Analytical solution of the Klein–Gordon equation under the Coulomb-like scalar plus vector potential with the Laplace transforms approach

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Abstract: In this paper, analytical solution of the Klein–Gordon equation with Coulomb-like scalar plus vector potentials is obtained exactly. We considered the Laplace transform approach in our calculation. The exact bound state energy eigenvalues and the corresponding eigenfunctions are reported for various values of the quantum numbers n and l.

Key words: Coulomb-like potential, Klein–Gordon equation, Laplace transforms approach
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1. Introduction

In recent years, the problem of exact solutions of nonrelativistic and relativistic wave equations for a number of special potentials has been of great interest. Among various wave equations in both nonrelativistic and relativistic regions, the Klein–Gordon equation has been receiving much theoretical and phenomenological attention as it allows us to study spin-zero particles. The most appealing choices for the considered potentials are perhaps the spherically symmetric ones because of their wide applications in many branches of physics including particle and nuclear physics. These potentials, due to their wide applications in theoretical physics, have been receiving increasing interest within recent decades. This statement is true in both relativistic and nonrelativistic regimes. Some authors, by using different methods, studied the bound states solutions of relativistic wave equations [1–6]. For example, Jia et al. [7] obtained the exact solution of the Klein–Gordon equation under the scalar and vector kink-like potentials. Dong et al. [8] studied the Klein–Gordon equation with a Coulomb potential in D dimensions and represented the energy eigenvalues and the corresponding eigenfunctions for this system. The Klein–Gordon equation in the presence of Woods–Saxon potential was investigated by Badalov et al. [9]; the authors provided an exact expression for energy eigenvalues and corresponding eigenfunctions. Here, we shall attempt to solve the Klein–Gordon equation under the Coulomb-like scalar plus vector potential by using the Laplace transform approach (LTA). The LTA is an integral transform and is comprehensively useful in physics and engineering [10]. The LTA is a powerful method that helps us to solve second-order differential equations. In this method, a second-order equation can be converted into a simpler form whose solutions may be obtained easily. As a result, the LTA describes a simple way of solving radial and one-dimensional differential equations [11,12]. The organization of this manuscript is as follows: in section 2 we investigate the solution of the Klein–Gordon equation under the Coulomb-like potential with the LTA. The conclusion is given in section 3.

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2. Klein–Gordon equation

The Klein–Gordon equation with Coulomb-like scalar potential $S(r)$ and vector one $V(r)$ can be written as:

$$\left\{-\nabla^2 + [M + S(r)]^2\right\} \psi_{n,l}(r) = [E_{n,l} - V(r)]^2 \psi_{n,l}(r) \tag{1}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}, S(r) = -\frac{Z_s}{r}, V(r) = -\frac{Z_\nu}{r}(Z_s, Z_\nu > 0) \tag{2}$$

Substituting Eq. (2) into Eq. (1),

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + [M^2 - \frac{2MZ_s}{r} + \frac{Z_s^2}{r^2}]\right\} \psi_{n,l}(r) = \left[E_{n,l}^2 + \frac{2E_{n,l}Z_\nu}{r} + \frac{Z_\nu^2}{r^2}\right] \psi_{n,l}(r) \tag{3}$$

Defining a wave function as $\psi_{n,l}(r) = r^A f(r)$ with A as a constant and inserting it into Eq. (3) leads to

$$\frac{d^2 f(r)}{dr^2} + \frac{2(A+1)}{r}\frac{df(r)}{dr} + \frac{1}{r^2}\{(E_{n,l}^2 - M^2)r^2 + 2(MZ_s + E_{n,l}Z_\nu)r$$

$$+ (A(A+1) - \ell(\ell+1) - (Z_s^2 - Z_\nu^2))\}f(r) = 0 \tag{4}$$

and A can be obtained from

$$A^2 + A + C = 0 \rightarrow A = \frac{-1 - \sqrt{1-4C}}{2} \tag{5}$$

where

$$C = -\ell(\ell+1) - (Z_s^2 - Z_\nu^2) \tag{6}$$

By using Eq. (6) and the following abbreviations,

$$D = 2(MZ_s + E_{n,l}Z_\nu) \tag{7-a}$$

$$P = (E_{n,l}^2 - M^2) \tag{7-b}$$

Therefore, we can rewrite Eq. (4) as follows:

$$r\frac{d^2 f(r)}{dr^2} + 2(A+1)\frac{df(r)}{dr} + \frac{1}{r}\left[Pr^2 + Dr\right] = 0 \tag{8}$$

