Turkish Journal of Physics
http://journals.tubitak.gov.tr/physics/
Research Article

Turk J Phys
(2013) 37: 268 – 274

doi:10.3906/fiz-1207-21

Bound states and the oscillator strengths for the Klein–Gordon equation under Möbius square potential

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Received: 31.07.2012 • **Accepted:** 31.10.2012 • **Published Online:** 19.06.2013 • **Printed:** 12.07.2013

Abstract: The Klein–Gordon equation under the equal scalar and vector Möbius square potentials in D-dimensions is solved by using the Nikiforov–Uvarov method. The energy eigenvalues and the corresponding eigenfunctions are obtained and numerically calculated. The oscillator strengths are determined and discussed in terms of parameters of the system.

Key words: Klein–Gordon equation, Möbius square potential, Nikiforov–Uvarov method, oscillator strengths

PACS: 03.65.Fd, 03.65.pm, 0365.ca, 03.65.Ge

1. Introduction

In relativistic quantum mechanics, solution of the Klein–Gordon equation under different potentials plays an important role because one can understand the physics that can be brought by such solutions. The Klein–Gordon equation is one of the most frequently used wave equations that describes spin-zero particles. Among the most successful methods that have been used to solve the Schrödinger, Dirac, Duffin–Kemmer–Petiau, and Klein–Gordon equations, the Nikiforov–Uvarov (NU) and supersymmetric quantum mechanics (SUSYQM) methods have received great attention [1–5]. In the present work, we have approximately solved the Klein–Gordon equation under equal scalar and vector Möbius square potentials, which is the more general case of both Hulthén and Morse potentials. In addition the Klein–Gordon equation has been solved and investigated with different potentials [6,7]. For instance, Egrifes and Sever obtained bound-state solutions of the Klein–Gordon equation for the generalized PT-symmetric Hulthén potential [8], and Soylu et al. considered the Klein–Gordon equation under Rosen–Morse-type potentials [9]. Here, for the sake of generality, we investigate the Klein–Gordon equation in D-dimensional space. This paper is organized as follows: in Section 2, we introduce the radial part of the Klein–Gordon equation and solve this equation under the Möbius square potential. The oscillator strengths are obtained in Section 3. Finally, our conclusion is given in Section 4.

2. Radial part of the Klein–Gordon equation in D-dimensions

The radial part of the Klein–Gordon equation in the presence of vector and scalar potentials in the D-dimensional space is written as [10,11]:

$$[\frac{d^2}{dr^2}+E_{n,l}^2+V^2(r)-2E_{n,l}V(r)-m^2-S^2(r)-2mS(r)-\frac{(D+2l-1)(D+2l-3)}{4r^2}]u_{n,l}(r)=0. \quad (1)$$

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Here, we consider the Möbius square potential as below [12]:

$$V(r) = V_0\left(\frac{A + B\exp(-\alpha r)}{C + D'\exp(-\alpha r)}\right)^2. \tag{2}$$

Here V_0, A, B, C, D', and α are constant coefficients. The good approximation for the centrifugal barrier is taken as [13]:

$$\frac{1}{r^2} \approx \alpha^2\left(\frac{C}{C + D'\exp(-\alpha r)}\right)^2, \tag{3}$$

where $C = -D'$ and Eq. (3) is a quite logical alternative for $\alpha < 0.1$ (see Figure 1).

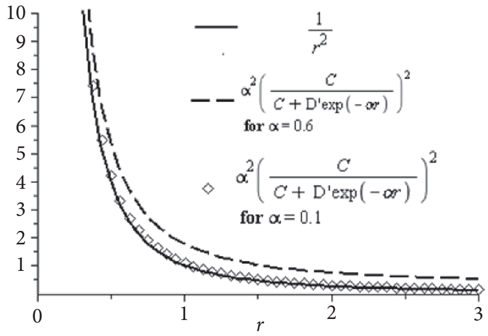

Figure 1. $\frac{1}{r^2}$ and its approximation ($C = 1, D' = -1$).

