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Spin-one DKP equation in the presence of nonminimal vector Woods–Saxon potential in $(1+1)$ dimensions

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Abstract: The Duffin–Kemmer–Petiau equation in the presence of a nonminimal vector Woods–Saxon potential is solved for spin-one particles via the Nikiforov–Uvarov method. The reflection and transmission coefficients of the general form of the DKP equation in the presence of a nonminimal vector interaction are discussed.

Key words: DKP equation, nonminimal vector Woods–Saxon potential, Nikiforov–Uvarov method

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1. Introduction

The Duffin–Kemmer–Petiau (DKP) equation is a first-order relativistic wave equation that considers spin-zero and spin-one bosons [1–4]. It is similar to the Dirac equation in structure, but, as the forthcoming Eq. (4) reveals, the γ matrices are replaced by the so-called β matrices [5]. DKP matrices have 3 irreducible representations: a 1-dimensional representation that is trivial, a 5-dimensional representation that is for spin-zero particles, and a 10-dimensional formulation that is for spin-one particles [6]. Recently, some articles have been published that investigate the equation with different types of potentials [7–13]. Such studies are motivated by the successful outcomes of DKP theory in various physical fields including particle and nuclear physics [1–5]. The main purpose of the present article is to investigate the DKP equation with a nonminimal vector Woods–Saxon (WS) potential in $(1+1)$-dimensions for spin-one particles. The mean field WS potential is a short-range interaction that has been successfully tested within the framework of the nuclear shell model [14–17]. It has also been successfully applied to investigate the interaction of a nucleon with a heavy nucleus [18]. The outline of this paper is as follows. In section (2), we review the essentials of the DKP equation. Section (3) introduces the equation in $(1+1)$ dimensional space-time. The next section is devoted to the powerful analytical Nikiforov–Uvarov (NU) method by which we solve the differential equation obtained. We report the energy eigenvalues and the wave functions in section (5). We derive the asymptotic behavior of the wave functions and thereby comment on the reflection and transmission coefficients. In the last section, we give our concluding remarks.

2. Duffin–Kemmer–Petiau equation

The DKP equation in Minkowski space-time is $(in\, natural\, units\ \hbar = C = 1)$ [1–3]

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$$(i\beta^{\mu}\partial_{\mu} - m)\Psi = 0, \tag{1}$$

and β^{μ} matrices satisfy in the following DKP algebra $\beta^{\alpha}\beta^{\mu}\beta^{\nu} + \beta^{\nu}\beta^{\mu}\beta^{\alpha} = \beta^{\alpha}\eta^{\mu\nu} + \beta^{\nu}\eta^{\mu\alpha}$, where $\eta^{\mu\nu}$ is the metric tensor with signature $(+---)$. The DKP equation is determined from the following lagrangian density [19]:

$$L = \frac{i}{2}\bar{\Psi}\beta^{\mu}\partial_{\mu}\Psi - \frac{i}{2}\partial_{\mu}\bar{\Psi}\beta^{\mu}\Psi - m\bar{\Psi}\Psi, \tag{2}$$

in which $\bar{\Psi} = \Psi^{+}\eta^{0}$, $\eta^{0} = 2(\beta^{0})^{2} - 1$, and $(\eta^{0})^{2} = 1$. The β^{μ} matrices for spin-one particles are given by [20]

$$\beta^{0} = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^{T} & 0 & I & 0 \\ \bar{0}^{T} & I & 0 & 0 \\ \bar{0}^{T} & 0 & 0 & 0 \end{pmatrix}, \qquad \beta^{i} = \begin{pmatrix} 0 & \bar{0} & e_{i} & \bar{0} \\ \bar{0}^{T} & 0 & 0 & -iS_{i} \\ -e_{i}^{T} & 0 & 0 & 0 \\ \bar{0}^{T} & -iS_{i} & 0 & 0 \end{pmatrix} \tag{3}$$

with S_i matrices being 3×3 ones, $(S_i)_{jk} = -i\varepsilon_{ijk}$, where ε_{ijk} takes the values 1, –1, and 0 for an even permutation, an odd permutation, and repeated indices, respectively. e_i Matrices are 1×3 ones, $(e_i)_{1j} = \delta_{ij}$, i.e. $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$, I and *0* respectively represent the identity and zero 3×3 matrices and $\bar{0}$s are 1×3 ones.

