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Octonion symmetric Dirac–Maxwell equations

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Abstract: We know that the octonion algebra is the largest division algebra. Therefore, we have discussed the octonion 8-dimensional space as the combination of 2 (external and internal) 4-dimensional spaces. The octonion wave equations in terms of 8 components has been written in terms of an 8×8 matrix. Octonion forms of potential as well as fields equations of dyons in terms of an 8×8 matrix are discussed in a consistent manner. At last, we have obtained the generalized Dirac–Maxwell equations of dyons in two 4-dimensional spaces, from the 8×8 matrix representation of octonion wave equations. Generalized Dirac–Maxwell equations are fully symmetric Maxwell's equations and allow for the possibility of magnetic charges and currents, i.e. analogous to electric charges and currents.

Key words: Octonions, monopole, dyons, symmetric wave equations, 8×8 matrix representation, generalized Dirac–Maxwell equations

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1. Introduction

The role of hypercomplex numbers is an important factor for understanding the various theories of physics from macroscopic to microscopic levels. Decomposition of 4 algebras, in view of the celebrated Hurwitz theorem, has been characterized from the Cayley–Dickson process over the field of real numbers of dimensions $\mathcal{D} = 1$, $\mathcal{D} = 2$, $\mathcal{D} = 4$, and $\mathcal{D} = 8$, respectively for real, complex, quaternion, and octonion algebras. There has been a revival in the formulation of natural laws so that there exist [1] 4-division algebras consisting of the algebra of real numbers, complex numbers, quaternions, and octonions. All 4 algebras are alternative with totally antisymmetric associators. Quaternions [2, 3] were the very first example of hypercomplex numbers and have been widely used [4, 5, 6, 7, 8, 9] in various applications of mathematics and physics. Since octonions [10, 11] share with complex numbers and quaternions many attractive mathematical properties, one might expect that they would be equally as useful. Octonion analysis was widely discussed by Baez [12]. It has also played an important role in the context of various physical problems [13, 14, 15, 16] of higher-dimensional supersymmetry, super gravity and super strings, etc. In recent years, it has also drawn the interest of many [17, 18, 19, 20] towards the developments of wave equation and octonion forms of Maxwell's equations. We have also studied octonion quantum chromodynamics [21], generalized octonion electrodynamics [22], generalized split-octonion electrodynamics [23], and octonionic non-Abelian gauge theory [24] and obtained the corresponding field equations (Maxwell's equations) and the equation of motion in a compact and simpler formulation. Keeping in view the recent update of theoretical physics and the role of octonions in quantum physics, we have expressed

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the octonion 8-dimensional representation in terms of 8×8 matrices. Here we may consider the 8-dimensional space as the combination of 2 (external and internal) 4-dimensional spaces. The octonion wave equations in terms of 8 components has been rewritten in terms of an 8×8 matrix. Octonion forms of generalized potential and current equations of dyons in terms of an 8×8 matrix are discussed in a consistent manner. It has been shown that due to the nonassociativity of octonion variables it is necessary to impose certain constraints to describe generalized octonion field equations in a manifestly covariant and consistent manner. Thus, we have obtained the generalized Dirac–Maxwell (GDM) equations of dyons from the 8×8 matrix representation of the octonion wave equation in a simple, compact, and consistent manner. GDM equations are fully symmetric Maxwell equations and allow for the possibility of magnetic charges and currents, analogous to electric charges and currents. The octonion variable of dyons reproduces the dynamics of electric (magnetic) charge in the absence of magnetic (electric) charge.

2. The octonion algebra

According to the celebrated Hurwitz theorem, there exist only 4 different composition algebras over the real or complex number fields. These are real numbers $\mathbb{R}$ of dimension $\mathbf{1}$, complex numbers $\mathbb{C}$ of dimension $\mathbf{2}$, quaternions $\mathbb{H}$ of dimension $\mathbf{4}$, and octonions $\mathcal{O}$ of dimension $\mathbf{8}$. Of these algebras, the quaternions $\mathbb{H}$ are not commutative and the octonions $\mathcal{O}$ are neither commutative nor associative. The composition algebra [25] is defined as an algebra $\mathbb{A}$ with identity and with a nondegenerate quadratic form Q defined over it such that Q permits composition, i.e.

$$Q(XY)=Q(X)\,Q(Y)\qquad\forall\,X,Y\in\mathbb{A}. \tag{1}$$

A composition algebra is said to be a division algebra if the quadratic form Q is anisotropic i.e. if $Q(X)=0$ implies that $X=0$. Otherwise the algebra is called split [26, 27].

