Temperature dependence for the synthetic magnetization of a rotating Bose gas in harmonic trap

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Abstract: In this paper, the temperature dependence for synthetic magnetization $\bar{M}$ of a rotating Bose gas in harmonic trap and the magnetic field dependence of the critical temperature are investigated. A simple semiclassical approximation for the density of state approach is used in this study. The suggested approach simultaneously gives corrections to quantities which arise from the finite size and interatomic interaction effects. The calculated results show that the critical temperature decreases due to the increase of the synthetic magnetic field. The synthetic magnetization also exhibits a characteristic temperature dependence. It is proportional to T^3 if $\bar{T} < \bar{T}_{\mathrm{c}}$, while it is proportional to T if $\bar{T} > \bar{T}_{\mathrm{c}}$. The calculated results provide a solid theoretical foundation of the current experiments.

Key words: Synthetic magnetization of neutral boson; Static properties of condensates; Semiclassical theories and applications; Boson systems

1. Introduction

Rotating Bose-Einstein condensation (BEC) offers several promising ways to simulate a number of models, Hamiltonians, and physical systems. Owing to the mathematical similarity between Coriolis and Lorentz forces, rotating neutral gases are indeed the exact analogue of an assembly of charged particles plunged in a magnetic field [1, 2, 3]. This scenario can be seen by exploiting the equivalence of the Lorentz force $q\mathbf{v} \times B$ in a magnetic field $\mathbf{B} = 2(m/q)\Omega$ and the Coriolis force $2m\dot{\mathbf{r}} \times \Omega$ in a rotating condensate at the frequency Ω. In this case, the Coriolis force associated with the vector potential is chosen such that $\mathbf{A} = -\frac{1}{2}\mathbf{B} \times \mathbf{r}$. Recently, an alternative approach to spin-up neutral atomic Bose gases is introducing a synthetic magnetic field instead of rotating the frame [4, 5, 6].

Physically, both approaches mentioned above lead to trick the neutral condensate into acting as if electrically charged. However, in a study of charged boson gas, it was indicated that an arbitrarily small value of the magnetic field can eliminate BEC in 3-dimensions [7]. Furthermore, the orbital motion result in extremely large Landau diamagnetism and even leads to Meissner effect at low temperatures [8].

Several experimental [9, 10] and theoretical [11, 12] studies have considered the thermodynamic properties of ideal rotating boson gas as a function of the rotation rate. The studies indicate that the BEC critical temperature is irrelevant to the magnetic field, conflicting with the established intuition that the critical temperature decreases with the field increasing. However, in these studies the critical temperature was obtained

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by making a coordinate transformation for the radial harmonic trap frequency to include the rotation effect. Since the coordinate transformation cannot induce real rotation, rotation effect in this case is a transformation of the chemical potential to include the centrifugal force effects without affecting the critical temperature. Another important quantity to discuss is the magnetization. It was pointed out that the magnetization keeps invariant at all temperatures [13]. Therefore, an alternative modified approach to the standard semiclassical approximation is needed.

Whereas the preceding papers elaborate on the thermodynamic properties of such a rotating BEC with finite size and indirect interaction effects, including the condensate fraction, critical temperature and heat capacity [14], and effective widths, effective area, and the expansion energy [15], the present paper investigates the temperature dependence for synthetic magnetization $\bar{M}$ of a rotating Bose gas in harmonic trap. The semiclassical density of states approach is used [16, 17].

Outline of this paper is as follows. Section Two includes the quantum mechanics of a Bose particle in a synthetic magnetic field and the derivation of the DOS. The dependence of the critical temperature on the synthetic magnetization is given in Section Three. Temperature dependence for the magnetization is given in Section Four. Conclusion is given in the last section.