3. DKP equation with the Woods–Saxon potential

The DKP equation in $(1+1)$-dimensions is written as

$$(i\beta^{0}\partial_{0} + i\beta^{1}\partial_{1} - m - U)\Psi(x,t) = 0. \tag{4}$$

where the well-known ansatz $\Psi(x,t) = \exp(-iE_n t)\psi_n(x)$ removes the time component. We represent the 10-component stationary spinor as

$$\psi_n^{T}(x) = (\phi_n^{(1)}(x), \phi_n^{(2)}(x), \phi_n^{(3)}(x), \phi_n^{(4)}(x), \phi_n^{(5)}(x), \phi_n^{(6)}(x), \phi_n^{(7)}(x), \phi_n^{(8)}(x), \phi_n^{(9)}(x), \phi_n^{(10)}(x))^{T}. \tag{5}$$

The potential matrix U can be written in terms of well-defined Lorentz structures. For the spin-zero version of the DKP equation there are 2 scalar, 2 vector and 2 tensor terms Within the spin-one formulation, there are 2 scalar, 2 vector, 1 pseudoscalar, 2 pseudovector and 8 tensor terms. Because of noncausal effects the tensor terms have been avoided in applications [21,22] U is in the general form $U = \beta^{\mu}A^{\mu} + i[P,\beta^{\mu}]H_{\mu}$, where $[P,\beta^{\mu}] = P\beta^{\mu} - \beta^{\mu}P$ conserves the 4-current [23], $\beta^{\mu}A^{\mu}$ is minimally coupled, and $i[P,\beta^{\mu}]H_{\mu}$ is the nonminimal vector term. Therefore, we are considering the term $i[P,\beta^{\mu}]H_{\mu}$. Moreover, we study the term $U = i[P,\beta^{1}]H_{1}$, where $P = (\beta^{\mu}\beta_{\mu} - 2) = diag(1,1,1,1,0,0,0,0,0,0)$ is the projection operator [24]. For a deeper knowledge about the terms engaged, we refer to the work of Glass (1971), who constructed a basis for the algebra in terms of tensors constructed from the β_{μ}. On this basis, the 100 elements pertinent to the 10-dimensional (spin-1) representation appear as components of 15 irreducible tensors.

$$\begin{aligned}
&B = \beta_\mu\beta_\mu, \Gamma = \varepsilon_{k\lambda\mu\nu}\beta_k\beta_\lambda\beta_\mu\beta_\nu \\
&\alpha_\mu = \varepsilon_{\mu\lambda k\nu}\beta_\lambda\beta_k\beta_\nu, \sigma_{\mu\nu} = [\beta_\mu, \beta_\nu],\ B_{\mu\nu} = \{\beta_\mu, \beta_\nu\} \\
&S_{\mu\nu} = \{\beta_\mu, \alpha_\nu\} + \{\beta_\nu, \alpha_\mu\} \\
&T_{\mu\nu\rho} = i\{\beta_\mu, \sigma_{\nu\rho}\} + \tfrac{2}{3}\varepsilon_{\mu\nu\rho k}\alpha_k - \tfrac{1}{3}i\delta_{\mu\nu}[B, \beta_\rho] + \tfrac{1}{3}i\delta_{\mu\rho}[B, \beta_\nu], \\
&Y_{k\lambda\mu\nu} = \{\sigma_{k\mu}, \sigma_{\lambda\nu}\} - \{\sigma_{k\nu}, \sigma_{\lambda\mu}\} - \tfrac{1}{2}\delta_{\kappa\mu}\{\sigma_{\rho\nu}, \sigma_{\rho\lambda}\} - \tfrac{1}{2}\delta_{\lambda\nu}\{\sigma_{\rho\mu}, \sigma_{\rho\kappa}\} + \\
&\tfrac{1}{2}\delta_{\kappa\nu}\{\sigma_{\rho\mu}, \sigma_{\rho\lambda}\} + \tfrac{1}{2}\delta_{\lambda\mu}\{\sigma_{\rho\nu}, \sigma_{\rho\kappa}\} + \tfrac{1}{6}(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu})\{\sigma_{\rho\alpha}, \sigma_{\rho\alpha}\}.
\end{aligned} \tag{6a}$$

where the summation convention is assumed. With these definitions, the DKP equation can now include all possible nonminimal and minimal interactions:

(scalar)	$1; B$
(pseudoscalar)	Γ
(vector);	$\beta_\mu : i[B, \beta_\mu]$
(*pseudo vector*);	$\alpha_\mu : i[B, \alpha_\mu]$
(symmetric traceless tensor);	$B_{\mu\nu} - \frac{1}{2}g_{\mu\nu}B : \{B, B_{\mu\nu}\} - g_{\mu\nu}B^2$ (6b)
(symmetric traceless pseudo tensor);	$S_{\mu\nu}$
(antisymmetric tensor);	$i[\beta_\mu, \beta_\nu] : i\{B, [\beta_\mu, \beta_\nu]\}$
	$T_{\mu\nu\rho} : i\{B, T_{\mu\nu\rho}\}$
	$Y_{\kappa\lambda\mu\nu}$

Substitutions of the latter in Eq. (4) give the following coupled equations:

$$i\frac{\partial\phi_n^{(5)}(x)}{\partial x} - m\phi_n^{(1)}(x) - iH_1\phi_n^{(5)}(x) = 0, \qquad E_n\phi_n^{(3)}(x) - m\phi_n^{(6)}(x) = 0 \tag{7}$$

$$E_n\phi_n^{(5)}(x) - m\phi_n^{(2)}(x) = 0, \qquad E_n\phi_n^{(4)}(x) - m\phi_{n,7}(x) = 0 \tag{8}$$

$$E_n\phi_n^{(6)}(x) - i\frac{\partial\phi_n^{(10)}(x)}{\partial x} - m\phi_n^{(3)}(x) + iH_1\phi_n^{(10)}(x) = 0, \qquad m\phi_n^{(8)}(x) = 0 \tag{9}$$

$$E_n\phi_{n,7}(x) + i\frac{\partial\phi_n^{(9)}(x)}{\partial x} - m\phi_n^{(4)}(x) - iH_1\phi_n^{(9)}(x) = 0, \qquad i\frac{\partial\phi_n^{(4)}(x)}{\partial x} + m\phi_n^{(9)}(x) + iH_1\phi_n^{(4)}(x) = 0 \tag{10}$$

$$E_n\phi_n^{(2)}(x) - i\frac{\partial\phi_n^{(1)}(x)}{\partial x} - m\phi_n^{(5)}(x) - iH_1\phi_n^{(1)}(x) = 0, i\frac{\partial\phi_n^{(3)}(x)}{\partial x} - m\phi_n^{(10)}(x) + iH_1\phi_n^{(3)}(x) = 0. \tag{11}$$

Combination of the above equations yields $\phi_n^{(3)}(x) = \phi_n^{(4)}(x)$ and we arrive at

$$\frac{d^2\phi_n^{(3)}(x)}{dx^2} + (E_n^2 - m^2 - H_1^2)\phi_n^{(3)}(x) + \frac{\partial H_1}{\partial x}\phi_n^{(3)}(x) = 0. \tag{12}$$

Let us now enter the WS potential (see Figure 1) [25–29]

$$H_1(x) = V_0[\frac{\theta(-x)}{1+\exp(\frac{x-a}{b})} + \frac{\theta(x)}{1+\exp(\frac{x-a}{b})}], \tag{13}$$

where V_0, a, and b are constant, to obtain

$$\frac{d^2\phi_n^{(3)}(x)}{dx^2} + (E_n^2 - m^2 - \frac{V_0^2}{(1+\exp(\frac{x-a}{b}))^2} - \frac{V_0 \exp(\frac{x-a}{b})}{b(1+\exp(\frac{x-a}{b}))^2})\phi_n^{(3)}(x) = 0 \tag{14}$$

