of A-QS, the WP can be chosen in $\left(2^5 - 1\right)$ different ways. To implement ET on A-QS, as a specific choice, we select er^{10} as the WP. The functional form of $g(r)$, obtained from (15), is

$$g(r) = \pm\left(-\frac{E_B}{e}\right)^{1/2} (6r)^{1/6}, \tag{25}$$

with the local property $g(0) = 0$. Taking the positive sign in equation (25) and utilizing equations (11), we get the following B-Sturmian quantum system potential:

$$V_B(r) = \alpha_1 r^{-5/3} + \alpha_2^{(n)} r^{-4/3} - \alpha_3^{(n)} r^{-1} + \alpha_4^{(n)} r^{-2/3} - \alpha_5^{(n)} r^{-1/3} \tag{26}$$

with $\alpha_1 = 6^{-5/3}\left(-\frac{E_B}{e}\right)^{1/6}(-E_A) = C_B^2, \alpha_2^{(n)} = -6^{-4/3} a\left(-\frac{E_B}{e}\right)^{1/3}, \quad \alpha_3^{(n)} = -6^{-1} b\left(-\frac{E_B}{e}\right)^{1/2}, \quad \alpha_4^{(n)} = -6^{-2/3} c\left(-\frac{E_B}{e}\right)^{2/3}$ and $\alpha_5^{(n)} = -6^{-1/3} d\left(-\frac{E_B}{e}\right)^{5/6}$.

To make the above potential normal, we take $a \to a_n$, $b \to b_n$, $c \to c_n$ and $d \to d_n$ of the A-QS parameter such that they become $a_n = 6^{4/3}\alpha_2\left(-\frac{E_B}{e}\right)^{-1/3}, b_n = 6\alpha_3\left(-\frac{E_B}{e}\right)^{-1/2}$, $c_n = 6^{2/3}\alpha_4\left(-\frac{E_B}{e}\right)^{-2/3}$ and $d_n = 6^{1/3}\alpha_5\left(-\frac{E_B}{e}\right)^{-5/6}$. Consequently, the normal form of $V_B(r)$ is given by

$$V_B(r) = \alpha_1 r^{-5/3} + \alpha_2 r^{-4/3} - \alpha_3 r^{-1} + \alpha_4 r^{-2/3} - \alpha_5 r^{-1/3}. \tag{27}$$

The energy eigenvalues of B-QS is

$$E_B = -\left[\frac{\alpha_5\left(6l_B+3D_B-5\right)^2\left(6l_B+3D_B-4\right)}{3\left\{9\alpha_1-2\alpha_2\left(6l_B+3D_B-5\right)\right\}}\right]. \tag{28}$$

The angular momentum quantum number l_B of B-QS is related to the angular momentum quantum number l_A of A-QS through equation (16) and is $l_A = 6l_B + 3D_B - \frac{11}{2}$.

The parameters of the potential energy eigenvalues and angular momentum quantum number of B-QS are connected by constraint equations as

$$\alpha_3 - \frac{\alpha_1}{\left(6l_B+3D_B-5\right)\left(6l_B+3D_B-4\right)}\left\{81\frac{\alpha_2^2}{\left(6l_B+3D_B-5\right)^2}-18\alpha_2\right\}$$

$$+\left(6l_B+D_B-1\right)\sqrt{E_B}=0 \tag{29}$$

$$\alpha_4 - \left[\frac{1}{\left(6l_B+3D_B-4\right)}\left\{\frac{27}{2}\frac{\alpha_1^2}{\left(6l_B+3D_B-5\right)^2}-\alpha_2\right\}\right]^2 - \frac{6\alpha_1}{\left(6l_B+3D_B-5\right)}\sqrt{-E_B}=0 \tag{30}$$

$$\alpha_5 - \frac{1}{\left(6l_B+3D_B-4\right)}\left\{27\frac{\alpha_1^2}{\left(6l_B+3D_B-5\right)^2}-6\alpha_2\right\}\sqrt{-E_B}=0. \tag{31}$$

