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A model of the multijunction ac Josephson effect in a superconductor

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Abstract: We present a model of the multijunction ac Josephson effect in a superconductor. Josephson predicted that at a finite applied voltage (V_o) an alternating supercurrent of frequency $\omega_J = 2eV_o/\hbar$ flows between 2 superconductors separated by an insulating layer, called the ac Josephson effect. Adding 2 or more Josephson junctions (so-called multijunction) with an applied voltage, we have shown that the resultant current (which is equivalent to the vector sum of the currents in each junction) has the same frequency as the single Josephson junction. The amplitude of the resultant current for the multijunction is increased with the increasing number of junctions. For maximum current, the phase and frequency follow the relation $\omega_J t + \delta_{0N} = (4n+1)\,\pi/2$. Furthermore, we have shown that in the absence of applied voltage this multijunction theory is similar to the dc SQUID theory for 2 junctions and satisfied all conditions for identical and nonidentical Josephson junctions.

Key words: Josephson effect (ac and dc) in superconductor, superconducting quantum interference device (SQUID), identical and nonidentical Josephson junctions

1. Introduction

The Josephson effect is an example of a macroscopic quantum phenomenon in superconductivity. In 1962, Brian Josephson [1] predicted that the Cooper pair could tunnel between superconductors separated by an insulating layer, with the same probability as that of ordinary electrons. There is a possibility of both of them (super electrons and normal electrons) tunneling across an insulating barrier. When a thin layer of insulating material separates 2 superconductors (1 and 2, shown in Figure 1), electron pairs are able to tunnel through the insulator from one superconductor to the other. This phenomenon is called the Josephson (tunnel) effect and is analogous to quantum mechanical tunneling.

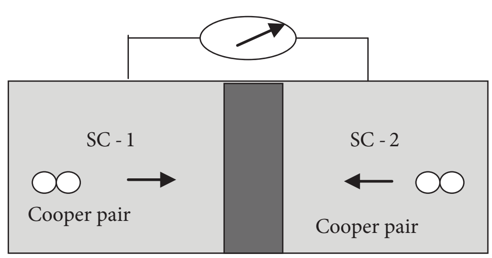

Figure 1. A schematic view of a dc Josephson junction.

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There are 4 possible modes of Cooper pair tunneling [1,2] through a Josephson junction that will produce: (i) the dc Josephson effect, in which a dc current can flow across the junction in the absence of electric field, i.e. without the need for an applied voltage, (ii) the ac Josephson effect, where an ac current can flow through the junction with an applied voltage across the junction, (iii) the inverse ac Josephson effect [3], whereby dc voltages are induced across an unbiased junction by an impressed rf current, and (iv) macroscopic quantum interference effects [4].

For a parallel connection, it is very easy to add or fabricate a number of junctions in the presence of applied voltage and the resultant current will be the vector sum of the current in each junction. We use simple mathematics for the junctions connected in parallel and constitute a model of the multijunction ac Josephson effect. The result shows that the amplitude of the resultant current is increased with the increasing number of junctions. For maximum current, the phase and frequency follow the universal relation $\omega_J t+\delta_{0N} = (4n+1)\,\pi/2$.

In section 2, we develop the theory of the multijunction ac Josephson effect. The resultant current equation is discussed for both identical and nonidentical junctions. In section 3, we discuss the maximum current first for 2 junctions and then generalize the result for N number of identical Josephson junctions. A numerical result is given showing that the amplitude of the resultant current increases with the increasing number of junctions. In section 4, we discuss the overall result compared with the dc SQUID equation. The result is discussed in the absence of applied voltage and for 2 nonidentical Josephson junctions. Finally, the important conclusion is given in section 5.

2. Theoretical model for the multijunction ac Josephson effect

Let us consider 2 Josephson junctions (I and II) connected in parallel at point ‘a’ and ‘b’. There is a constant dc voltage (V_o) across both junctions. The circuit connection is shown in Figure 2.