2. Quantum mechanics of a Bose particle in a synthetic magnetic field

Let us consider a system of an ideal Bose gas with mass m and charge q is placed in anisotropic harmonic potential and a uniform magnetic field $\mathbf{B}$. For a constant magnetic field the Hamiltonian is given by [2, 3]

$$\hat{H} = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A})^2 + \frac{1}{2}m\omega_\perp^2(x^2 + y^2) + \frac{1}{2}m\omega_z^2 z^2. \tag{1}$$

Here, $\{\omega_\perp (\equiv \omega_{x,y}), \omega_z\}$ are the effective trapping frequencies of the harmonic potential. For a constant magnetic field the vector potential $\mathbf{A}$ can be written as $\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$. Consequently, expanding $(\hat{\mathbf{p}} - q\mathbf{A})^2$ leads to

$$\hat{H} = \frac{-\hbar^2}{2m}\nabla^2 + \frac{q}{2m}\mathbf{B} \cdot \hat{\mathbf{L}} + \frac{1}{2}m\left(\omega_\perp^2 + \frac{q^2}{4m}B^2\right)(x^2 + y^2) + \frac{1}{2}m\omega_z^2 z^2. \tag{2}$$

The term $\frac{q}{2m}\hat{\mathbf{L}} \cdot \mathbf{B}$ is known as the paramagnetic term representing the Lorentz force; $\hat{\mathbf{L}}$ is the angular momentum operator and follows the Coulomb gauge condition; $\nabla \cdot \mathbf{A} = 0$; and the term $\frac{q^2 B^2}{4m}(x^2 + y^2)$, which is known as the diamagnetic term, is obtained by choosing the magnetic field to lie along the z-axis, i.e. $\mathbf{B} = B\,\hat{e}_z$.

In a reference frame rotating at frequency Ω about the z-axis, the non-interacting single particle Hamiltonian for Bose gas is given by $\hat{H}^{\mathrm{rot}} = p^2/(2m) + \frac{1}{2}m\omega_\perp^2(x^2 + y^2) + \frac{1}{2}m\omega_z^2 z^2 - \mathbf{\Omega} \cdot \mathbf{r} \times \hat{\mathbf{p}}$. Completing the square leads to

$$
\begin{aligned}
\hat{H}^{\mathrm{rot}} &= \frac{1}{2m}(\hat{\mathbf{p}} - m\mathbf{\Omega} \times \mathbf{r})^2 + \frac{1}{2}m(\omega_\perp^2 - \Omega^2)(x^2 + y^2) + \frac{1}{2}m\omega_z^2 z^2 \\
&\equiv \frac{-\hbar^2}{2m}\nabla^2 + \mathbf{\Omega} \cdot \hat{\mathbf{L}} + \frac{1}{2}m(\omega_\perp^2 - \Omega^2)(x^2 + y^2) + \frac{1}{2}m\omega_z^2 z^2
\end{aligned} \tag{3}
$$

A direct comparison between equation (2) and equation (3) showed the two Hamiltonian, have the same mathematical structure. The kinetic term in the rotating Hamiltonian is equivalent to that of a particle with

charge $q^* = 1$ placed in a uniform *synthetic* magnetic field $\mathbf{B} = \frac{2m}{q^*}\mathbf{\Omega}$ and confined in a harmonic potential of frequencies $\sqrt{\omega_\perp^2 - \Omega^2}$, ω_z. Thus the synthetic magnetic field is proportional to the rotation frequency Ω. The Coriolis force in the rotating frame, which is $m\mathbf{\Omega} \times \mathbf{r}$, plays the same role as the Lorentz force on a moving charged particle in a uniform magnetic field. One difference arises from the centrifugal potential $-\frac{m}{2}\Omega^2(x^2 + y^2)$ in the rotation frame which has no analogue in a magnetic field. The centrifugal force acts to reduce the harmonic confinement potential. However, the stability of a rotating harmonically trapped gas requires that $\Omega \leq \omega_\perp$.

When $\Omega < \omega_\perp$, the eigenvalues of the Hamiltonian given in equation (3) in the rotating frame are given by [18, 19]

$$
\begin{aligned}
E(n_z, m_z, n_m) &= (n_z + \frac{1}{2})\hbar\omega_z + (2n_m + |m_z| + 1)\hbar\omega_\perp - m_z\hbar\Omega \\
&= n_z\hbar\omega_z + (2n_m + |m_z|)\hbar\omega_\perp - m_z\hbar\Omega + E_0,
\end{aligned}
\tag{4}
$$

where $E_0 = (\frac{1}{2}\hbar\omega_z + \hbar\omega_\perp)$ is the ground state energy, while $n_z = 0, 1, \ldots$ is the vibrational quantum number along z; and $n_m = 0, 1, \ldots$ characterizes the excitation energy in the xy-plane for a state with a given m_z, where m_z is the angular momentum quantum number about the rotation axis z.

