

Space-time scaling invariant traveling wave solutions of some nonlinear fractional equations

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Abstract

In this paper, a homogeneous principle is proposed to seek the space-time scaling invariant traveling wave solutions expressed by power functions for some fractional differential equations. Applying this principle to generalized fractional Benjamin-Ono equations and generalized fractional Zakharov-Kuznetsov equations, the traveling wave solutions expressed by power functions have been obtained under some parameter conditions.

Key Words: Homogeneous principle; space-time scaling invariant traveling wave solution; generalized fractional Benjamin-Ono equations; generalized fractional Zakharov-Kuznetsov equations

1. Introduction

Fractional differential equations (FDEs) play an outstanding role in physics, chemistry and engineering. However, effective general method for solving them can not be found even in the most useful works on fractional derivatives and integrals [1]. Fortunately, the Adomian decomposition method (ADM) [2, 3], the homotopy perturbation method (HPM) [4–8], Homotopy analysis method (HAM) [9–14], the variational iteration method (VIM) [15–17] are efficient for solving some FDEs.

Recently, Djordjevic and Atanackovic [18] found similarity solutions of nonlinear conduction and Burgers/Korteweg-de Vries fractional equations by using Lie-group scaling transformation. They considered heat conduction fractional equations

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{\partial}{\partial x} \left[(k + mT^n) \frac{\partial T}{\partial x} \right], x \in (0, \infty), t > 0, 0 < \alpha \leq 1, \quad (1.1)$$

and obtained its traveling wave solutions of the form

$$T(x, t) = \left[\frac{nc^\alpha \Gamma(1 + \frac{2-\alpha}{n})}{m(2-\alpha)\Gamma(2-\alpha + \frac{2-\alpha}{n})} \right]^{\frac{1}{n}} (ct - x)^{\frac{2-\alpha}{n}}$$

under $k = 0$. Motivated by [18], we seek the space-time scaling invariant (called STSI shortly) traveling wave solutions expressed by power functions of the following more generalized fractional equations:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + L((u^{n_1})_{N_1 x M_1 y}, (u^{n_2})_{N_2 x M_2 y}, \dots, (u^{n_i})_{N_i x M_i y}) = 0, \tag{1.2}$$

where L is a polynomial function, $u = u(x, y, t)$, α is the order of the fractional derivative ($p - 1 < \alpha \leq p, p \in N$), $n_i \in R$ and $N_i x M_i y$ denotes N_i^{th} -order derivative with respect to x and M_i^{th} -order derivative with respect to y . Equation (1.2) includes the special case of equation (1.1) as $k = 0$. Indeed, if we take

$$L = r_1(u^{n_1})_{N_1 x M_1 y}((u^{n_2})_{N_2 x M_2 y})^2 + r_2(u^{n_3})_{N_3 x M_3 y}(u^{n_4})_{N_4 x M_4 y},$$

and $p = 1, r_1 = -mn, n_1 = n - 1, N_1 = M_1 = 0, n_2 = 1, N_2 = 1, M_2 = 0, r_2 = -m, n_3 = n, N_3 = M_3 = 0, n_4 = 1, N_4 = 2, M_4 = 0$, then equation (1.2) is reduced to equation (1.1).

Remainder of this paper is organized as follows. A basic principle of finding STSI traveling wave solutions for equation (1.2) is presented in the next section. Two examples are given to demonstrate the effectiveness of this principle in Section 3. Finally, our findings are summarized in Section 4, Summary and Conclusions.

2. A basic principle: homogeneous principle

The three most commonly used definitions in Fractional Calculus are the Riemann-Liouville, Grunwald-Letnikov and Caputo definitions [1]. In this paper, we generally use Riemann-Liouville fractional derivative as follows.

Definition. Let $p - 1 < \alpha \leq p, p \in N$. The Riemann-Liouville fractional derivative of order α of any function $f(t)$ is defined as

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(p - \alpha)} \frac{d^p}{dt^p} \int_0^t \frac{f(q) dq}{(t - q)^{\alpha + 1 - p}}, \tag{2.1}$$

where Γ is the Gamma function.