For the equal scalar and vector Möbius square potentials, by substituting Eqs. (2) and (3) into Eq. (1), we obtain:

$$\begin{aligned}&[\frac{d^2}{dr^2} + (E_{n,l}^2 - m^2) - 2V_0(E_{n,l} + m)\left(\frac{A + B\exp(-\alpha r)}{C + D'\exp(-\alpha r)}\right)^2 \\ &-\frac{\alpha^2 C^2 (D + 2l - 1)(D + 2l - 3)}{4}\frac{1}{(C + D'\exp(-\alpha r))^2}]u_{n,l}(r) = 0.\end{aligned} \tag{4}$$

By a change of variable of the form

$$z = \exp(-\alpha r), \tag{5}$$

Eq. (4) is written as:

$$\begin{aligned}&\{\frac{d^2}{dz^2} + \frac{1 + \frac{D'}{C}z}{z(1 + \frac{D'}{C}z)}\frac{d}{dz} + \frac{1}{[z(1 + \frac{D'}{C}z)]^2}[(\frac{E_{n,l}^2 - m^2}{\alpha^2}\frac{D'^2}{C^2} - \frac{2V_0 B^2}{\alpha^2 C^2}(m + E_{n,l}))z^2 \\ &+(\frac{2(E_{n,l}^2 - m^2)}{\alpha^2}\frac{D'}{C} - \frac{4V_0 AB}{\alpha^2 C^2}(m + E_{n,l}))z + (\frac{E_{n,l}^2 - m^2}{\alpha^2} - \frac{(D + 2l - 1)(D + 2l - 3)}{4} - \frac{2V_0 A^2}{C^2\alpha^2}(m + E_{n,l}))]\}u_{n,l}(z) = 0.\end{aligned} \tag{6}$$

Bearing in mind Eq. (1), and comparing Eq. (6) with Eq. (I) (Appendix), one can find

$$\xi_1 = -(\frac{E_{n,l}^2 - m^2}{\alpha^2}\frac{D'^2}{C^2} - \frac{2V_0 B^2}{\alpha^2 C^2}(m + E_{n,l})),\ \xi_2 = \frac{2(E_{n,l}^2 - m^2)}{\alpha^2}\frac{D'}{C} - \frac{4V_0 AB}{\alpha^2 C^2}(m + E_{n,l}) \tag{7a}$$

$$\xi_3 = -\frac{E_{n,l}^2 - m^2}{\alpha^2} + \frac{(D + 2l - 1)(D + 2l - 3)}{4} + \frac{2V_0 A^2}{C^2\alpha^2}(m + E_{n,l}),\ \alpha_1 = 1,\ \alpha_2 = \alpha_3 = -\frac{D'}{C} \tag{7b}$$

where:

$$\alpha_4=0,\ \alpha_5=\tfrac{D'}{2C},\ \alpha_6=\tfrac{D'^2}{4C^2}-\tfrac{E_{n,l}^2-m^2}{\alpha^2}\tfrac{D'^2}{C^2}+\tfrac{2V_0B^2}{\alpha^2C^2}(m+E_{n,l}),$$

$$\alpha_7=-\tfrac{2(E_{n,l}^2-m^2)}{\alpha^2}\tfrac{D'}{C}+\tfrac{4V_0AB}{\alpha^2C^2}(m+E_{n,l}),\ \alpha_8=-\tfrac{E_{n,l}^2-m^2}{\alpha^2}+\tfrac{(D+2l-1)(D+2l-3)}{4}+\tfrac{2V_0A^2}{C^2\alpha^2}(m+E_{n,l}),$$