The Cayley numbers or the octonions are the largest division algebra, with 8 dimensions, double the number of the quaternions from which they are an extension. The octonions can be thought of as octets (or 8-tuples) of real numbers [28, 29, 30, 31, 32]. An octonion $\mathcal{O}$ is expressed [33, 34, 35, 36, 37] as a real linear combination of the unit octonions $(e_0,e_1,e_2,e_3,e_4,e_5,e_6,e_7)$, i.e.

$$\mathcal{O}=\mathcal{O}_0e_0+\sum_{A=1}^{7}\mathcal{O}_Ae_A\,, \tag{2}$$

where e_0 and $e_A(A=1,2,.....,7)$ are octonion units. The octet is defined as $e_0,e_1,e_2,e_3,e_4,e_5,e_6,e_7$, which is known as the octonion basis, and its elements satisfy the following multiplication rules:

$$\begin{aligned}
e_0 &= 1,\\
e_A^2 &= -\,1,\\
e_0e_A = e_Ae_0 &= e_A,\ (\forall\,A=1,2,.......7)\,,\\
e_Ae_B &= -\,\delta_{AB}e_0+f_{ABC}\,e_C\,.\quad (A,B,C=1,2,......7).
\end{aligned} \tag{3}$$

The structure constants f_{ABC} are completely antisymmetric and take the value 1, i.e.

$$f_{ABC}=+1;\ \forall\,(ABC)\equiv(123),\,(471),\,(257),\,(165),\,(624),\,(543),\,(736). \tag{4}$$

As such, the elements e_A satisfy the following multiplication rules:

$$(e_1 e_2 = e_3)\,,\ (e_4 e_7 = e_1)\,,\ (e_2 e_5 = e_7)\,,$$
$$(e_1 e_6 = e_5)\,,\ (e_6 e_2 = e_4)\,,\ (e_5 e_4 = e_3)\,,\ (e_7 e_3 = e_6)\,. \tag{5}$$

The above multiplication rules for the octonion basis are given in the Table.

Table. Octonion multiplication table.

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

Thus, we get the following relations among octonion basis elements:

$$\begin{aligned}[e_A,\ e_B] &= 2f_{ABC}e_C;\\ \{e_A,\ e_B\} &= -\delta_{AB}e_0;\\ e_A(e_B e_C) &\neq (e_A e_B)e_C;\end{aligned} \tag{6}$$

where δ_{AB} is the usual Kronecker delta-Dirac symbol. The octonion conjugate is thus defined as:

$$\begin{aligned}\bar{\mathcal{O}} &= \mathcal{O}_0 e_0 - \mathcal{O}_1 e_1 - \mathcal{O}_2 e_2 - \mathcal{O}_3 e_3 - \mathcal{O}_4 e_4 - \mathcal{O}_5 e_5 - \mathcal{O}_6 e_6 - \mathcal{O}_7 e_7\\ &= \mathcal{O}_0 e_0 - \sum_{A=1}^{7} \mathcal{O}_A e_A\,.\end{aligned} \tag{7}$$

An octonion can be decomposed in terms of its scalar $(\mathbf{Sc}(\mathcal{O}))$ and vector $(\mathbf{Vec}(\mathcal{O}))$ parts as

$$\mathbf{Sc}(\mathcal{O}) = \frac{1}{2}(\mathcal{O} + \bar{\mathcal{O}}) = \mathcal{O}_0\,, \quad \mathbf{Vec}(\mathcal{O}) = \frac{1}{2}(\mathcal{O} - \bar{\mathcal{O}}) = \sum_{A=1}^{7} \mathcal{O}_A e_A\,. \tag{8}$$

Conjugates of the product of 2 octonions and its own are described as

$$(\overline{\mathcal{O}_1 \mathcal{O}_2}) = \overline{\mathcal{O}_2}\,\overline{\mathcal{O}_1}\,, \quad \overline{(\bar{\mathcal{O}})} = \mathcal{O}. \tag{9}$$