For equation (1.2), we seek the STSI traveling wave solutions expressed by power functions as follows:

$$u(x, y, t) = \begin{cases} \varphi_1(ct - k_1x - k_2y)^a & , \text{ if } k_1x + k_2y \leq ct, \\ 0 & , \text{ if } k_1x + k_2y > ct, \end{cases} \tag{2.2}$$

where φ_1 and a are undetermined constants, c is a wave speed, $x, y \in R$ and $k_1 = 1, k_2 = 0 (k_1 = 1, k_2 = 1)$ correspond to $u = u(x, t) (u = u(x, y, t))$, respectively.

Now let us give a basic principle (homogeneous principle) accompanied by the main steps for seeking the STSI traveling wave solutions expressed by power functions for equation (1.2) as follows:

Step 1: Taking the space-time scaling transformation

$$t = kt_1, \quad x = kx_1, \quad y = ky_1, \quad k > 0, \tag{2.3}$$

then

$$u(x, y, t) = k^a \begin{cases} \varphi_1(ct_1 - k_1x_1 - k_2y_1)^a & , \text{ if } k_1x_1 + k_2y_1 \leq ct_1, \\ 0 & , \text{ if } k_1x_1 + k_2y_1 > ct_1. \end{cases} \tag{2.4}$$

When $ct_1 - k_1x_1 - k_2y_1 \geq 0$, substituting equation (2.4) into equation (1.2), we obtain an ordinary differential equation involving fractional derivative:

$$\begin{aligned}
 & [k(ct_1 - k_1x_1 - k_2y_1)]^{a-\alpha} c^\alpha \varphi_1 \frac{\Gamma(a+1)}{\Gamma(a+1-\alpha)} + L([k(ct_1 - k_1x_1 - k_2y_1)]^{an_1-N_1-M_1} k_1^{N_1} k_2^{M_1} \varphi_1^{n_1} an_1(an_1 - 1) \\
 & \cdots (an_1 - N_1 - M_1 + 1), \cdots, [k(ct_1 - k_1x_1 - k_2y_1)]^{an_i-N_i-M_i} k_1^{N_i} k_2^{M_i} \varphi_1^{n_i} an_i(an_i - 1) \\
 & \cdots (an_i - N_i - M_i + 1)) = 0.
 \end{aligned} \tag{2.5}$$

From equation (2.5), the conditions of the existence for STSI traveling wave solutions (2.4) are that equation (2.5) is homogenous with respect to k .

Step 2: Under the above conditions of invariance, φ_1 is determined from the relation

$$\begin{aligned}
 & c^\alpha \varphi_1 \frac{\Gamma(a+1)}{\Gamma(a+1-\alpha)} + L(k_1^{N_1} k_2^{M_1} \varphi_1^{n_1} an_1(an_1 - 1) \cdots (an_1 - N_1 - M_1 + 1), \\
 & \cdots, k_1^{N_i} k_2^{M_i} \varphi_1^{n_i} an_i(an_i - 1) \cdots (an_i - N_i - M_i + 1)) = 0.
 \end{aligned} \tag{2.6}$$

For the convenience of statement, let equation (1.2) be of the form

$$\begin{aligned}
 & \frac{\partial^\alpha u}{\partial t^\alpha} + r_1 u^{n_{01}} \left(\frac{\partial u}{\partial x}\right)^2 + r_2 u^{n_{02}} \frac{\partial^2 u}{\partial x^2} + l_{11}(u^{n_{11}})_x + l_{12}(u^{n_{12}})_y \\
 & + l_{21}(u^{n_{21}})_{xx} + l_{22}(u^{n_{22}})_{xy} + l_{23}(u^{n_{23}})_{yy} \\
 & + l_{31}(u^{n_{31}})_{xxx} + l_{32}(u^{n_{32}})_{xxy} + l_{33}(u^{n_{33}})_{xyy} + l_{34}(u^{n_{34}})_{yyy} \\
 & + l_{41}(u^{n_{41}})_{xxxx} + \cdots + l_{45}(u^{n_{45}})_{yyyy} = 0.
 \end{aligned} \tag{2.7}$$