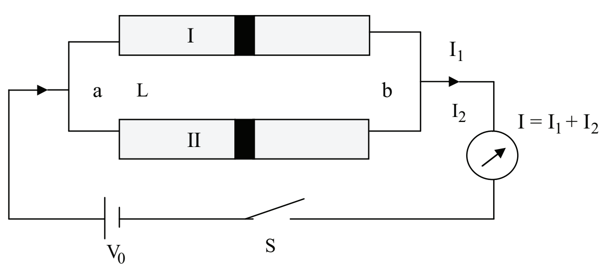

Figure 2. Schematic diagram of 2 Josephson junctions connected in parallel with a constant dc voltage source (V_o).

Since the connection is parallel the voltage is the same across both junctions, but different currents pass through junctions I and II. Now the current equations [1,2] for 2 junctions in the presence of dc voltage (V_o) are written as

$$I_1 = I_{o1}\sin(\omega_J t + \delta_1). \tag{1}$$

$$I_2 = I_{o2}\sin(\omega_J t + \delta_2). \tag{2}$$

Here I_{o1} and I_{o2} are the maximum current flowing through junctions I and II; $\delta_1 = \theta_{I1} \sim \theta_{I2}$ and $\delta_2 = \theta_{II1} \sim \theta_{II2}$ are the phase differences across junctions I and II, respectively. For each junction, $\omega_J = 2eV_o/\hbar$ represents the frequency of oscillation of the ac current.

Now the resultant (total) current flowing through the circuit is the vector sum of the current in each junction:

$$\begin{aligned} I_{total} &= I_1 + I_2 \\ &= I_{01}\sin(\omega_J t + \delta_1) + I_{02}\sin(\omega_J t + \delta_2) \\ &= I_{00}\sin(\omega_J t + \phi), \end{aligned} \tag{3}$$

where

$$\phi = \tan^{-1}\frac{I_{o1}\sin\delta_1 + I_{o2}\sin\delta_2}{I_{o1}\cos\delta_1 + I_{o2}\cos\delta_2}, \tag{4}$$

$$I_{00} = \sqrt{I_{o1}^2 + I_{o2}^2 + 2I_{o1}I_{o2}\cos(\delta_1 - \delta_2)}. \tag{5}$$

Eq. (3) is similar to the ac Josephson current, i.e. current in Eqs. (1) and (2), but maximum current and phase differences are different. For identical Josephson junctions, we may assume that $I_{o1} = I_{o2}$ and, using some algebra, we have

$$\phi = \frac{\delta_1 + \delta_2}{2}. \tag{6}$$

This equation represents the total phase difference across both junctions in the presence of dc voltage (V_o).

Using a similar argument, we also have

$$I_{00} = 2I_0\cos\left(\frac{\delta_1 - \delta_2}{2}\right). \tag{7}$$

Using Eqs. (6) and (7) in Eq. (3), the total result current for 2 Josephson junctions in the presence of dc voltage is, therefore, written as

$$I_{total} = 2I_o\cos\left(\frac{\delta_1 - \delta_2}{2}\right)\sin\left[\omega_J t + \left(\frac{\delta_1 + \delta_2}{2}\right)\right]. \tag{8}$$

This is the ac Josephson effect for 2 identical Josephson junctions.

If we assume that the phase differences across the junctions are the same $(i.e. \delta_1 = \delta_2)$ and denoting $I_{total} = I_{2t}$ (i.e. the total current for 2 Josephson junctions), Eq. (8) implies

$$I_{2t} = 2I_o\sin[\omega_j t + \delta_2]. \tag{9}$$

This is the ac Josephson effect for 2 identical Josephson junctions when the phase differences across the junctions are the same.

For 2 nonidentical Josephson junctions [i.e. for $I_{o1} \neq I_{o2}$], the total resultant current equation can be written as

$$I_{total} = \sqrt{(I_{o1}^2 + I_{o2}^2 + 2I_{o1}I_{o2}\cos(\delta_1 - \delta_2))} \times \sin\left[\frac{2eV_o}{\hbar}t + \tan^{-1}\left(\frac{I_{o1}\sin\delta_1 + I_{o2}\sin\delta_2}{I_{o1}\cos\delta_1 + I_{o2}\cos\delta_2}\right)\right]. \tag{10}$$

This is the ac Josephson effect for 2 nonidentical Josephson junctions.