As we are searching for the scattering states of the equation for a WS potential barrier, we study the wave functions for $x > 0$ and $x < 0$. For the region $x \succ 0$, we choose $z = \frac{1}{1+e^{\frac{x-a}{b}}}$, and Eq. (14) appears as

$$\frac{d^2\phi_n^{(3,R)}(z)}{dz^2} + \frac{1-2z}{z(1-z)}\frac{d\phi_n^{(3,R)}(z)}{dz} + \frac{1}{z^2(1-z)^2}[b^2(E_n^2 - m^2) + (-V_0^2 b^2 + V_0 b)z^2 - V_0 bz]\phi_n^{(3,R)}(z) = 0, \tag{15}$$

For $x \prec 0$, we use a change of variable of the form $y = \frac{1}{1+e^{\frac{-x-a}{b}}}$ and arrive at

$$\frac{d^2\phi_n^{(3,L)}(y)}{dy^2} + \frac{1-2y}{y(1-y)}\frac{d\phi_n^{(3,L)}(y)}{dy} + \frac{1}{y^2(1-y)^2}[b^2(E_n^2 - m^2) + (-V_0^2 b^2 + V_0 b)y^2 - V_0 by]\phi_n^{(3,L)}(y) = 0. \tag{16}$$

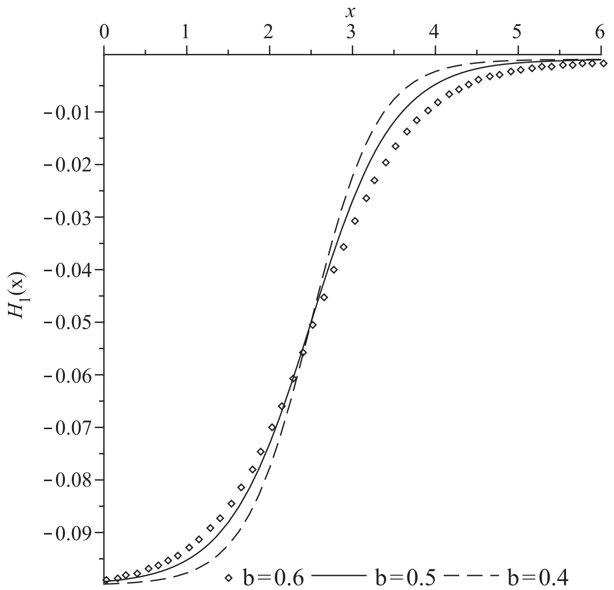

Figure 1. Woods–Saxon potential vs. x.

4. Nikiforov–Uvarov method

4.1. The idea

The NU method enables us to solve second-order differential equations. We include some lines, namely Eqs. (17) to (26), from [30] to illustrate the underlying idea. The NU method rewrites a second-order differential equation as

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{17}$$

where prime shows the derivative with respect to s. $\sigma(s)$ and $\tilde{\sigma}(s)$ must be polynomials of at most second degree and $\tilde{\tau}(s)$ is a polynomial of at most first degree. Using the below transformation

$$\psi(s) = W(s)\Phi(s), \tag{18}$$

Eq. (17) is reduced to the hypergeometric-type equation

$$\sigma(s)\Phi''(s) + \tau(s)\Phi'(s) + \lambda\Phi(s) = 0, \tag{19}$$

and

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s). \qquad \text{with} \qquad \tau'(s) < 0 \tag{20}$$

Hence, Eq. (17) has a particular solution of degree n when the following relation is satisfied

$$\lambda = \lambda_n = -n\,\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \qquad \text{with} \qquad n = 0, 1, 2, ... \tag{21}$$

In order to obtain the energy eigenvalues equation, the definitions given below are required in the NU method

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{(\frac{\sigma'(s) - \tilde{\tau}(s)}{2})^2 - \tilde{\sigma}(s) + k\sigma(s)}, \tag{22}$$

$$\lambda = k + \pi'(s), \tag{23}$$

where λ is a constant. The expression under the square root in the polynomial in $\pi(s)$ must be the square of a polynomial of first degree as $\pi(s)$ is the first-degree polynomial. Then one can derive the k values by considering that the discriminant of the square root has to be zero in Eq. (22). Now the energy equation is obtained by comparing Eq. (23) with Eq. (21). The function $\Phi(s)$ given in Eq. (20) is a hypergeometric-type function and its solution can be written in terms of polynomials, which are given by the Rodrigues relation