$$\alpha_9=\tfrac{D'}{C}\xi_2+\tfrac{D'^2}{C^2}(-\tfrac{E_{n,l}^2-m^2}{\alpha^2}+\tfrac{(D+2l-1)(D+2l-3)}{4}+\tfrac{2V_0A^2}{C^2\alpha^2}(m+E_{n,l}))-(\tfrac{E_{n,l}^2-m^2}{\alpha^2}\tfrac{D'^2}{C^2}-\tfrac{2V_0B^2}{\alpha^2C^2}(m+E_{n,l}))+\tfrac{D'^2}{4C^2},$$

$$\alpha_{10}=1+2\sqrt{-\tfrac{E_{n,l}^2-m^2}{\alpha^2}+\tfrac{(D+2l-1)(D+2l-3)}{4}+\tfrac{2V_0A^2}{C^2\alpha^2}(m+E_{n,l})},$$

$$\alpha_{11}=\tfrac{-2D'}{C}+2(\sqrt{\alpha_9}-\tfrac{D'}{C}\sqrt{-\tfrac{E_{n,l}^2-m^2}{\alpha^2}+\tfrac{(D+2l-1)(D+2l-3)}{4}+\tfrac{2V_0A^2}{C^2\alpha^2}(m+E_{n,l})}),$$

$$\alpha_{12}=\sqrt{-\tfrac{E_{n,l}^2-m^2}{\alpha^2}+\tfrac{(D+2l-1)(D+2l-3)}{4}+\tfrac{2V_0A^2}{C^2\alpha^2}(m+E_{n,l})},$$

$$\alpha_{13}=\tfrac{D'}{2C}-(\sqrt{\alpha_9}-\tfrac{D'}{C}\sqrt{-\tfrac{E_{n,l}^2-m^2}{\alpha^2}+\tfrac{(D+2l-1)(D+2l-3)}{4}+\tfrac{2V_0A^2}{C^2\alpha^2}(m+E_{n,l})}).\tag{8}$$

Thus, from Eq. (II) (Appendix), the eigenfunction of the system is:

$$u_{n,l}(r)=N_{n,l}\exp(-\alpha\alpha_{12}r)(1+\frac{D'}{C}\exp(-\alpha r))^{-\alpha_{12}+\frac{\alpha_{13}C}{D'}}P_n^{(\alpha_{10}-1,-\frac{C\alpha_{11}}{D'}-\alpha_{10}-1)}(1+\frac{2D'}{C}\exp(-\alpha r)).\tag{9}$$

The energy spectrum of the system, by using Eq. (III) (Appendix), satisfies

$$-\frac{D'}{C}n-\frac{D'}{2C}(2n+1)+(2n+1)(\sqrt{\alpha_9}-\frac{D'}{C}\sqrt{\alpha_8})-\frac{D'}{C}n(n-1)+\alpha_7-\frac{2D'}{C}\alpha_8+2\sqrt{\alpha_8\alpha_9}=0.\tag{10}$$

In the Table, we report some numerical results for some values of n and l. The behavior of $E_{n,l}$ versus V_0,α,A and B is plotted in Figures 2–5. From Figures 2–4, we see that the energy of the system has an increasing behavior as V_0,α , and A increase. As B increases, the energy of the system decreases and tends to a constant value.

3. Oscillator strengths

Here, we want to calculate the oscillator strength. Absorption of light yields a transition from one quantum state to another. The spectra of stars are an important source of transition data. Calculation of the transition probabilities is important because one can determine the chemical abundances in the sun and other stars. The oscillator strength gives additional information on the fine structure and selection rules of the optical absorption. This quantity is, in many cases, an important part of scientific reports. In transition from a lower state ψ_i to an upper state ψ_f, we have the following forms for the absorption oscillator strength [14]:

$$f_{ij}^l=\frac{2}{3}\frac{m}{\hbar^2}(E_j-E_i)\left|\langle\psi_j\left|r\right|\psi_i\rangle\right|^2,\tag{11a}$$

$$f_{ij}^v=\frac{2}{3}\frac{1}{m(E_j-E_i)}\left|\langle\psi_j\left|p\right|\psi_i\rangle\right|^2.\tag{11b}$$

These are respectively called the length and velocity strengths in the jargon. In Figures 6 and 7, we have plotted the variation of the length and velocity oscillator strengths versus V_0, respectively.