Now we can proceed for 3 junctions, as shown in Figure 3. For this case the current equation for the third junction is written as

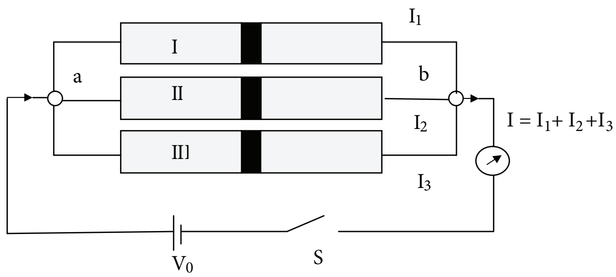

Figure 3. Schematic diagram of 3 Josephson junctions connected in parallel with a constant dc voltage source (V_o).

$$I_3 = I_{o3} \sin(\omega_J t + \delta_3). \tag{11}$$

The total resultant current flowing through the circuit is the vector sum of the current in each junction, written as

$$I_{total} = I_1 + I_2 + I_3 = I_{000} \sin\left(\omega_J t + \psi\right), \tag{12}$$

where

$$I_{000} = \sqrt{I_{o1}^2 + I_{o2}^2 + I_{o3}^2 + 2I_{o1}I_{o2}\cos\left(\delta_1 - \delta_2\right) + 2I_{02}I_{03}\cos\left(\delta_2 - \delta_3\right) + 2I_{03}I_{01}\cos\left(\delta_3 - \delta_1\right)} \tag{13}$$

and

$$\psi = \tan^{-1}\left(\frac{I_{01}\sin\delta_1 + I_{02}\sin\delta_2 + I_{03}\sin\delta_3}{I_{01}\cos\delta_1 + I_{02}\cos\delta_2 + I_{03}\cos\delta_3}\right). \tag{14}$$

Let us assume that for 3 identical Josephson junctions $I_{01} = I_{02} = I_{03} = I_0$; then Eqs. (13) and (14) imply

$$I_{ooo} = \sqrt{[3I_o^2 + 2I_o^2\left\{\cos(\delta_1 - \delta_2) + \cos(\delta_2 - \delta_3) + \cos(\delta_3 - \delta_1)\right\}]} \tag{15}$$

$$\psi = \tan^{-1}\left(\frac{\sin\delta_1 + \sin\delta_2 + \sin\delta_3}{\cos\delta_1 + \cos\delta_2 + \cos\delta_3}\right). \tag{16}$$

Using Eqs. (15) and (16) in Eq. (12), the total resultant current for 3 Josephson junctions in the presence of dc voltage (V_o) is

$$\begin{aligned} I_{total} \quad &= \quad \sqrt{[3I_0^2 + 2I_0^2\left\{\cos\left(\delta_1 - \delta_2\right) + \cos\left(\delta_2 - \delta_3\right) + \cos\left(\delta_3 - \delta_1\right)\right\}]} \\ &\times \sin\left[\omega_J t + \tan^{-1}\left(\tfrac{\sin\delta_1 + \sin\delta_2 + \sin\delta_3}{\cos\delta_1 + \cos\delta_2 + \cos\delta_3}\right)\right]. \end{aligned} \tag{17}$$

This is the ac Josephson effect for 3 identical Josephson junctions, when the phase differences across the junctions are not the same.

When phase differences across the junctions are the same (i.e. $\delta_1 = \delta_2 = \delta_3$), then Eq. (17) becomes

$$I_{3t} = 3I_o \sin[\omega_j t + \delta_3]. \tag{18}$$

This is the ac Josephson effect for 3 identical Josephson junctions, when the phase differences across the junctions are the same.

Extending the result to 4 Josephson junctions, then the total current is

$$I_{4t} = 4I_o \sin[\omega_j t + \delta_4], \tag{19}$$

when junctions are identical (i.e. $I_1 = I_2 = I_3 = I_4 = I_o$) and phase differences across the junctions are the same ($\delta_1 = \delta_2 = \delta_3 = \delta_4$).