$$\Phi_n(s) = \frac{C_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)], \tag{24}$$

in which C_n is the normalization constant and the weight function $\rho(s)$ satisfies the condition

$$[\sigma(s)\rho(s)]' = \tau(s)\rho(s), \tag{25}$$

Furthermore, the other factor $W(s)$ satisfies the logarithmic equation

$$\frac{d}{ds}\ln W(s) = \frac{\sigma(s)}{\pi(s)}. \tag{26}$$

4.2. Parametric generalization of the NU method

The Nu parametric NU method, which is the ready-to-use version of the basic idea, solves the form [31]

$$\frac{d^2}{ds^2}\psi_n(s) + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds}\psi_n(s) + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{[s(1-\alpha_3 s)]^2}\psi_n(s) = 0. \tag{27}$$

By comparing Eq. (27) with Eq. (17), one can see that

$$\tilde{\tau}(s) = \alpha_1 - \alpha_2 s, \tag{28}$$

$$\sigma(s) = s(1-\alpha_3 s), \tag{29}$$

$$\tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3, \tag{30}$$

Inserting the above equations into Eq. (22) leads to [31]

$$\pi(s) = \alpha_4 + \alpha_5 s \pm \sqrt{(\alpha_6 - k\alpha_3)s^2 + (\alpha_7 + k)s + \alpha_8}, \tag{31}$$

where

$$\alpha_4 = \frac{1}{2}(1 - \alpha_1), \tag{32}$$

$$\alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3), \tag{33}$$

$$\alpha_6 = \alpha_5^2 + \xi_1, \tag{34}$$

$$\alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \tag{35}$$

$$\alpha_8 = \alpha_4^2 + \xi_3, \tag{36}$$

Considering that the discriminant of the square root in Eq. (22) has to be zero, we obtain

$$k_{1,2} = -(\alpha_7 + 2\alpha_3\alpha_8) \pm 2\sqrt{\alpha_8\alpha_9}, \tag{37}$$

with

$$\alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6, \tag{38}$$

From Eq. (31), one can easily see that different k values lead to different $\pi(s)$. If we take

$$k = -(\alpha_7 + 2\alpha_3\alpha_8) - 2\sqrt{\alpha_8\alpha_9}, \tag{39}$$

$\pi(s)$ becomes

$$\pi(s) = \alpha_4 + \alpha_5 s - [(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}), \tag{40}$$

and then we find

$$\tau(s) = \alpha_1 + 2\alpha_4 - (\alpha_2 - 2\alpha_5)s - [(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}]. \tag{41}$$

The energy equation can be readily obtained by using Eqs. (21) and (22) as

$$\alpha_2 n - (2n+1)\alpha_5 + (2n+1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) + n(n-1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0. \tag{42}$$

In order to obtain the wave functions, one uses the relations [31]

$$\rho(s) = s^{\alpha_{10}-1}(1 - \alpha_3 s)^{(\alpha_{11}/\alpha_3)-\alpha_{10}-1}, \tag{43}$$

$$\Phi_n(s) = P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1 - 2\alpha_3 s), \tag{44}$$

$$W(s) = s^{\alpha_{12}}(1 - \alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)}, \tag{45}$$

$$\Psi_n(s) = s^{\alpha_{12}}(1 - \alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)} P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1 - 2\alpha_3 s). \tag{46}$$

where $P_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1 - 2\alpha_3 s)$ is a Jacobi polynomial and

$$\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \tag{47}$$

$$\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}), \tag{48}$$

$$\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \tag{49}$$

$$\alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}). \tag{50}$$

Furthermore, in some cases we can use [31]

$$\psi_n(s) = s^{\alpha^*_{12}}(1-\alpha_3 s)^{-\alpha^*_{12}-\frac{\alpha^*_{13}}{\alpha_3}} P_n^{(\alpha^*_{10}-1,\frac{\alpha^*_{11}}{\alpha_3}-\alpha^*_{10}-1)}(1-2\alpha_3 s), \tag{51}$$

$$\alpha^*_{10} = \alpha_1 + 2\alpha_4 - 2\sqrt{\alpha_8}, \tag{52}$$

$$\alpha^*_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} - \alpha_3\sqrt{\alpha_8}), \tag{53}$$

$$\alpha^*_{12} = \alpha_4 - \sqrt{\alpha_8}, \tag{54}$$

$$\alpha^*_{13} = \alpha_5 - (\sqrt{\alpha_9} - \alpha_3\sqrt{\alpha_8}). \tag{55}$$