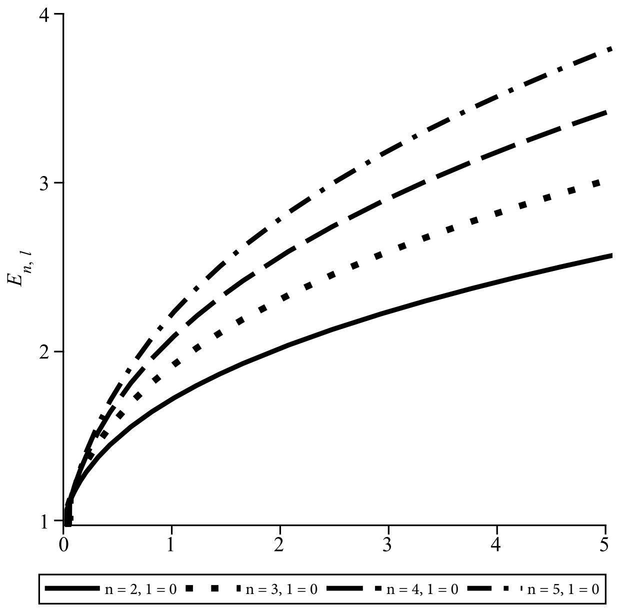

Figure 2. En, l versus V_0 for $A=-1,\ B=2,\ C=1,\ D'=-1,\ D=3,\ m=1,\ \alpha=0.1$.

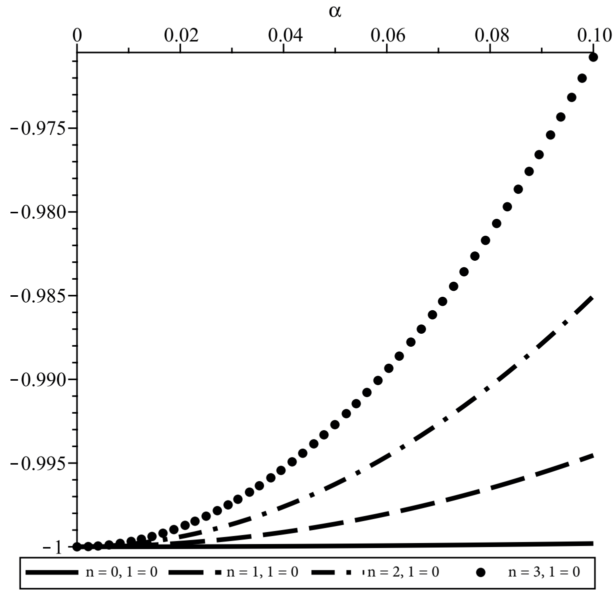

Figure 3. En, l versus α for $A=-1,\ B=2,\ C=1,\ D'=-1,\ D=3,\ m=1,\ V_0=3$.

Table. Energy eigenvalues of the system for $A=-1, B=2, C=1, D'=-1, \alpha=0.1, m=1, V_0=3$.

$\lvert n,l\rangle$ / **D**	$\lvert 1,0\rangle$	$\lvert 1,1\rangle$	$\lvert 1,2\rangle$	$\lvert 1,3\rangle$
0	1.831514077	1.820142563	1.831514077	1.865088323
1	1.822994127	1.822994127	1.845600166	1.889761315
2	1.820142563	1.831514077	1.865088323	1.919359381
3	1.822994127	1.845600166	1.889761315	1.953591721
4	1.831514077	1.865088323	1.919359381	1.992147820
5	1.845600166	1.889761315	1.953591721	2.034707800
6	1.865088323	1.919359381	1.992147820	2.080951273
7	1.889761315	1.953591721	2.034707800	2.130564471
8	1.919359381	1.992147820	2.080951273	2.183245604
9	1.953591721	2.034707800	2.130564471	2.238708601
10	1.992147820	2.080951273	2.183245604	2.296685450
11	2.034707800	2.130564471	2.238708601	2.356927405
12	2.080951273	2.183245604	2.296685450	2.419205319
13	2.130564471	2.238708601	2.356927405	2.483309332
14	2.183245604	2.296685450	2.419205319	2.549048115
15	2.238708601	2.356927405	2.483309332	2.616247838