Let

$$u_1(x, y, t) = \begin{cases} \varphi_1(ct_1 - k_1x_1 - k_2y_1)^a, & \text{if } k_1x_1 + k_2y_1 \leq ct_1, \\ 0, & \text{if } k_1x_1 + k_2y_1 > ct_1, \end{cases} \tag{2.8}$$

and when $k_1x_1 + k_2y_1 < ct_1$, substituting equation (2.8) into equation (2.7), equation (2.7) can be changed into

$$\begin{aligned}
 & k^{a-\alpha} \frac{\partial^\alpha u_1}{\partial t_1^\alpha} + k^{an_{01}+2(a-1)} r_1 u_1^{n_{01}} \left(\frac{\partial u_1}{\partial x_1}\right)^2 + k^{an_{02}+a-2} r_2 u_1^{n_{02}} \frac{\partial^2 u_1}{\partial x_1^2} \\
 & + k^{an_{11}-1} l_{11}(u_1^{n_{11}})_{x_1} + k^{an_{12}-1} l_{12}(u_1^{n_{12}})_{y_1} \\
 & + k^{an_{21}-2} l_{21}(u_1^{n_{21}})_{x_1x_1} + \cdots + k^{an_{23}-2} l_{23}(u_1^{n_{23}})_{y_1y_1} \\
 & + k^{an_{31}-3} l_{31}(u_1^{n_{31}})_{x_1x_1x_1} + \cdots + k^{an_{34}-3} l_{34}(u_1^{n_{34}})_{y_1y_1y_1} \\
 & + k^{an_{41}-4} l_{41}(u_1^{n_{41}})_{x_1x_1x_1x_1} + \cdots + k^{an_{45}-4} l_{45}(u_1^{n_{45}})_{y_1y_1y_1y_1} = 0.
 \end{aligned} \tag{2.9}$$

Then the conditions of invariance read:

$$a - \alpha = an_{01} + 2(a - 1) = an_{02} + a - 2 = an_{11} - 1 = \cdots = an_{45} - 4. \tag{2.10}$$

When $k_1x_1 + k_2y_1 < ct_1$, one may easily verify that

$$\begin{aligned} \frac{\partial^\alpha u_1}{\partial t_1^\alpha} &= \varphi_1 \frac{d^\alpha (ct_1 - k_1x_1 - k_2y_1)^\alpha}{dt_1^\alpha} = [ct_1 - k_1x_1 - k_2y_1]^{a-\alpha} \varphi_1 c^\alpha \frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)}, \\ u_1^{n_{01}} \left(\frac{\partial u_1}{\partial x_1}\right)^2 &= [ct_1 - k_1x_1 - k_2y_1]^{an_{01}+2(a-1)} \varphi_1^{n_{01}+2} a^2 k_1^2, \\ u_1^{n_{02}} \frac{\partial^2 u_1}{\partial x_1^2} &= [ct_1 - k_1x_1 - k_2y_1]^{an_{02}+a-2} \varphi_1^{n_{02}+1} k_1^2 a(a-1), \\ (u_1^{n_{11}})_{x_1} &= -[ct_1 - k_1x_1 - k_2y_1]^{an_{11}-1} \varphi_1^{n_{11}} k_1 an_{11}, \dots \\ (u_1^{n_{45}})_{4y_1} &= [ct_1 - k_1x_1 - k_2y_1]^{an_{45}-4} (k_2)^4 an_{45} (an_{45} - 1)(an_{45} - 2)(an_{45} - 3) \varphi_1^{n_{45}}. \end{aligned}$$

Substituting the above equations into equation (2.9), under the invariant conditions (2.10), we obtain

$$\begin{aligned} c^\alpha \varphi_1 \frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} + r_1 \varphi_1^{n_{01}+2} a^2 k_1^2 + r_2 k_1^2 a(a-1) \varphi_1^{n_{02}+1} - l_{11} an_{11} k_1 \varphi_1^{n_{11}} \\ + \dots + l_{45} \varphi_1^{an_{45}} an_{45} (an_{45} - 1)(an_{45} - 2)(an_{45} - 3) k_2^4 = 0. \end{aligned} \tag{2.11}$$

This implies that φ_1 is determined by equation (2.11).