Invoking equations (27) and (28) in equation (14), we get the standard Schrodinger GF equation in D_B dimensional Euclidean space as

$$\left[\partial_r^2+\frac{D_B-1}{r}\partial_r+E_N^B-\left(\alpha_1 r^{-5/3}+\alpha_2 r^{-4/3}-\alpha_3 r^{-1}+\alpha_4 r^{-2/3}-\alpha_5 r^{-1/3}\right)\right.$$

$$\left.-\frac{l_B\left(l_B+D_B-2\right)}{r^2}\right]G_B\left(r,r_0,E_N^B,V_B\left(r\right),C_B\right)=\frac{\delta\left(r-r_0\right)}{r_0^{D_B-1}}. \tag{32}$$

From equation (17), the new radial eigenfunction is read off as

$$\psi_B\left(r\right)=N_B r^{l_B}\exp\left[\frac{9\alpha_1 r^{1/3}}{\left(6l_B+3D_B-5\right)}-\right.$$

$$\left.-\frac{9}{2}\frac{1}{\left(6l_B+3D_B-4\right)}\left\{\frac{9}{2}\frac{\alpha_1^2}{\left(6l_B+3D_B-5\right)^2}-\alpha_2\right\}r^{2/3}+\sqrt{-E_B}r\right], \tag{33}$$

where we have selected er^{10} as the WP. The functional form of $g(r)$ obtained from (15) is

$$g(r)=\pm\left(-\frac{E_B}{e}\right)^{1/2}\left(6r\right)^{1/6}, \tag{25}$$

with the local property $g(0)=0$. Taking the positive sign in equation (25) and utilizing the equations (11), we get the following potential of B-Sturmian quantum system:

$$V_B\left(r\right)=\alpha_1 r^{-5/3}+\alpha_2^{(n)}r^{-4/3}-\alpha_3^{(n)}r^{-1}+\alpha_4^{(n)}r^{-2/3}-\alpha_5^{(n)}r^{-1/3} \tag{26}$$

with $\alpha_1 = 6^{-5/3}\left(-\frac{E_B}{e}\right)^{1/6}(-E_A) = C_B^2, \alpha_2^{(n)} = -6^{-4/3}a\left(-\frac{E_B}{e}\right)^{1/3}$, $\alpha_3^{(n)} = -6^{-1}b\left(-\frac{E_B}{e}\right)^{1/2}$, $\alpha_4^{(n)} = -6^{-2/3}c\left(-\frac{E_B}{e}\right)^{2/3}$ and $\alpha_5^{(n)} = -6^{-1/3}d\left(-\frac{E_B}{e}\right)^{5/6}$.

The potential (26) is a Sturmian. To make it normal, we take $a \to a_n$, $b \to b_n$, $c \to c_n$ and $d \to d_n$ of the A-QS parameter such that $a_n = 6^{4/3}\alpha_2\left(-\frac{E_B}{e}\right)^{-1/3}, b_n = 6\alpha_3\left(-\frac{E_B}{e}\right)^{-1/2}$, $c_n = 6^{2/3}\alpha_4\left(-\frac{E_B}{e}\right)^{-2/3}$ and $d_n = 6^{1/3}\alpha_5\left(-\frac{E_B}{e}\right)^{-5/6}$. Consequently the normal form of $V_B(r)$ is given by

$$V_B(r) = \alpha_1 r^{-5/3} + \alpha_2 r^{-4/3} - \alpha_3 r^{-1} + \alpha_4 r^{-2/3} - \alpha_5 r^{-1/3}. \tag{27}$$

The energy eigenvalues of B-QS is

$$E_B = -\left[\frac{\alpha_5(6l_B + 3D_B - 5)^2(6l_B + 3D_B - 4)}{3\{9\alpha_1 - 2\alpha_2(6l_B + 3D_B - 5)\}}\right]. \tag{28}$$

The angular momentum quantum number l_B of B-QS is related to the angular momentum quantum number l_A of A-QS through equation (24) and is $l_A = 6l_B + 3D_B - \frac{11}{2}$.