5. The solutions and the transmission and reflection coefficients

In this section, we seek exact solutions of our obtained equation via the parametric NU method. A comparison of Eq. (15) with Eq. (17) reveals

$$\xi_1 = -V_0 b + V_0^2 b^2, \qquad \xi_2 = -V_0 b, \qquad \xi_3 = b^2(m^2 - E_n^2), \tag{56}$$

and

$$\alpha_4 = 0, \qquad \alpha_5 = 0, \qquad \alpha_6 = -V_0 b + V_0^2 b^2, \qquad \alpha_7 = V_0 b, \qquad \alpha_8 = b^2(m^2 - E_n^2), \tag{57}$$

$$\alpha_9 = V_0^2 b^2 + b^2(m^2 - E_n^2), \tag{58}$$

$$\alpha_{10} = 1 + 2b\sqrt{m^2 - E_n^2}, \tag{59}$$

$$\alpha_{11} = 2 + 2(b\sqrt{V_0^2 + m^2 - E_n^2} + b\sqrt{m^2 - E_n^2}), \tag{60}$$

$$\alpha_{12} = b\sqrt{m^2 - E_n^2}, \tag{61}$$

$$\alpha_{13} = -(b\sqrt{V_0^2 + m^2 - E_n^2} + b\sqrt{m^2 - E_n^2}). \tag{62}$$

Moreover,

$$\alpha^*_{10} = 1 - 2b\sqrt{m^2 - E_n^2}, \tag{63}$$

$$\alpha^*_{11} = 2 + 2(b\sqrt{V_0^2 + m^2 - E_n^2} - b\sqrt{m^2 - E_n^2}), \tag{64}$$

$$\alpha^*_{12} = -b\sqrt{m^2 - E_n^2}, \tag{65}$$

$$\alpha^*_{13} = -(b\sqrt{V_0^2 + (m^2 - E_n^2)} - b\sqrt{m^2 - E_n^2}). \tag{66}$$

Thus the energy eigenvalues (Table) satisfy

$$n(n+1) + V_0 b + (2n+1)b(\sqrt{m^2 + V_0^2 - E_n^2} - \sqrt{m^2 - E_n^2}) + 2b^2(m^2 - E_n^2) - 2b^2\sqrt{(m^2 - E_n^2)(m^2 + V_0^2 - E_n^2)} = 0. \tag{67}$$

Table. Energy eigenvalues for $(m = 1,\ a = 1.1,\ b = 10)$.

$\|n\rangle$	$E_n,\ V_0 = 5$	$\|n\rangle$	$E_n,\ V_0 = 5.5$	$\|n\rangle$	$E_n,\ V_0 = 6$	$\|n\rangle$	$E_n,\ V_0 = 6.5$
$\|0\rangle$	–0.994884	$\|0\rangle$	–0.994894	$\|0\rangle$	–0.994902	$\|0\rangle$	–0.994909
$\|1\rangle$	–0.978936	$\|1\rangle$	–0.979018	$\|1\rangle$	–0.979086	$\|1\rangle$	–0.979143
$\|2\rangle$	–0.950875	$\|2\rangle$	–0.951174	$\|2\rangle$	–0.951420	$\|2\rangle$	–0.951626
$\|3\rangle$	–0.908727	$\|3\rangle$	–0.909508	$\|3\rangle$	–0.910147	$\|3\rangle$	–0.910679
$\|4\rangle$	–0.849382	$\|4\rangle$	–0.851102	$\|4\rangle$	–0.852500	$\|4\rangle$	–0.853658
$\|5\rangle$	–0.767617	$\|5\rangle$	–0.771083	$\|5\rangle$	–0.773879	$\|5\rangle$	–0.776182