Remark. When $p = 1, r_1 = -mn, n_{01} = n - 1, r_2 = -m, n_{02} = n, k_1 = 1, k_2 = 0, l_{11} = \dots = l_{45} = 0$, we obtain $a = \frac{2-\alpha}{n}$ from invariant conditions and

$$\begin{aligned} \varphi_1 &= \left[\frac{c^\alpha \Gamma(a+1)}{(an+a-1)\Gamma(a-\alpha+1)ma} \right]^{\frac{1}{n}} = \left[\frac{c^\alpha \Gamma(a+1)}{(a+1-\alpha)\Gamma(a-\alpha+1)ma} \right]^{\frac{1}{n}} \text{ from equation (2.11), i.e.} \\ \varphi_1 &= \left[\frac{c^\alpha \Gamma(a+1)}{\Gamma(a-\alpha+2)ma} \right]^{\frac{1}{n}} = \left[\frac{c^\alpha n \Gamma(\frac{2-\alpha}{n}+1)}{\Gamma(\frac{2-\alpha}{n}-\alpha+2)m(2-\alpha)} \right]^{\frac{1}{n}}. \end{aligned}$$

The results agree with the results presented in [18].

3. Applications

In this section, the homogeneous principle proposed in Section 2 shall be demonstrated via application to two examples.

3.1. Space-time scaling invariant traveling wave solutions of generalized fractional Benjamin-Ono equations

If we take $u = u(x, t), \alpha = \beta, 1 < \beta \leq 2, l_{21} = -l_2, n_{21} = n, l_{41} = -l_1, n_{41} = m$ and other coefficients as zero, then equation (2.7) is reduced to the generalization of the fractional Benjamin-Ono (called GFB(m, n) shortly) equation,

$$\frac{\partial^\beta u}{\partial t^\beta} - l_1 (u^m)_{xxxx} - l_2 (u^n)_{xx} = 0, \tag{3.1}$$

where $mn \neq 0, m, n$ are constants. When $\beta = 2$, its traveling wave solutions are obtained by dynamical system method [19]. If we take $k_1 = 1$ and $k_2 = 0$ in (2.2), then the STSI traveling wave solutions of the GFB(m, n) equations are

$$u(x, y, t) = \begin{cases} \varphi_1(ct - x)^a & , \text{ if } x \leq ct, \\ 0 & , \text{ if } x > ct. \end{cases} \tag{3.2}$$

By applying the homogeneous principle to GFB(m, n) equation, we obtain a and φ_1 , which are determined by the conditions of invariance,

$$a - \beta = an - 2 = am - 4, \tag{3.3}$$

and

$$c^\beta \varphi_1 \frac{\Gamma(a+1)}{\Gamma(a+1-\beta)} = l_1 am(am-1)(am-2)(am-3)\varphi_1^m + l_2 an(an-1)\varphi_1^n. \tag{3.4}$$

From (3.3) and (3.4), we obtain the following conclusions.

(i) If $\beta = 2$ then $n = 1, a = \frac{2}{m-1}$ and $\varphi_1 = \left[\frac{(m-1)^2(c^2-l_2)}{2l_1m(m+1)} \right]^{\frac{1}{m-1}}$

(ii) If $1 < \beta < 2$, then $a = \frac{4-\beta}{m-1} = \frac{2-\beta}{n-1}$ and in this case we have the following results:

(A) If one of the cases satisfies (a) $a = \frac{3}{m}$, (b) $a = \frac{2}{m}$, (c) $a = \frac{1}{m}$, (d) $a = \frac{1}{n}$, then $\varphi_1 = 0$.

(B) If $l_1 = 0$, then $\varphi_1 = \left[\frac{c^\beta \Gamma(\frac{n+1-\beta}{n-1})}{l_2 \Gamma(\frac{3n-n\beta-1}{n-1})} \right]^{\frac{1}{n-1}}$.