In equation (4) the Landau level structure appears when $\omega_\perp \to \Omega$. The Landau levels are spaced by the energy $\hbar(\omega_\perp + \Omega)$, and the lowest Landau level contains states of angular momentum $m_z = 0, 1, 2, \ldots$, the first Landau level states with $m_z = -1, 0, 1, \ldots$, and so on. Two adjacent states within a Landau level are separated by the energy $\hbar(\omega_\perp - \Omega)$.

2.1. The precise DOS of the synthetic magnetic field

The precise DOS for the synthetic magnetic field can be parametrized using a normalized single particle energy eigenvalues in terms of $\hbar\omega_z$. Thus from equation (4) one has

$$
\begin{aligned}
\bar{\epsilon}_{n_z, m_z, n_m} &= \frac{E(n_z, m_z, n_m)}{\hbar\omega_z} - \frac{E_0}{\hbar\omega_z} \\
&= \bar{E}(n_z, m_z, n_m) - \bar{E}_0 \\
&= n_z + (2n_m + |m_z|)\lambda - m_z\bar{B},
\end{aligned}
\tag{5}
$$

where $\lambda = \frac{\omega_\perp}{\omega_z}$ is the harmonic oscillate aspect ratio and $\bar{B} = \frac{\Omega}{\omega_z}$ is proportional to the synthetic magnetic field. In terms of $\bar{\epsilon}_{n_z, m_z, n_m}$, the single-particle partition function is given by $\bar{Q}(\beta) = \sum_{n_z, m_z, n_m} e^{-\beta\bar{\epsilon}_{n_z, m_z, n_m}}$, with $\beta = (1/K_B T)$ [20], thus

$$
\begin{aligned}
\bar{Q}(\beta) &= \sum_{n_z, n_m = 0; \, m_z = -n_m}^{\infty} e^{-\beta n_z} \, e^{-2\beta n_m \lambda} \, e^{-\beta(|m_z|\lambda - m_z\bar{B})} \\
&= \sum_{n_z, n_m, m_z = 0}^{\infty} e^{-\beta n_z} \, e^{-\beta n_m(\lambda + \bar{B})} \, e^{-\beta m_z(\lambda - \bar{B})} \\
&= \frac{1}{\beta^3(\lambda^2 - \bar{B}^2)}\left\{1 + \frac{\beta}{2}(2\lambda + 1) + \frac{\beta^2}{8}\{(2\lambda + 1)^2 - \frac{1}{3}[2\lambda^2 + 2\bar{B}^2 + 1]\} + \cdots\right\},
\end{aligned}
\tag{6}
$$

where the asymptotic high-temperature expansion (small β) is used here.

On the other hand, the single-particle partition function can be calculated from the relation

$$\bar{Q}(\beta) = \int_0^\infty \bar{\rho}(\bar{\epsilon}) e^{-\beta\bar{\epsilon}} d\bar{\epsilon}, \tag{7}$$

where $\bar{\rho}(\bar{\epsilon})$ is the DOS. From a direct comparison of equation (7) with equation (6) an approximated formula for the DOS for the single particle system is obtained as

$$\bar{\rho}(\bar{\epsilon}) = (\lambda^2 - \bar{B}^2)^{-1} \left\{ \frac{1}{2} \ \bar{\epsilon}^2 + \frac{1}{2}(2\lambda + 1) \ \bar{\epsilon} + \frac{1}{8}\left\{ (2\lambda+1)^2 - \frac{1}{3}[2\lambda^2 + 2\bar{B}^2 + 1] \right\} \right\} \tag{8}$$

The last term in the DOS parametrized in equation (8) produces $\zeta(1)$ function when the DOS is used in calculating the thermodynamic quantities. Since, $\zeta(1) = \infty$, then T_c (critical temperature for BEC) is undefined at the onset of the condensation. This divergence can be removed by using Robinson's formula for Bose function, $g_l(e^{-a}) = \frac{(-a)^{l-1}}{(l-1)!}\left\{ \sum_{k=1}^{l-1} \frac{1}{k} - \ln a \right\} + \sum_{k=0,k\neq l-1}^{k=\infty} \frac{(-a)^k}{k!}\zeta(l-k)$. Following Klünder and Pelster [21] for $\lambda/K_B\bar{T} < 1$, one has $\zeta(1) = \ln[\frac{2K_B\bar{T}}{\lambda}] \cong \ln[2\frac{T}{T_0}(N/\zeta(3))^{1/3}]$, with T_0 being the BEC transition temperature for the ideal gas. However, in this approximation $\zeta(1)$ corresponds to logarithmic corrections in the particle number. More accurate treatments for the problem can be considered.