By applying a Laplace transform defined as

$$L\{f(r)\} = f(t) = \int_0^\infty dr e^{-tr} f(t) \tag{9}$$

we arrive at a first-order differential equation from Eq. (8) as

$$(t^2 + P)f'(t) - (2At + D)f(t) = 0 \tag{10}$$

The solution of Eq. (10) is

$$\ln f(t) = \ln(t + P)^{2A} + \ln\left(\frac{t - P^{1/2}}{t + P^{1/2}}\right)^{\frac{D}{2P^{1/2}} + A} \tag{11}$$

To obtain single-valued wave functions we should have

$$n = A + \frac{D}{2P^{1/2}} \tag{12}$$

Considering this condition and applying a simple series expansion to Eq. (11) gives

$$f_{n,l}(t) = \sum_{m=0}^{n} \frac{(-1)^m n!}{(n-m)!m!} (t + P^{1/2})^{2A-m} (2P^{1/2})^m \tag{13}$$

Using the inverse Laplace transformation [10] in Eq. (13) we obtain

$$f_{n,l}(r) = N_{n,l} r^{-2A-1} e^{-rP^{1/2}} \sum_{m=0}^{n} \frac{(-1)^m n!}{(n-m)!m!} \frac{\Gamma(-2A)}{\Gamma(m-2A)} (2rP^{1/2})^m \tag{14}$$

or equivalently

$$f_{n,l}(r) = N_{n,l} r^{-2A-1} e^{-rP^{1/2}} {}_1F_1(-n, -2A, (2rP^{1/2})) \tag{15}$$

where $N_{n,l}$ is a normalization constant and we use the following definition of hypergeometric function [13]:

$${}_1F_1(-n, \sigma, x) = \sum_{m=0}^{n} \frac{(-1)^m n!}{(n-m)!m!} \frac{\Gamma(\sigma)}{\Gamma(\sigma + m)} x^m \tag{16}$$

From the relation of Laguerre polynomials and confluent hypergeometric [13],

$$L_n^{\eta}(x) = \frac{\Gamma(n + \eta + 1)}{n!\Gamma(\eta + 1)} {}_1F_1(-n, \eta + 1, x) \tag{17}$$

We obtain $f_{n,l}(r)$ as

$$f_{n,l}(r) = \frac{n!\Gamma(-2A)}{\Gamma(n-2A)} N_{n,l} r^{-2A-1} e^{-rP^{1/2}} L_n^{-2A-1}(2rP^{1/2}) \tag{18}$$

Therefore, from Eq. (11) the equation of the energy of the system can be found as

$$n + \frac{1 + \sqrt{1 + 4\ell(\ell+1) + 4\left(Z_s^2 - Z_\nu^2\right)}}{2} = \frac{(MZ_s + E_{n,l}Z_\nu)}{\sqrt{\left(E_{n,l}^2 - M^2\right)}} \tag{19}$$

The Table contains some numerical results for the energy of the system. The wave function of the system for different states is shown in Figure 1. The behavior of energy of the system versus Z_1 and Z_2 for different values of n and l is represented in Figures 2 and 3.
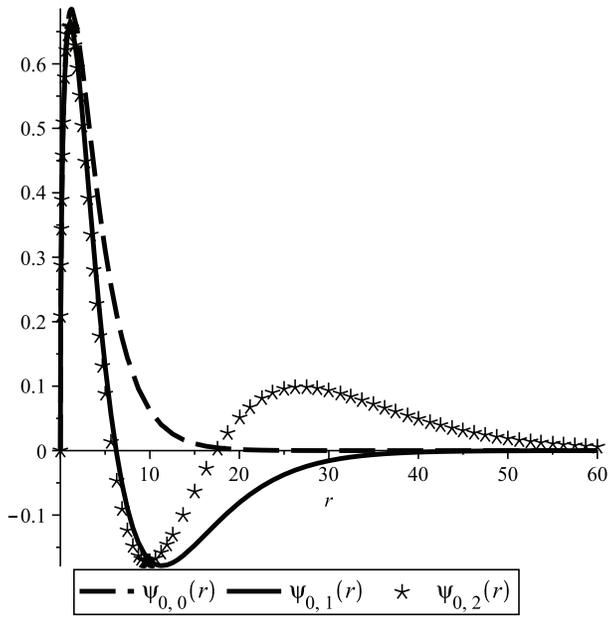
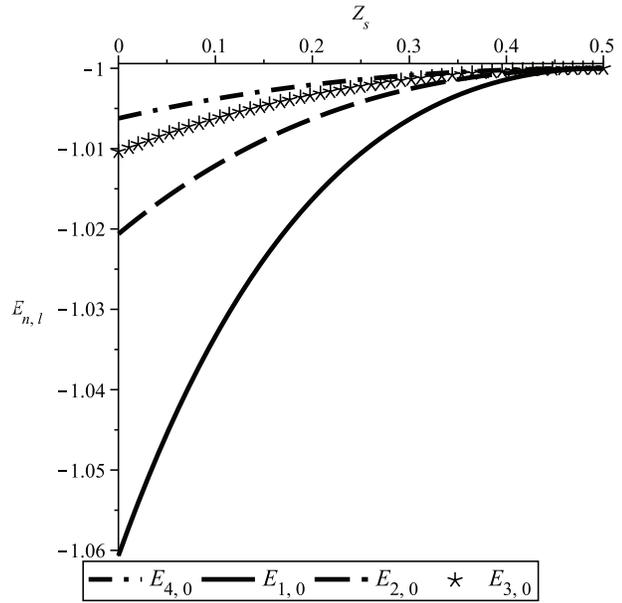**Figure 1.** Wave function of the system for different states and $Z_s = 0.9, Z_v = 0.3, M = 1$.