(C) If $l_2 = 0$, then $\varphi_1 = \left[\frac{c^\beta \Gamma(\frac{m+3-m\beta}{m-1})}{l_1 \Gamma(\frac{5m-m\beta-1}{m-1})} \right]^{\frac{1}{m-1}}$

To ascertain the effect of parameters on the wave shape of STSI traveling wave solution in the general cases, we plot (3.2) for the case where $\beta = 1.8, c = l_1 = l_2 = 1$ and for different values of n and t (see Figure 1).

In (3.2), φ_1 represents the amplitude of a wave, which propagates along x -axis with the speed c (See Figures 1 and 2). In Figure 1(a), we show the wave profiles for $n = 1.1$. It is seen that the wave profiles are convex. When n is increased to 12, i.e. $\frac{2-\beta}{n-1} = 1$, the dependence of u on $ct - x$ in (3.2) is linear (See Figure 1(b)). Further increasing n to 2 makes the wave profiles be concave (See Figure 1(c)).

Now we examine the influence of the parameter β on the wave shape for fixed time. In the special cases where $c = l_1 = l_2 = 1, n = 1.4$ ($n = 2$), we solve (3.2) for four different values of β and show the wave profiles for $t = 0.3$ in Figure 2.

In Figure 2 (a), $n = 1.4$, when $\beta = 1.6$, the dependence of u on $ct - x$ is linear. If we increase the order of the derivative β , then the wave profile changes from convex ($\beta = 1.5$) to concave ($\beta = 1.8$).

In Figure 2(b), $n = 2$, the wave profile is concave for the different values of β .

3.2. Space-time scaling invariant traveling wave solutions of generalized fractional Zakharov-Kuznetsov equations

If we take $u = u(x, y, t), 0 < \alpha \leq 1, r_1 = r_2 = 0, l_{11} = l_1, n_{11} = m, l_{12} = l_{21} = \dots = l_{23} = 0, l_{31} = l_2, n_{31} = n, l_{32} = 0, l_{33} = l_3, n_{33} = r, l_{34} = l_{41} = \dots = l_{45} = 0$, then equation (2.7) is reduced to the generalization of fractional Zakharov-Kuznetsov equation (abbreviated GFZK) in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + l_1(u^m)_x + l_2(u^n)_{xxx} + l_3(u^r)_{yyx} = 0, \tag{3.5}$$

where l_1, l_2, l_3, m, n, r are constants and $mnr \neq 0$ governs the behavior of weakly nonlinear ion acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [20,