Indeed, one can overcome this limitation by multiplying the last term by $\frac{E_0}{\frac{1}{2}\hbar(2\omega_\perp + \omega_z)} \equiv \frac{\bar{E}_0}{\frac{1}{2}(2\lambda+1)}(\equiv 1)$. Here, $\bar{E}_0 \equiv \frac{E_0}{\hbar\omega_z}$. (Any other choice in terms of higher energy E_n is not acceptable.) Moreover, using the essential criterion for BEC phenomenon, which is $\bar{\mu}(\Omega) \equiv \frac{\mu(\Omega)}{\hbar\omega_z} = \bar{E}_0$, the DOS will include the chemical potential for this system. Thus equation (8) becomes

$$\bar{\rho}(\bar{\epsilon}) = (\lambda^2 - \bar{B}^2)^{-1} \left\{ \frac{1}{2} \ \bar{\epsilon}^2 + \frac{2\lambda+1}{2} \ \bar{\epsilon} + \frac{1}{8}\left\{ (2\lambda+1)^2 - \frac{1}{3}[2\lambda^2 + 2\bar{B}^2 + 1] \right\} \frac{\bar{\mu}(\Omega)}{\frac{1}{2}(2\lambda+1)} \right\}. \tag{9}$$

The dependence of the chemical potential on the rotation frequency is given by [18]

$$\bar{\mu}(\mathbf{\Omega}) = \bar{\mu}(0)\left(1 - \frac{\mathbf{\Omega^2}}{\omega_\perp^2}\right)^{2/5}, \tag{10}$$

where $\bar{\mu}(0) \equiv \frac{\mu(0)}{\hbar\omega_z}$ is the chemical potential for the non-rotating condensate. The latter can be expressed in terms of the dimensionless Dalfovo's interaction parameter η [17]. This parameter is determined by the ratio between the chemical potential in the absent of the rotation and the transition temperature $\bar{T}_0(0)$ for the non-interacting particles in the same trap,

$$\bar{T}_0(0) = \frac{\lambda^{2/3}}{K_B}\left(\frac{N}{\zeta(3)}\right)^{1/3}. \tag{11}$$

In terms of η and the synthetic magnetic field, the chemical potential is given by

$$\bar{\mu}(\bar{B}) = \eta \ K_B\bar{T}_0(0) \left(1 - \frac{\bar{B}^2}{\lambda^2}\right)^{2/5} \tag{12}$$

with $K_B\bar{T}_0(0) = K_B T_0(0)/\hbar\omega_z$. In the thermodynamic limit η goes to zero [22].

Generalization to the many particle system is straightforward. The accurate DOS for the many particles system is given by [16]

$$\bar{\rho}(\bar{\epsilon}) = (\lambda^2 - \bar{B}^2)^{-1} \left[\frac{1}{2} \ \bar{\epsilon}^2 + \bar{\epsilon} \ \left\{ \frac{2\lambda + 1}{2} \ + \frac{1}{8} \left\{ (2\lambda + 1)^2 - \frac{1}{3}[2\lambda^2 + 2\bar{B}^2 + 1] \right\} \frac{\bar{\mu}(\bar{B})}{\lambda(2\lambda + 1)} \right\} \right]. \tag{13}$$