The parameters of the potential energy eigenvalues and angular momentum quantum number of B-QS are connected by some constraint equations as

$$\alpha_3 - \frac{\alpha_1}{(6l_B + 3D_B - 5)(6l_B + 3D_B - 4)}\left\{81\frac{\alpha_2^2}{(6l_B + 3D_B - 5)^2} - 18\alpha_2\right\}$$
$$+(6l_B + D_B - 1)\sqrt{E_B} = 0, \tag{29}$$

$$\alpha_4 - \left[\frac{1}{(6l_B + 3D_B - 4)}\left\{\frac{27}{2}\frac{\alpha_1^2}{(6l_B + 3D_B - 5)^2} - \alpha_2\right\}\right]^2 - \frac{6\alpha_1}{(6l_B + 3D_B - 5)}\sqrt{-E_B} = 0, \tag{30}$$

$$\alpha_5 - \frac{1}{(6l_B + 3D_B - 4)}\left\{27\frac{\alpha_1^2}{(6l_B + 3D_B - 5)^2} - 6\alpha_2\right\}\sqrt{-E_B} = 0. \tag{31}$$

Invoking the equations (27) and (28) in equation (14) we get the standard Schrodinger GF equation in D_B dimensional Euclidean space as

$$\left[\partial_r^2 + \frac{D_B - 1}{r}\partial_r + E_N^B - \left(\alpha_1 r^{-5/3} + \alpha_2 r^{-4/3} - \alpha_3 r^{-1} + \alpha_4 r^{-2/3} - \alpha_5 r^{-1/3}\right)\right.$$
$$\left.-\frac{l_B(l_B + D_B - 2)}{r^2}\right] G_B\left(r, r_0, E_N^B, V_B(r), C_B\right) = \frac{\delta(r - r_0)}{r_0^{D_B - 1}}. \tag{32}$$

From equation (17), the new radial eigenfunction can be read off as

$$\psi_B(r) = N_B r^{l_B} \exp\left[\frac{9\alpha_1 r^{1/3}}{(6l_B + 3D_B - 5)} - \right.$$
$$\left.-\frac{9}{2}\frac{1}{(6l_B + 3D_B - 4)}\left\{\frac{9}{2}\frac{\alpha_1^2}{(6l_B + 3D_B - 5)^2} - \alpha_2\right\} r^{2/3} + \sqrt{-E_B r}\right]. \tag{33}$$

In the similar procedure we have find the EASs, taking $-dg^8, cg^6$, $-bg^4$ and ag^2 as WP from equation (17) Tables 1 and 2.

Table 1. List of ESP, E_B and constraint equation generated from decemvirate power potential $V_A(r) = ar^2 - br^4 + cr^6 - dr^8 + er^{10}$ taking the WPs: $-dg^8, cg^6$, $-bg^4$ and ag^2.