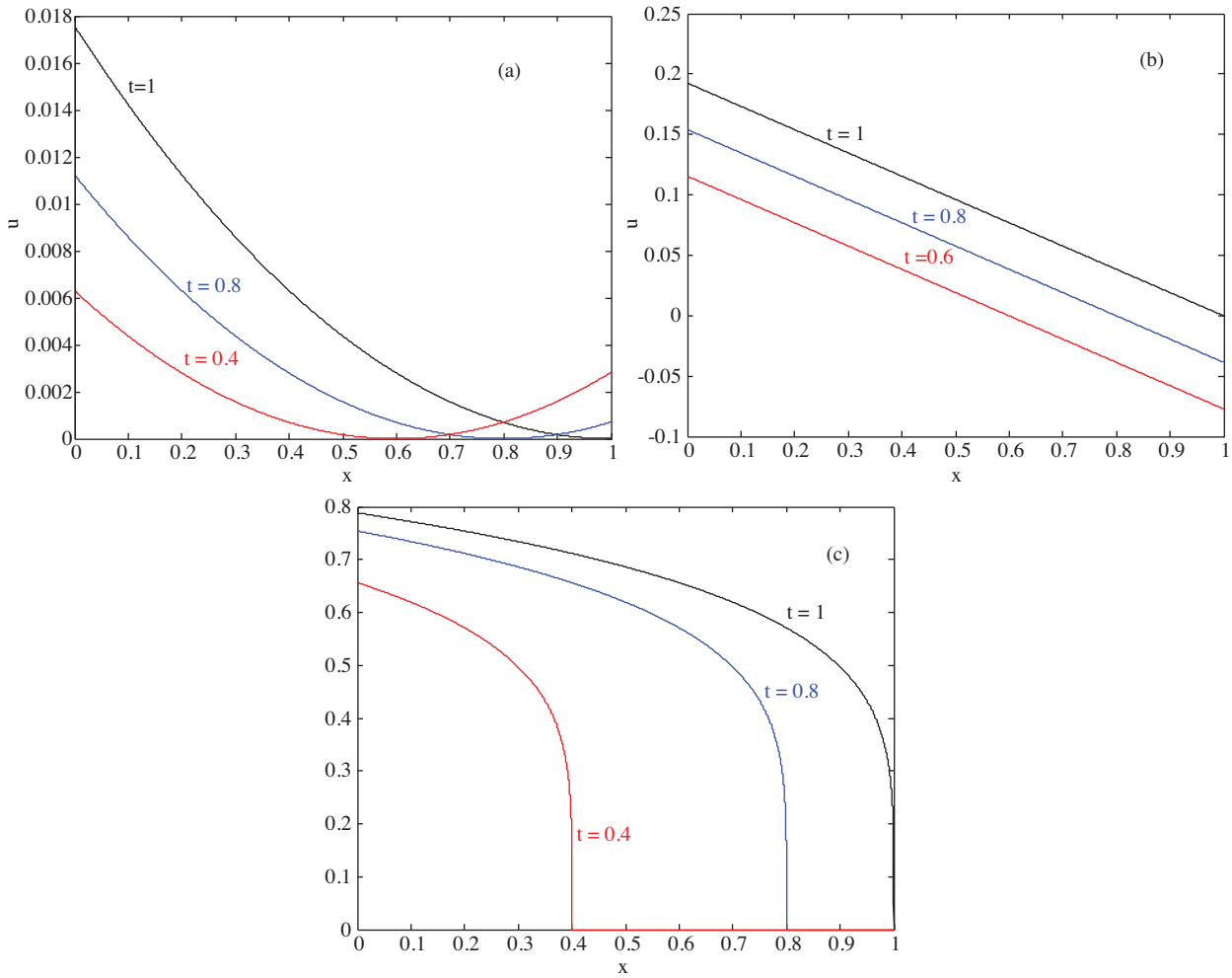


Figure 1. Wave profiles of GFB(m, n) equations for $\beta = 1.8, c = l_1 = l_2 = 1$: (a) $n = 1.1$, (b) $n = 1.2$, and (c) $n = 2$.

21]. Some solutions of the GFZK equation have been obtained by VIM [22] and HPM [23]. But traveling wave solutions have not been considered.

In (22), if we take $k_1 = k_2 = 1$, then the STSI traveling wave solution is

$$u(x, y, t) = \begin{cases} \varphi_1(ct - x - y)^a & , \text{ if } x + y \leq ct, \\ 0 & , \text{ if } x + y > ct. \end{cases} \quad (3.6)$$

By applying the homogeneous principle to GFZK equation, we obtain

$$a - \alpha = am - 1 = an - 3 = ar - 3, \quad (3.7)$$

and

$$\frac{\Gamma(1 + a)}{\Gamma(1 + a - \alpha)} c^\alpha - l_1 \varphi_1^{m-1} am - (l_2 + l_3) \varphi_1^{r-1} ar(ar - 1)(ar - 2) = 0. \quad (3.8)$$

From (3.7) and (3.8), we obtain $r = n$ and the following conclusions:

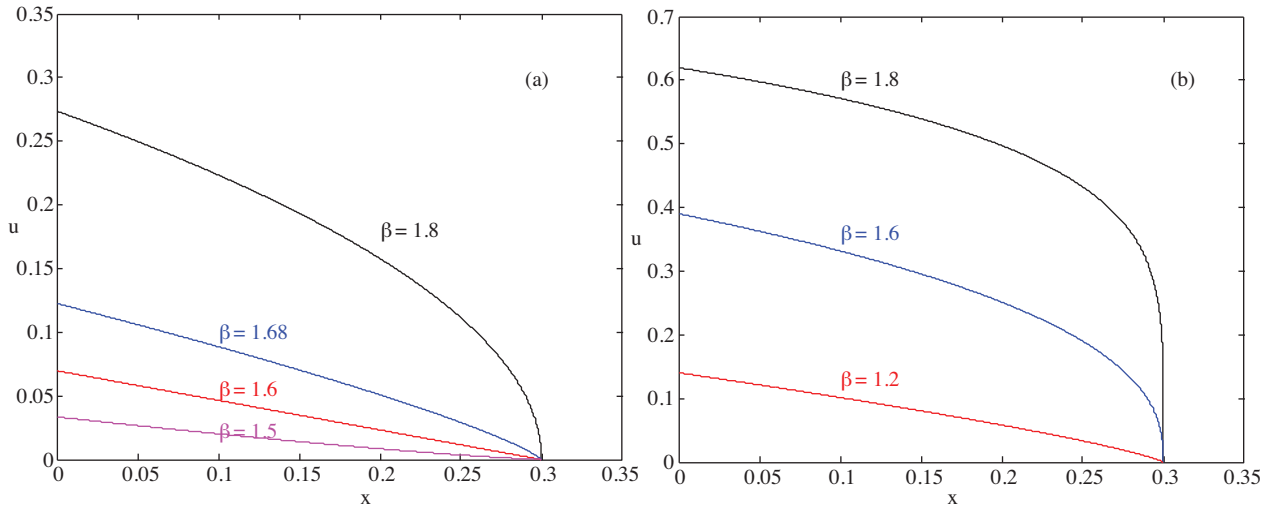


Figure 2. Wave profiles of GFB(m, n) equations for $t = 0.3, c = l_1 = l_2 = 1$ and different values of β : (a) $n = 1.4$, and (b) $n = 2$.

- (i) If $\alpha = 1$, then $m = 1$ and $a = \frac{2}{r-1}$.
- (ii) If $0 < \alpha < 1$, then $a = \frac{1-\alpha}{m-1} = \frac{3-\alpha}{r-1}$. In this case, the value of φ_1 is as follows:
 - (A) If $l_2 = -l_3$, then $\varphi_1 = \left[\frac{c^\alpha (m-1) \Gamma(\frac{m-\alpha}{m-1})}{m(1-\alpha) l_1 \Gamma(\frac{m(1-\alpha)}{m-1})} \right]^{\frac{1}{m-1}}$.
 - (B) If one of the cases is satisfied: (a) $a = \frac{1}{r}$, (b) $a = \frac{2}{r}$, then $\varphi_1 = 0$.
 - (C) If $l_1 = 0$, then $\varphi_1 = \left[\frac{c^\alpha \Gamma(\frac{2+r-\alpha}{r-1})}{(l_2+l_3) \Gamma(\frac{4r-r\alpha-1}{r-1})} \right]^{\frac{1}{r-1}}$.

In (3.6), φ_1 represents the amplitude of a wave, which propagates along xy plane with the speed c . If we regard $x + y$ as a coordinate axis, then the wave profiles of STSI traveling wave solutions are similar to those defined by GFB(m, n) (see Figure 1 and Figure 2).

4. Summary and conclusions

In this paper, we have considered the space-time scaling invariant traveling wave solutions of equation (1.2). A basic principle (homogeneous principle) is proposed to seek the traveling wave solutions expressed by power functions which are invariant under space-time scaling transformations. The effectiveness of this method is confirmed by applying to GFB(m, n) equations and GFZK equations. The traveling wave solutions expressed by power functions have been obtained under some given parameter conditions for GFB(m, n) equations and GFZK equations. These solutions may be helpful to describe waves features for some fractional partial differential equations in physics. It is worth mentioning that the homogeneous principle proposed in Section 2 is also applicable to the case in which L is a polynomial function with respect to powers of $u = u(x_1, x_2, \dots, x_n, t)$ and their partial derivatives of any order.

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