3. BEC critical temperature

In the DOS approach, most of the thermodynamical quantities can be calculated from the partial derivatives of the q potential, which is given by the logarithm of the canonical partition function as [23]

$$\bar{q} = -\sum_{n=0}^{\infty} \ln(1 - ze^{-\bar{\beta}\bar{E}_n}), \tag{14}$$

where $z = e^{\bar{\beta}(\bar{\mu}(\Omega) - \bar{E}_0)}$ is the fugacity and $\bar{\beta} = \hbar\omega_z/K_B T$. Using the semiclassical approximation (in which the summation over n is converted into an integral weighted by an accurate DOS) to calculate q requires expanding the logarithm to express the q potential as a sum over Bose-Einstein distribution:

$$
\begin{aligned}
\bar{q} &= \bar{q}_0 + \sum_{n=1}^{\infty} \frac{ze^{-\beta\bar{\epsilon}_n}}{1 - ze^{-\beta\bar{\epsilon}_n}} \\
&= \bar{q}_0 + \sum_{j=1}^{\infty} \frac{z^j}{j} \sum_{n=1}^{\infty} e^{-j\bar{\beta}\bar{\epsilon}_n} \\
&= \bar{q}_0 + \sum_{j=1}^{\infty} \frac{z^j}{j} \int_0^{\infty} \bar{\rho}(\bar{\epsilon}) e^{-j\beta\bar{\epsilon}} d\bar{\epsilon}.
\end{aligned} \tag{15}
$$

Here, $\bar{q}_0 = -\ln(1 - ze^{-\bar{\beta}\bar{E}_0})$. The first term in q is the part which accounts for a possible condensate, and the second term is the standard expression for the q potential from statistical mechanics. Substituting equation (13) in (15) leads to

$$\bar{q} = \bar{q}_0 + (\lambda^2 - \bar{B}^2)^{-1} \left\{ (K_B\bar{T})^3 g_4(z) + (K_B\bar{T})^2 g_3(z) R(\bar{B}) \right\}, \tag{16}$$

where $g_k(z) = \sum_{j=1}^{\infty}(z^j/j^k)$ is the usual Bose-Einstein function, and the parameter $R(\Omega)$ is given by

$$R(\bar{B}) = \frac{2\lambda + 1}{2} \ + \frac{\eta}{8} \frac{K_B\bar{T}_0(0)}{\lambda(2\lambda + 1)} \ \left(1 - \frac{\bar{B}^2}{\lambda^2}\right)^{2/5} \left\{ (2\lambda + 1)^2 - \frac{1}{3}[2\lambda^2 + 2\bar{B}^2 + 1] \right\}. \tag{17}$$

The synthetic magnetic field dependent critical temperature can be calculated from the total number of particles $N = z \left(\frac{\partial q}{\partial z} \right)_T$,

$$N = N_0 + (\lambda^2 - \bar{B}^2)^{-1} \left\{ (K_B\bar{T})^3 g_3(z) + (K_B\bar{T})^2 g_2(z) R(\bar{B}) \right\}, \tag{18}$$

with N_0 being the number of condensate particles in the ground state.

The critical temperature $\bar{T}_c(\bar{B})$ is the value of the temperature at which the chemical potential is equal to the ground state energy, formally $\bar{\epsilon}_0 - \bar{\mu}(\Omega) \to 0$. However, this condition is required to vanish N_0 at $\bar{T} = \bar{T}_c$ [14]. Thus

$$\bar{T}_c(\bar{B}) = \bar{T}_0(0) \left(1 - \frac{\bar{B}^2}{\lambda^2}\right)^{1/3} \left[1 - \frac{1}{3} \ R_1(\bar{B})\right] \tag{19}$$

In absence of the synthetic magnetic field the critical temperature is just the BEC critical temperature for ideal Bose gas of N particles confined in anisotropic trap with the frequencies $\omega_\perp$ in x, y plane and ω_z in the z direction. As it should be, when $\bar{B} = \lambda$, the trapping force is canceled by the centrifugal effect. In this case BEC does not take place.