In a similar way, for N−junctions, we have

$$I_{Nt} = NI_o \sin[\omega_j t + \delta_N], \tag{20}$$

when the phase differences across the junctions are the same (i.e. $\delta_1 = \delta_2 = \delta_3 = \delta_4 ..... = \delta_N$). This completes the current equation for the multijunction ac Josephson effect in a superconductor.

3. Numerical analysis

For numerical analysis, we considered first 2 Josephson junctions connected in parallel. In the presence of applied voltage and for the identical Josephson junction, we may assume that the initial phase differences of both junctions are the same, i.e. for $\delta_1 = \delta_2 \equiv \delta_{02}$ and for maximum current we can set $\sin(\omega_J t + \delta_{02}) = 1$. The resultant current is maximum and Eq. (8) implies

$$I_{2t}^{\max} = 2I_o; \tag{21}$$

$$\omega_J t + \delta_{02} = (4n+1)\frac{\pi}{2}, (n = 0, 1, 2, 3....). \tag{22}$$

Eq. (22) represents the phase and frequency relation. A similar relation holds for N number of junctions.

Analogously, the total maximum current [using Eq. (17)] for 3 identical junctions is found to be

$$I_{3t}^{\max} = 3I_o. \tag{23}$$

Therefore, the total maximum current for N identical junctions is written as

$$I_{Nt}^{\max} = NI_o, \text{ for } \delta_1 = \delta_2 = \delta_3 = \delta_4 ... \delta_N. \tag{24}$$

We see from the above equations [i.e. Eqs. (21), (23), and (24)] that the amplitude of the resultant current increases with the increase in the number of junctions.

For numerical analysis, the resultant current equation for N−junctions is written as

$$\begin{aligned} I_{Nt} &= NI_o \sin[\tfrac{2eV_o}{\hbar} t + \delta_N], \\ &= NI_o \sin[\tfrac{4\pi eV_o}{h} t + \delta_N]. \end{aligned} \tag{25}$$

Here, $h = 2\pi\hbar$ and $4\pi eV_0/h$ is a constant. Using $e = 1.6 \times 10^{-19} C$, and let [5,6] $V_o = 10^{-15} V$, $h = 6.63 \times 10^{-34} js$, then $4eV_0/h = 0.96 \approx 1$ and we have fixed $\delta_N = \pi/2$, for junction N=1, 2, 3, 4 etc. The numerical analysis is shown in Figure 4.

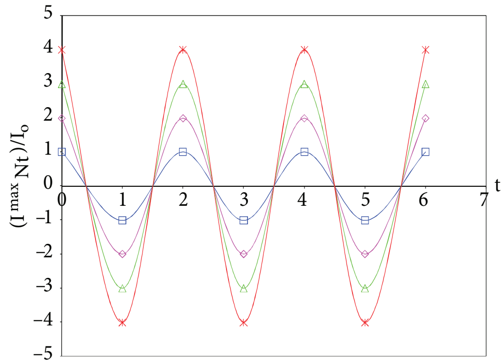

Figure 4. Graphical representation of the current in multijunction ac Josephson effect, showing the increase in the amplitude of the resultant current with the increasing number of junctions.

4. Discussion

The result we have found here is discussed in detail. First we have shown that in the absence of applied voltage the junction equation [i.e. Eq. (3)] is equivalent to the dc SQUID equation. Then the result is discussed in the absence of applied voltage and for nonidentical junctions.

From Eq. (8), we may conclude that in the presence or absence of voltage magnetic flux must flow though the loop containing 2 Josephson junctions or passes through the loop due to the current I_{total}. Let us consider the relation between phase differences and total magnetic flux [4,7] through the loop, which is written as

$$\delta_1 - \delta_2 = \left(\frac{2e}{\hbar c}\right)\varphi. \tag{26}$$

Now in the absence of voltage (i.e. for $V_o = 0$), Eq. (8) implies

$$I_{total} = 2I_o \cos\left(\frac{\delta_1 - \delta_2}{2}\right)\sin\left[\left(\frac{\delta_1 + \delta_2}{2}\right)\right]. \tag{27}$$

Eq. (27) is nothing but the dc SQUID equation [4–7] provided that

$$\left(\frac{\delta_1 - \delta_2}{2}\right) = \left(\frac{e}{\hbar c}\right)\varphi \text{ and } \left(\frac{\delta_1 + \delta_2}{2}\right) = \delta_o. \tag{28}$$

The parameter δ_o introduced earlier in the SQUID equation,

$$I_{total} = 2I_o \cos\left(\frac{\pi\varphi}{\varphi_o}\right)\sin\left(\delta_o\right), \tag{29}$$

is an adjustable parameter and it represents the phase difference $\delta_1 = \delta_2$ across both junctions when the magnetic flux $\varphi = 0$.