$V_A^{(W)}(r)$	$V_B(r)$	E_B	Constraint equation
$-dg^8$	$\beta_1 r^{-\frac{8}{5}} - \beta_2 r^{-\frac{6}{5}} - \beta_3 r^{-\frac{4}{5}}$ $+ \beta_4 r^{-\frac{2}{5}} + \beta_5 r^{\frac{2}{5}}$	$\frac{(10.l_B + 5D_B - 8)}{\beta_1} \{ \frac{1}{5} \beta_2 \sqrt{\beta_5}$ $+ \beta_5 (10 l_B + 5D_B - 4) \}$	$\beta_2 - 25 \frac{\beta_1}{(10 l_B + 5D_B - 4)} + (10.l_B + 5D_B - 4)$ $\left\{ \frac{1}{10} \frac{\beta_3}{\beta_1} + \frac{\sqrt{\beta_5}}{\beta_3} (10.l_B + 5D_B - 4) \right\} \left(2l_B + D_B - \frac{6}{5} \right) = 0$ $\beta_4 \left[(10 l_B + 5D_B - 8) \left\{ \frac{1}{10} \frac{\beta_3}{\beta_1} + \frac{1}{2} (10 l_B + 5D_B - 4) \right\} \right]^2$ $- 10 \frac{\beta_1}{(10 l_B + 5D_B - 8)} \sqrt{\beta_5} = 0$
cg^6	$\gamma_1 r^{-\frac{3}{2}} + \gamma_2 r^{-1} - \gamma_3 r^{-\frac{1}{2}}$ $- \gamma_4 r^{1/2} + \gamma_5 r^2$	$- [\frac{1}{4} \frac{\gamma_4^2}{\gamma_5} + \frac{4\gamma_1 \sqrt{\gamma_5}}{4l_B + 2D_B -}$	$\gamma_2 - \left(\frac{2\gamma_1}{4l_B + 2D_B - 3} \right)^2 + \frac{\gamma_4}{2\sqrt{\gamma_5}} (2l_B + D_B - 1)$ $\gamma_3 - \frac{2\gamma_1}{4l_B + 2D_B - 3} \frac{\gamma_4}{\sqrt{\gamma_5}} - \sqrt{\gamma_5} \left(2l_B + D_B - \frac{3}{2} \right) = 0$
$-bg^4$	$\sigma_1 r^{-\frac{4}{3}} + \sigma_2 r^{-\frac{2}{3}} + \sigma_3 r^{\frac{2}{3}}$ $- \sigma_4 r^{\frac{4}{3}} + \sigma_5 r^2$	$- \left[\frac{\sigma_1 \sigma_4}{(6l_B + 3D_B - 4)\sqrt{\sigma_5}} \right.$ $\left. + \sqrt{\sigma_5} (2l_B + D_B) \right]$	$\sigma_2 - \frac{9\sigma_1^2}{(6l_B + 3D_B - 4)^2} + \frac{\sigma_4}{6\sqrt{\sigma_5}} - (6l_B + 3D_B - 1 = 0)$ $\sigma_3 - \frac{1}{4} \frac{\sigma_4^2}{\sigma_5} - \frac{6\sigma_1 \sqrt{\sigma_5}}{6l_B + 3D_B - 4} = 0$
ag^2	$\rho_1 r^{-1} - \rho_2 r + \rho_3 r^2$. $- \rho_1 r^3 + \rho_3 r^4$	$\frac{\rho_4}{2\sqrt{\rho_5}} (2l_B + D_B)$ $- \frac{\rho_1^2}{(2l_B + D_B - 1)^2}$	$\rho_2 - \frac{\rho_1 \rho_4}{\sqrt{\rho_5} (2l_B + D_B - 1)} + \sqrt{\rho_5} (2l_B + D_B + 1) = 0$ $\rho_3 - \frac{\rho_4^2}{4\rho_5} - \frac{2\rho_1 \sqrt{\rho_5}}{2l_B + D_B - 1} = 0$

where

$$\beta_1 = 5^{-8/5} \left(\frac{E_B}{d} \right)^{1/5} (-E_A), \quad \beta_2 = 5^{-6/5} a \left(\frac{E_B}{d} \right)^{2/5}, \quad \beta_3 = 5^{-4/5} b \left(\frac{E_B}{d} \right)^{3/5},$$

$$\beta_4 = 5^{-2/5} c \left(\frac{E_B}{d} \right)^{4/5}, \quad \beta_5 = 5^{2/5} e \left(\frac{E_B}{d} \right)^{6/5}$$

$$\gamma_1 = \frac{1}{8}\left(-\frac{E_B}{c}\right)^{1/4}(-E_A), \quad \gamma_2 = \frac{1}{2}a\left(-\frac{E_B}{c}\right)^{1/2}, \quad \gamma_3 = \frac{1}{2}b\left(-\frac{E_B}{c}\right)^{3/4},$$

$$\gamma_4 = 2d\left(-\frac{E_B}{c}\right)^{5/4}, \quad \gamma_5 = 4e\left(-\frac{E_B}{d}\right)^{3/2}$$

$$\sigma_1 = 3^{-4/3}\left(\frac{E_B}{b}\right)^{1/3}(-E_A), \quad \sigma_2 = 3^{-2/3}a\left(\frac{E_B}{b}\right)^{2/3}, \quad \sigma_3 = 3^{2/3}c\left(\frac{E_B}{b}\right)^{4/3},$$

$$\sigma_4 = 3^{4/3}d\left(\frac{E_B}{b}\right)^{5/3}, \quad \sigma_5 = 9e\left(\frac{E_B}{b}\right)^{2}$$

and

$$\rho_1 = \frac{1}{2}\left(-\frac{E_B}{a}\right)^{1/2}(-E_A), \rho_2 = 2b\left(-\frac{E_B}{a}\right)^{3/2}, \rho_3 = 4c\left(-\frac{E_B}{a}\right)^{2},$$

$$\rho_4 = 8d\left(-\frac{E_B}{a}\right)^{5/2}, \quad \rho_5 = 16e\left(-\frac{E_B}{a}\right)^{3}.$$