In the thermodynamic limit $R_1(\bar{B}) \to 0$ [16, 22], while for finite number of atoms the parameter R_1 provides the correction to the result of the ideal gas due to finite size, interaction, and synthetic magnetic field effects:

$$R_1(\bar{B}) = \frac{\zeta(2)}{\zeta(3)} \left[\frac{2\lambda + 1}{2[\lambda^2 - \bar{B}^2]^{1/3}} \left(\frac{\zeta(3)}{N}\right)^{1/3} + \frac{\eta}{8} \frac{(1 - \bar{B}^2/\lambda^2)^{2/5}}{\lambda(2\lambda + 1)} \left\{(2\lambda + 1)^2 - \frac{1}{3}[2\lambda^2 + 2\bar{B}^2 + 1]\right\}\right]. \tag{20}$$

In the bracket the first term gives the finite size effect while the second term accounts for the interaction effects. It is interesting to notice that the effect of the synthetic magnetic field is coupled with the finite size and interaction effects. This entails that within the semiclassical approximation the effect of Coriolis force on the critical temperature, which is associated with the vector potential $\mathbf{A}$, can be seen as a correction term to the ideal gas results given in references [11, 19].

Figure 3 graphically shows the calculated results from equation (19) for different values of η. This figure reveals that the normalized critical temperature $\bar{T}_c(\bar{B})/\bar{T}_0(0)$ decreases monotonically with increase of the synthetic magnetic field $\bar{B}$. This behavior is consistent with superconductivity properties when exposed to a magnetic field, where the transition temperature decreases as the magnetic field is strengthened. At higher values of $\bar{B}$ there is no difference between the normalized critical temperature curves for all η values. Thus the interaction effect is strongly quenched for higher values of $\bar{B}$.

4. Synthetic magnetization

In the semiclassical approach the synthetic magnetization is calculated using the relation $\bar{M} = -\partial \bar{q}/\partial \bar{B}$, which yields two different behaviors as functions of magnetization temperature:

1. When the temperature $\bar{T}$ is less than BEC transition temperature for the ideal gas $\bar{T}_0(\bar{B}) = \frac{[(\lambda^2 - \bar{B}^2)\frac{N}{\zeta(3)}]^{1/3}}{K_B}$, one has the magnetization per particle,

$$\begin{aligned} \bar{M} &= - \left\{\bar{M}_0 - \frac{2\bar{B}}{(\lambda^2 - \bar{B}^2)^2} \left\{(K_B\bar{T})^3 g_4(z) + (K_B\bar{T})^2 g_3(z) \left[R(\bar{B}) - \frac{(\lambda^2 - \bar{B}^2)}{2\bar{B}} \frac{\partial R(\bar{B})}{\partial \bar{B}}\right]\right\}\right\} \\ &= - \left\{\bar{M}_0 - \bar{M}_c \ \chi_1 \ t^2\right\}. \end{aligned} \tag{21}$$

Here, $\bar{M}_c$ denotes the critical magnetization at the transition temperature; and $t = \bar{T}/\bar{T}_0(\bar{B})$ is the
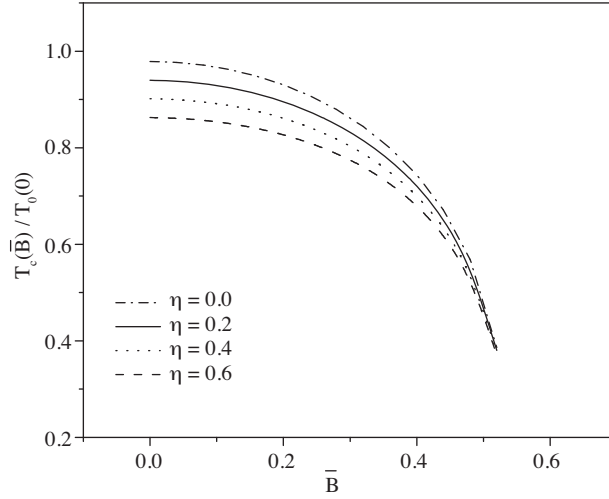
Figure 1. The normalized critical temperature $\bar{T}_c(\bar{B})/\bar{T}_0(0)$ as a function of the synthetic magnetic field $\bar{B}$.

reduced temperature:

$$\bar{M}_c = \frac{\zeta(4)}{\zeta(3)} \frac{2\bar{B}}{\lambda^2 - \bar{B}^2}; \quad \chi_1 = t + R_2(\bar{B})$$

$$R_2(\bar{B}) = \frac{\zeta(3)}{\zeta(4)} \left(\frac{\zeta(3)}{N}\right)^{1/3} \frac{1}{(\lambda^2 - \bar{B}^2)^{1/3}} \left\{ R(\bar{B}) - \frac{(\lambda^2 - \bar{B}^2)}{2\bar{B}} \frac{\partial R(\bar{B})}{\partial \bar{B}} \right\} \tag{22}$$