Compared to the dc SQUID equation, we have

$$\frac{\delta_1 - \delta_2}{2} = \left(\frac{\pi\varphi}{\varphi_o}\right), \tag{30}$$

where φ is the total flux through the device loop containing 2 Josephson junctions.

Again, Eq. (27) is satisfied for φ, if we substitute

$$\delta_1 = \delta_0 + \left(\frac{\pi\varphi}{\varphi_o}\right), \tag{31}$$

and

$$\delta_2 = \delta_0 - \left(\frac{\pi\varphi}{\varphi_o}\right). \tag{32}$$

What happens in the absence of applied voltage and for nonidentical junctions? For 2 nonidentical Josephson junctions [i.e. for $I_{o1} \neq I_{o2}$] and in absence of applied voltage V_0, Eq. (10) can be written as

$$I_{total} = \sqrt{I_{01}^2 + I_{02}^2 + 2I_{01}I_{02}\cos(\delta_1 - \delta_2)} \times \sin\left[\tan^{-1}\frac{I_{01}\sin\delta_1 + I_{02}\sin\delta_2}{I_{01}\cos\delta_1 + I_{02}\cos\delta_2}\right]. \tag{33}$$

Consider the relation between the phase differences and total magnetic flux passing through the entire device containing 2 Josephson junctions; we have

$$\delta_1 - \delta_2 = \frac{2e}{\hbar c}\varphi = \frac{2\pi\varphi}{\varphi_o}. \tag{34}$$

Using Eq. (34) in Eq. (33), we have

$$I_{total} = \sqrt{I_{01}^2 + I_{02}^2 + 2I_{01}I_{02}\cos\left(\frac{2\pi\varphi}{\varphi_0}\right)} \times \sin\left[\tan^{-1}\left(\frac{I_{01}\sin\delta_1 + I_{02}\sin\left(\delta_1 - \frac{2\pi\varphi}{\varphi_0}\right)}{I_{01}\cos\delta_1 + I_{02}\cos\left(\delta_1 - \frac{2\pi\varphi}{\varphi_0}\right)}\right)\right],$$

or

$$\begin{aligned} I_{total} &= \sqrt{I_{01}^2 + I_{02}^2 + 2I_{01}I_{02}\cos\left(\frac{2\pi\varphi}{\varphi_0}\right)} \\ &\times \sin\left[\tan^{-1}\left(\frac{I_{01}\sin\delta_1 + I_{02}\sin\delta_1\cos(2\pi\varphi/\varphi_0) - I_{02}\cos\delta_1\sin(2\pi\varphi/\varphi_0)}{I_{01}\cos\delta_1 + I_{02}\cos\delta_1\cos(2\pi\varphi/\varphi_0) + I_{02}\sin\delta_1\sin(2\pi\varphi/\varphi_0)}\right)\right]. \end{aligned} \tag{35}$$

Eq. (35) is a special case of Eq. (27) showing that the maximum current has 2 parts:

Case I: $\varphi = n\varphi_o$; $n =$ 0, 1, 2, 3 ...., then Eq. (35) becomes

$$\begin{aligned} I_t^{\max} &= \sqrt{(I_{o1}^2 + I_{o2}^2 + 2I_{o1}I_{o2})} \times \sin\left[\tan^{-1}\left(\frac{I_{01}\sin\delta_1 + I_{02}\sin\delta_1}{I_{01}\cos\delta_1 + I_{02}\cos\delta_1}\right)\right], \\ &= (I_{01} + I_{02}) \times \sin\left[\tan^{-1}\left(\frac{(I_{01}+I_{02})\sin\delta_1}{(I_{01}+I_{02})\cos\delta_1}\right)\right], \\ &= (I_{01} + I_{02})\sin(\delta_1)\ . \end{aligned} \tag{36}$$