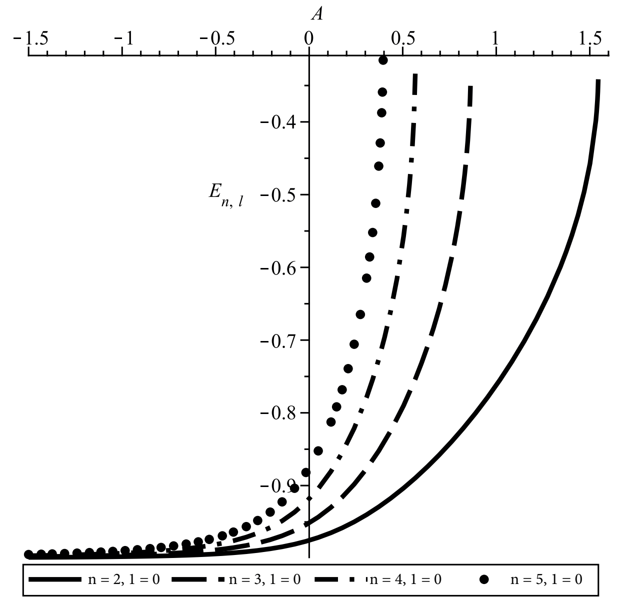

Figure 4. En, l versus A for $\alpha = 0.1$, $B = 2, C = 1$, $D' = -1$, $D = 3$, $m = 1$, $V_0 = 3$.

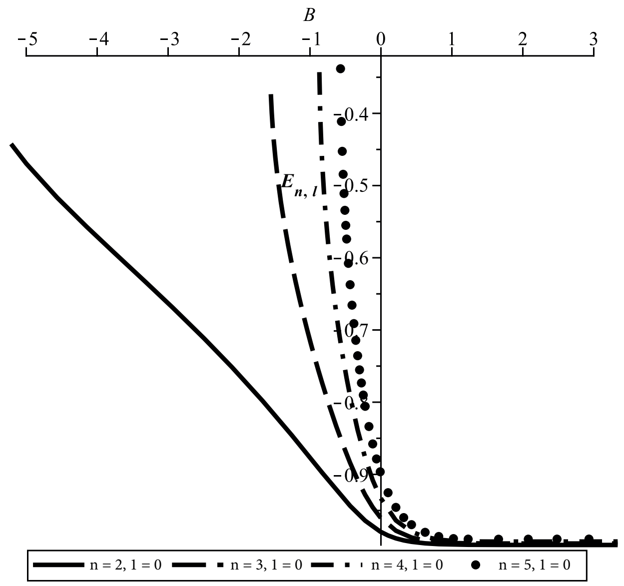

Figure 5. En, l versus B for $\alpha = 0.1$, $A = -1$, $C = 1$, $D' = -1$, $D = 3$, $m = 1$, $V_0 = 3$.