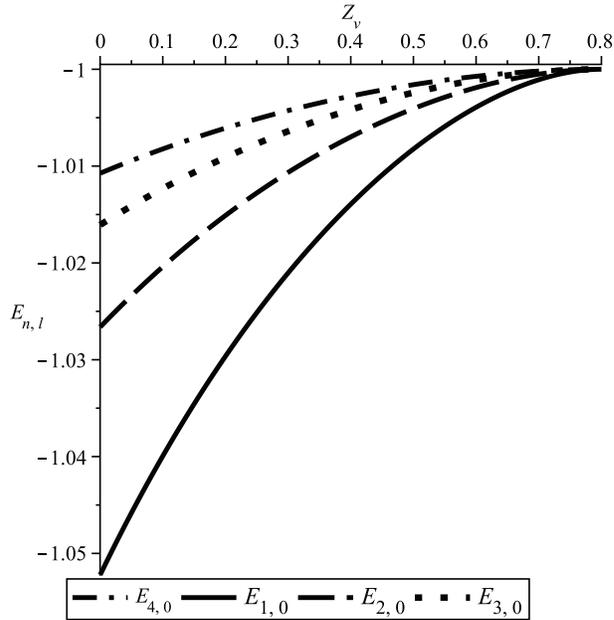**Figure 2.** Energy of the system versus Z_s for different states and $Z_v = 0.5, M = 1$.

Figure 3. Energy of the system versus Z_v for different states and $Z_s = 0.8, M = 1$.

Table. The energy of the system for different states and $M = 1$.

$|n, l\rangle$	$Z_s = 0.9, Z_v = 0.3$	$|n, l\rangle$	$Z_s = -0.1, Z_v = -0.5$
$|0, 0\rangle$	−1.073105626	$|0, 0\rangle$	−1.434964517
$|0, 1\rangle$	−1.034567174	$|0, 1\rangle$	−1.022748389
$|0, 2\rangle$	−1.017774434	$|0, 2\rangle$	−1.009355509
$|0, 3\rangle$	−1.010532782	$|0, 3\rangle$	−1.005139633
$|0, 4\rangle$	−1.006905001	$|0, 4\rangle$	−1.003255591
$|1, 0\rangle$	−1.027951633	$|1, 0\rangle$	−1.033353850
$|1, 1\rangle$	−1.016890162	$|1, 1\rangle$	−1.009577621
$|1, 2\rangle$	−1.010339850	$|1, 2\rangle$	−1.005176533
$|1, 3\rangle$	−1.006845945	$|1, 3\rangle$	−1.003265774
$|1, 4\rangle$	−1.004835196	$|1, 4\rangle$	−1.002252274
$|2, 0\rangle$	−1.014502285	$|2, 0\rangle$	−1.012122208
$|2, 1\rangle$	−1.009941970	$|2, 1\rangle$	−1.005267147
$|2, 2\rangle$	−1.006744309	$|2, 2\rangle$	−1.003284409
$|2, 3\rangle$	−1.004800502	$|2, 3\rangle$	−1.002258125
$|2, 4\rangle$	−1.003572431	$|2, 4\rangle$	−1.001650635

3. Conclusion

In this work we have studied the exact solution of the Klein–Gordon equation under unequal scalar and vector Coulomb-like potentials by using the LTA. The energy eigenvalues and the corresponding wave functions are computed. We have also given some figures to show the behavior of energy of the system versus the potential parameters.

Acknowledgment

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