2. The synthetic magnetization per a particle for $T \geq T_0(\bar{B})$ becomes

$$\bar{M} = - \left\{ \bar{M}_0 - \frac{2\bar{B}}{(\lambda^2 - \bar{B}^2)^2} \left\{ (K_B\bar{T})^3 g_4(z) + (K_B\bar{T})^2 g_3(z) \left[R(\bar{B}) - \frac{(\lambda^2 - \bar{B}^2)}{2\bar{B}} \frac{\partial R(\bar{B})}{\partial \bar{B}} \right] \right\} \right\}$$

$$\approx - \left\{ \bar{M}_0 - \bar{M}_c \, \chi_2 \right\} \tag{23}$$

where

$$\chi_2 = \alpha t + R_3(\bar{B})$$

$$R_3(\bar{B}) = \frac{\zeta(3)}{\zeta(4)} \left(\frac{g_3(z)}{N}\right)^{1/3} \frac{1}{(\lambda^2 - \bar{B}^2)^{1/3}} \left\{ R(\bar{B}) - \frac{(\lambda^2 - \bar{B}^2)}{2\bar{B}} \frac{\partial R(\bar{B})}{\partial \bar{B}} \right\} \tag{24}$$

and $\alpha = \frac{g_4(z)\ \zeta(3)}{g_3(z)\ \zeta(4)} \approx 1$ has a very weak dependence on T.

Now we come to our analysis for the obtained results in equations (21) and (23) for the synthetic magnetization. There are two cases we need to consider: the temperature dependence and the synthetic magnetic field dependence. For the temperature dependence, we note that the approximation made concerning the two dimensionless scales $K_B\bar{T}_0$ and λ. In most of the traps, the former is about 2 orders of magnitude greater than the latter, so semiclassical approximation is expected to work well in these systems in a wide range

of temperatures. Moreover, for high temperature $K_B \bar{T} >> \lambda$, as is normal in this case, the contribution from the condensate can be ignored. In the thermodynamic limit, equations (21) and (23) give

$$
\begin{aligned}
\frac{\bar{M}}{\bar{M}_c} &\propto t && \text{for } T > T_0(\bar{B}) \\
&\propto t^3 && \text{for } T < T_0(\bar{B}).
\end{aligned}
\tag{25}
$$

Thus the synthetic magnetization is proportional to t^3 for $T < T_0(\bar{B})$ and to t for $T > T_0(\bar{B})$. Note that in the thermodynamic limit $N \to \infty$, both $R_2(\bar{B}), R_3(\bar{B}) \to$ zero, and one has $\chi_1, \chi_2 \to t$.

When condensation is reached, $\bar{M}_0 \neq 0$, the synthetic magnetization $\bar{M}$ decreases faster as the temperature is lowered. This concluding remark is qualitatively in agreement with the recently experimental measured data of Pasquiou et al. [9].

Finally, the magnetization dependence on the synthetic magnetic field can be estimated from equations (21) and (23) as follows. Below the BEC transition temperature, the synthetic magnetization has maximum value $-\bar{M}_0$, at $t = 0$. When the temperature departs from zero, the magnetization decreases to its minimum value, $\bar{M} = 0$ at $\bar{T} = \bar{T}_0(\bar{B})$.

5. Conclusion

In this paper the semiclassical approximation is used to calculate the critical temperature and the synthetic magnetization for synthetically charged boson gas. The calculated results are modified significantly due to inclusion of finite size and interaction effects. The present method and the calculated results are applicable to the charged atomic Bose gas in external magnetic field. In both systems, the effect of synthetic (external) magnetic field is coupled with the finite size and interaction effects. The critical temperature decreases with increase in synthetic (external) magnetic field. The synthetic magnetization vanishes when the synthetic magnetic field is reduced to zero, implying that the Meissner-Ochsenfeld effect might not exist. Moreover, the magnetization is stronger and negative at lower temperatures; consequently the trapped rotating boson gas exhibits Landau diamagnetism. The synthetic magnetization is proportional to t^3 for $T < T_0(\bar{B})$ and to t for $T > T_0(\bar{B})$.

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