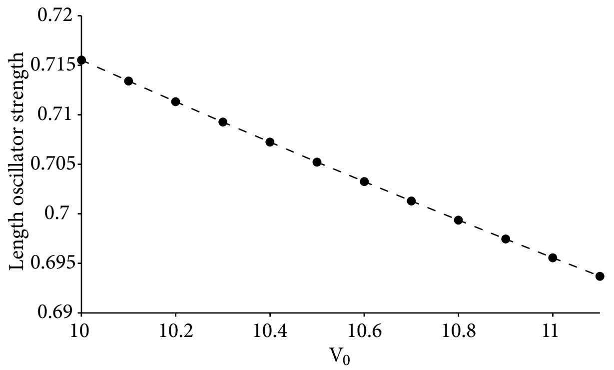

Figure 6. Length of oscillator strength versus V_0 for $A = -1, B = 2$, $C = 1$, $D' = -1$, $\alpha = 0.1$, $m = 1$, $D = 3$.

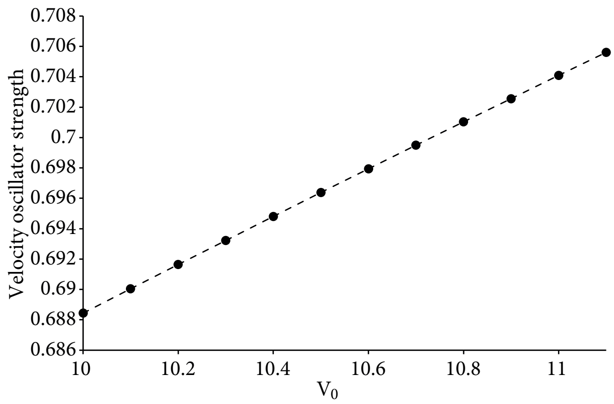

Figure 7. Velocity of oscillator strength versus V_0 for $A = -1$, $B = 2$, $C = 1$, $D' = -1$, $\alpha = 0.1$, $m = 1$, $D = 3$.

4. Conclusion

An approximately analytical solution of the Klein-Gordon equation in the case of equal scalar and vector potentials was obtained. The potential that we focused on was the Möbius square potential, which is the more general case of both Hulthén and Morse potentials.

As a further guide to interested readers, we have provided some numerical data that discuss the energy spectrum. As shown in the Table, for $l = 0$, $D = 1$ and $D = 3$ are symmetric w.r.t. $D = 2$. This symmetry exists between $D = 0$ and $D = 4$. Additionally, the energy shows a degenerate behavior when l increases to $l+1$ and D reduces by 2 units, i.e. to $D-2$, ($E_{n,l}^{D} = E_{n,l+1}^{D-2}$). We have also calculated the oscillator strengths for different values of V_0. From Figures 6 and 7, we understand that as V_0 increases the length and velocity strengths have decreasing and increasing behavior, respectively.

Acknowledgment

It is a great pleasure for the authors to thank the referees for their careful comments on the paper.

Appendix

The NU method solves many linear second-order differential equations by reducing them to a generalized equation of the hypergeometric type. Here, instead of the original formulation, we use the parametric version, which enables us to solve a second-order differential equation of the following form [4,5,15]:

$$\{\frac{d^2}{ds^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds}+\frac{1}{[s(1-\alpha_3 s)]^2}[-\xi_1 s^2+\xi_2 s-\xi_3]\}\psi=0. \tag{I}$$

According to the NU method, the eigenfunction is:

$$\psi(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s). \tag{II}$$

The energy of the system satisfies

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0, \tag{III}$$

where

$$\begin{gathered}\alpha_4=\frac{1}{2}(1-\alpha_1),\ \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\ \alpha_6=\alpha_5^2+\xi_1,\ \alpha_7=2\alpha_4\alpha_5-\xi_2,\ \alpha_8=\alpha_4^2+\xi_3\\ \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\ \alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8} \qquad \text{(IV)}\\ \alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),\ \alpha_{12}=\alpha_4+\sqrt{\alpha_8}\alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})\\ P_n^{(\alpha,\beta)}(x)=\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}\sum_{m=0}^{n}\left(\begin{array}{c}n\\ m\end{array}\right)\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)}(\frac{x-1}{2})^m.\end{gathered}$$

Here, $P_n^{(\alpha,\beta)}$ is a Jacobi polynomial.

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