For maximum value of the quantity $\sin(\delta_1)$=1, the resultant maximum current through the loop is

$$I_t^{\max} = (I_{o1} + I_{o2}). \tag{37}$$

Case II: $\varphi = \left(n + \frac{1}{2}\right)\varphi_o; n =$0, 1, 2, 3........., then Eq. (35) becomes

$$\begin{aligned} I_t^{\min} &= \sqrt{(I_{o1}^2 + I_{o2}^2 - 2I_{o1}I_{o2})} \times \sin\left[\tan^{-1}\left(\frac{I_{01}\sin\delta_1 - I_{02}\sin\delta_1}{I_{01}\cos\delta_1 - I_{02}\cos\delta_1}\right)\right], \\ &= (I_{02} - I_{01})\sin\left[\tan^{-1}\left(\frac{(I_{01}-I_{02})\sin\delta_1}{(I_{01}-I_{02})\cos\delta_1}\right)\right], \\ &= (I_{02} - I_{01})\sin(\delta_1). \end{aligned} \tag{38}$$

For minimum value of the quantity $\sin(\delta_1) = -1$, the resultant minimum current through the loop is

$$I_t^{\min} = (I_{o1} - I_{o2})\,. \tag{39}$$

The above maximum and minimum values of the current found in Eqs. (37) and (39) in the absence of applied voltage are similar to those of the dc SQUID current equations [7] for nonidentical Josephson junctions.

5. Conclusion

We have developed a model of the multijunction ac Josephson effect. From this we have concluded the following: (i) When 2 Josephson junctions are connected in parallel, the magnetic flux must flow through the entire device or loop in the absence or in the presence of dc voltage (V_o), i.e. magnetic flux that passes through the loop containing 2 Josephson junctions does not depend on the applied voltage. (ii) The magnetic flux that passes through the loop containing 2 Josephson junctions depends only on the phase difference $(\delta_1 \sim \delta_2)$ across both junctions. (iii) Amplitude of the resultant current increases with the increasing number of junction, which is the main and attractive information resulting from this work.

Since our derived relations, i.e. the current Eqs. (27) and (35) [corresponding to Eqs. (8) and (10)], verify all conditions of a dc SQUID for identical and nonidentical Josephson junctions, our multijunction theory is correct.

The proposed work may be applicable in power amplifiers [8], as well as power converters in superconducting magnetic energy storage (SMES) for electric utilities. For example, operation of electromagnetic rail launchers (EMRLs) [9] requires very high current ($\sim 10^3\text{A} - 10^6\text{A}$) and needs current multiplication. The idea of directly powering an EMRL with a SMES is under current investigation and appears to be promising. From the above analysis and discussion we can predict that the present theory may play an important role in the development of device technology in which a multijunction dc SQUID is needed.

References

[1] Josephson, B. D. *Phys. Lett.* **1962,** 1, 251-253; *Adv. Phys.* **1965**, *14*, 419–451.

[2] Poole, C. P.; Farach, H. A.; Creswick, R. J. *Superconductivity*; Academic Press, San Diego, CA, USA, 1995.

[3] Clarke, J. *Phil. Mag.* **1966**, *13*, 115–127.

[4] Clarke, J.; Goubau, W. M.; Ketchen, M. B. *J. Low Temp. Phys.* **1976**, *25*, 99–144.

[5] Fossheim, K.; Sudbø, A. *Superconductivity- Physics and Applications*; Wiley: Chichester, UK, 2004.

[6] Duzer, V.; Turner, C. W. *Principles of Superconductive Devices and Circuits*; Elsevier: New York, NY, USA, 1981.

[7] Ryhanen, T.; Seppa, H.; llmoniemi, R.; Knuutila, J. *J. Low Temp. Phys.* **1989**, *76*, 287–386.

[8] Kamal, A.; Clarke, J.; Devoret, M. H. *Phys. Rev. B* **2012**, *86*, 144510-12.

[9] Badel, A.; Tixador, P.; Arniet, M. *Supercond. Sci. Technol.* **2012**, *25*, 014006-14.