The wave functions, from Eq. (51), are

$$\begin{aligned}&\phi_n^{(3,R)}(z) = Cz^{b\sqrt{(m^2-E_n^2)}}(1-z)^{b\sqrt{m^2+V_0^2-E_n^2}}P_n^{(2b\sqrt{-E_n^2+m^2},2b\sqrt{-E_n^2+m^2+V_0^2})}(1-2z)+\\&Dz^{-b\sqrt{m^2-E_n^2}}(1-z)^{b\sqrt{V_0^2+(m^2-E_n^2)}}P_n^{(-2b\sqrt{m^2-E_n^2},2b\sqrt{V_0^2+m^2-E_n^2})}(1-2z),\end{aligned} \tag{68}$$

Before proceeding further, we have to determine the asymptotic form of the solutions as we search for the reflection and transmission coefficients When $x \to \infty$ or, equivalently, $z \to 0$ and $(1-z)^{b\sqrt{m^2+V_0^2-E_n^2}} \to 1$ and $z^{b\sqrt{(m^2-E_n^2)}} = e^{-\frac{ib\sqrt{(E_n^2-m^2)}(x-a)}{b}}$, and the Jacobi polynomials can be expressed in terms of the hypergeometric functions [32]:

$$P_n^{(A,B)}(1-2x) = \frac{\Gamma(n+1+A)}{n!\Gamma(1+A)}\,{}_2F_1(-n, n+A+B+1; A+1; x). \tag{69}$$

In the other limit where x tends to zero,

$${}_2F_1(\tau_1,\tau_2;\tau_3;x) \underset{x\to 0}{\to} 1$$

In this case we consider $C = 0$ and the acceptable solution is

$$\begin{aligned}&\phi_n^{(3,R)}(z) = Dz^{-b\sqrt{m^2-E_n^2}}(1-z)^{b\sqrt{V_0^2+(m^2-E_n^2)}}\\&{}_2F_1(-n, n-2b\sqrt{m^2-E_n^2}+2b\sqrt{V_0^2+m^2-E_n^2}+1; -2b\sqrt{m^2-E_n^2}+1; z),\end{aligned} \tag{70}$$

Let us now consider Eq. (16). In this case, the requisite parameters are

$$\xi_1 = -V_0 b + V_0^2 b^2, \qquad \xi_2 = -V_0 b, \qquad \xi_3 = b^2(m^2 - E_n^2), \tag{71}$$

$$\alpha_4 = 0, \qquad \alpha_5 = 0, \qquad \alpha_6 = -V_0 b + V_0^2 b^2, \qquad \alpha_7 = V_0 b, \qquad \alpha_8 = b^2(m^2 - E_n^2), \tag{72}$$

$$\alpha_9 = V_0^2 b^2 + b^2(m^2 - E_n^2), \tag{73}$$

$$\alpha_{10} = 1 + 2b\sqrt{m^2 - E_n^2}, \tag{74}$$

$$\alpha_{11} = 2 + 2(b\sqrt{V_0^2 + m^2 - E_n^2} + b\sqrt{m^2 - E_n^2}), \tag{75}$$

$$\alpha_{12} = b\sqrt{m^2 - E_n^2}, \tag{76}$$

$$\alpha_{13} = -(b\sqrt{V_0^2 + m^2 - E_n^2} + b\sqrt{m^2 - E_n^2}), \tag{77}$$

and

$$\alpha_{10}^{*} = 1 - 2b\sqrt{m^2 - E_n^2}, \tag{78}$$

$$\alpha_{11}^{*} = 2 + 2(b\sqrt{V_0^2 + m^2 - E_n^2} - b\sqrt{m^2 - E_n^2}), \tag{79}$$

$$\alpha_{12}^{*} = -b\sqrt{m^2 - E_n^2}, \tag{80}$$

$$\alpha_{13}^{*} = -(b\sqrt{V_0^2 + (m^2 - E_n^2)} - b\sqrt{m^2 - E_n^2}). \tag{81}$$