Table 2. List of B-QS transformation functions $g(r)$ and wave functions $\psi_B(r)$ generated from decemvirate power potential, taking the WP as given in the Table 1.

Sl No	$g(r)$	$\psi_B(r)$
1	$\pm\left(\frac{E_B}{d}\right)^{1/10}(5r)^{1/5}$	$N_B.r^{l_B}\exp.\left[\frac{25}{2}\frac{\beta_1 r^{2/5}}{10l_B+5D_B-8}+\frac{10l_B+5D_B-8}{8\beta_1}\left\{\beta_3+5\sqrt{\beta_5}10l_B+5D_B-4\right\}.r^{4/5}+\frac{5}{6}\sqrt{\beta_5}r^{6/5}\right]$
2	$\pm\left(-\frac{E_B}{c}\right)^{1/8}(4r)^{1/4}$	$N_B r^{l_B}\exp\left[\frac{4\gamma_1 r^{1/2}}{4l_B+2D_B-4}-\frac{1}{2}\frac{\gamma_1}{\sqrt{\gamma_5}}r+\frac{2}{3}\sqrt{\gamma_5}r^{3/2}\right]$
3	$\pm\left(\frac{E_B}{b}\right)^{1/6}(3r)^{1/3}$	$N_B r^{l_B}\exp\left[\frac{9}{2}\frac{\sigma_1 r^{2/3}}{(6l_B+3D_B-4)}-\frac{3}{8}\frac{\sigma_2}{\sqrt{\sigma_5}}r^{4/5}+\frac{1}{2}\sqrt{\sigma_5}r^2\right]$
4	$\pm\left(-\frac{E_B}{a}\right)^{1/4}(2r)^{1/2}$	$N_B r^{l_B}\exp\left[\frac{\rho_1}{(2l_B+D_B-1)}r-\frac{\rho_4}{\sqrt{\rho_5}}r^2+\frac{1}{3}\sqrt{\rho_5}r^3\right]$

4. Conclusions

We have generated a new class of exactly solved quantum systems in non-relativistic Schrodinger GF equation, using the extended transformation method in any arbitrary number of spatial D -dimensional Euclidean spaces from decemvirate power law anharmonic potential $V_A(r) = ar^2 - br^4 + cr^6 - dr^8 + er^{10}$, taking the working potentials $eg^{10}, -dg^8$, cg^6, $-bg^4$ and ag^2. The solutions consist of eigenfunction and the corresponding eigenvalues, which were obtained in a closed form. There is a distinct interrelation between the parameters of the potentials and the orbital momentum quantum number l. For quantum multi-term potentials it is possible to generate a finite number of different exactly solved quantum systems by selecting working potential, as mentioned earlier. We however restrict ourselves to taking one term WP. Two or multi-term WP as they offer the following practical difficulties: the indefinite integral specifying the transformation function $g(r)$ cannot be evaluated analytically in most of the cases and even if such integrals are found they are of the form $F(g(r)) = r + C$ and the analytical inverse function $F^{-1}(g(r))$ cannot be found.

This paper is an endeavor to find/construct/generate five new potentials which are exactly solvable. They may find application in various branches of science, such as physics, chemistry, biology, electronics etc. Four of the exactly solvable potentials generated are tabulated with associated properties. Even to find approximate solution of Schrodinger equation for a particular problem, these exactly solved quantum potentials may facilitate efficient approximate calculation when the potential happens to be "near" one of the exactly solved potential. Lower arbitrary dimensional potentials might help analysis and design of semi-conductor hetero-structures of nano-technology.

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