Therefore, by the same token of the previous case, the wave functions are

$$\begin{aligned}&\phi_n^{(3,L)}(y) = Ay^{b\sqrt{(m^2-E_n^2)}}(1-y)^{b\sqrt{m^2+V_0^2-E_n^2}}P_n^{(2b\sqrt{-E_n^2+m^2},2b\sqrt{-E_n^2+m^2+V_0^2})}(1-2y)\\ &+By^{-b\sqrt{m^2-E_n^2}}(1-y)^{b\sqrt{V_0^2+(m^2-E_n^2)}}P_n^{(-2b\sqrt{m^2-E_n^2},2b\sqrt{V_0^2+m^2-E_n^2})}(1-2y).\end{aligned} \tag{82}$$

In limit $x \to -\infty$, $y \to 0$, $(1-y)^{b\sqrt{m^2+V_0^2-E_n^2}} \to 1$, and $y^{b\sqrt{(m^2-E_n^2)}} \to e^{\frac{bi\sqrt{(E_n^2-m^2)}(x+a)}{b}}$. Thus, the solution for the left solution is

$$\begin{aligned}&\phi_n^{(3,L)}(y) = Ay^{b\sqrt{(m^2-E_n^2)}}(1-y)^{b\sqrt{m^2+V_0^2-E_n^2}}\\ &{}_2F_1(-n, n+b\sqrt{(m^2-E_n^2)}+2b\sqrt{V_0^2+m^2-E_n^2}+1; 2b\sqrt{m^2-E_n^2}+1; y)\\ &+By^{-b\sqrt{m^2-E_n^2}}(1-y)^{b\sqrt{V_0^2+(m^2-E_n^2)}}\\ &{}_2F_1(-n, n-b\sqrt{(m^2-E_n^2)}+2b\sqrt{V_0^2+m^2-E_n^2}+1; -2b\sqrt{m^2-E_n^2}+1; y)\end{aligned} \tag{83}$$

The wave function in Eq. (82) includes the incident and reflected waves. Eq. (68) contains the transmitted wave function. Thus, for deriving the explicit expressions for the coefficients we need to use the continuity conditions on the wave functions. These requirements give us the reflection and transmission coefficients as

$$R = \left|\frac{B}{A}\right|^2, \qquad T = \left|\frac{D}{A}\right|^2. \tag{84}$$

The expected balance equation $R + T = 1$ is shown in Figure 2.

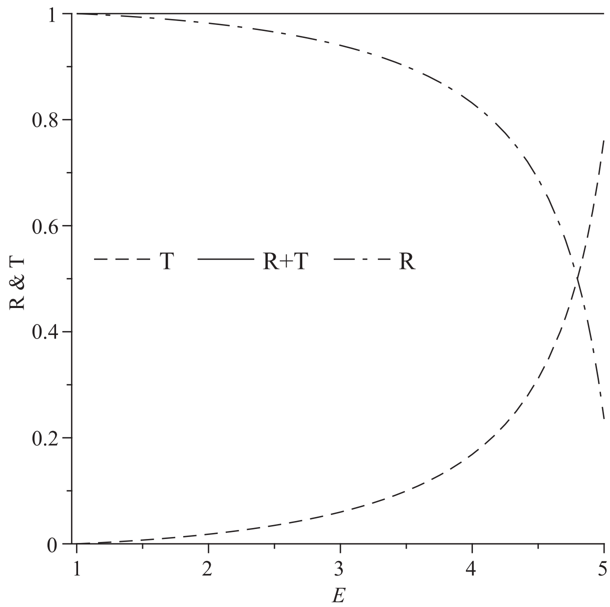

Figure 2. Transmission (T) and reflection (R) coefficients vs. E for $(m = 1,\ a = 2,\ b = 2,\ V_0 = 5,\ n = 0)$.

6. Conclusion

The motivation behind our study was 2-fold. The first was the high number of spin-one bosons frequently present in various physical fields including particle and nuclear physics. The second was the encouraging results of Woods–Saxon potential in the mentioned branches. Instead of a numerical procedure that hides the physics of the problem to a considerable extent, we used the powerful NU method in our calculations and thereby reported the exact analytical exact solutions. In addition, we discussed the reflection and transmission coefficient. To provide a potentially useful basis for possible further studies, we reported some numerical results on the spectrum of the system. We observe that the energy eigenvalues increase for increasing V_0. Our results can be particularly used in the study of spin-one bosons after the proper fits are performed.

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