The norm of the octonion $N(\mathcal{O})$ is defined as

$$N(\mathcal{O}) = \bar{\mathcal{O}}\,\mathcal{O} = \mathcal{O}\,\bar{\mathcal{O}} = \sum_{A=0}^{7} \mathcal{O}_A^2 e_0, \tag{10}$$

which is 0 if $\mathcal{O} = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra:

$$N(\mathcal{O}_1 \mathcal{O}_2) = N(\mathcal{O}_1)\,N(\mathcal{O}_2) = N(\mathcal{O}_2)\,N(\mathcal{O}_1). \tag{11}$$

Eq. (11) shows that octonions are not associative in nature and thus do not form the group in their usual form. Nonassociativity of octonion algebra is provided by the associator

$$(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) = (\mathcal{O}_1\mathcal{O}_2)\,\mathcal{O}_3 - \mathcal{O}_1\,(\mathcal{O}_2\mathcal{O}_3)\,, \tag{12}$$

where $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ are any 3 octonions. If the associator is totally antisymmetric for exchanges of any 3 variables, i.e.

$$(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) = -(\mathcal{O}_3, \mathcal{O}_2, \mathcal{O}_1) = -(\mathcal{O}_2, \mathcal{O}_1, \mathcal{O}_3) = -(\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_2)\,, \tag{13}$$

then the algebra is called alternative. We can obtain the real octonion algebra by a complex extension of the quaternions as [26, 27]

$$\begin{aligned}\mathcal{O} &= (\mathcal{O}_0 e_0 + \mathcal{O}_1 e_1 + \mathcal{O}_2 e_2 + \mathcal{O}_3 e_3) + e_7(\mathcal{O}_7 e_0 + \mathcal{O}_4 e_1 + \mathcal{O}_5 e_2 + \mathcal{O}_6 e_3)\\ &= \mathbb{Q}_1 + e_7\mathbb{Q}_2,\end{aligned} \tag{14}$$

where $\mathbb{Q}_1$, $\mathbb{Q}_2$ are the quaternion subalgebra $\mathbb{H}$ generated by the basis e_0, e_1, e_2, e_3. Similarly, the octonion conjugate of Eq. (14) may be defined as

$$\bar{\mathcal{O}} = \bar{\mathbb{Q}}_1 + e_7\bar{\mathbb{Q}}_2\,. \tag{15}$$

Here $\bar{\mathbb{Q}}_1 = \mathcal{O}_0 e_0 - \sum_{\mu=1}^{3} e_\mu \mathcal{O}_\mu$ and $\bar{\mathbb{Q}}_2 = \mathcal{O}_7 e_0 - e_7 \sum_{\mu=4}^{7} e_\mu \mathcal{O}_\mu$ are respectively the quaternion conjugate of $\mathbb{Q}_1$ and $\mathbb{Q}_2$.

The complex octonion $\mathbb{O}$ [36] can be expressed by a linear combination of 2 real octonions $\mathcal{O}$ and $\mathcal{O}'$ as [30, 32, 37]

$$\begin{aligned}\mathbb{O} &= \mathcal{O} + i\mathcal{O}'\\ &= \left(\mathcal{O}_0 e_0 + \sum_{A=1}^{7} \mathcal{O}_A e_A\right) + i\left(\mathcal{O}'_0 e_0 + \sum_{A=1}^{7} \mathcal{O}'_A e_A\right),\end{aligned} \tag{16}$$

which can be written as

$$\begin{aligned}\mathbb{O} &= \sum_{A=0}^{7} (\mathcal{O}_A + i\mathcal{O}'_A)\, e_A\\ &= (\mathcal{O}_0 + i\mathcal{O}'_0)\, e_0 + (\mathcal{O}_1 + i\mathcal{O}'_1)\, e_1 + (\mathcal{O}_2 + i\mathcal{O}'_2)\, e_2 + (\mathcal{O}_3 + i\mathcal{O}'_3)\, e_3\\ &+ (\mathcal{O}_4 + i\mathcal{O}'_4)\, e_4 + (\mathcal{O}_5 + i\mathcal{O}'_5)\, e_5 + (\mathcal{O}_6 + i\mathcal{O}'_6)\, e_6 + (\mathcal{O}_7 + i\mathcal{O}'_7)\, e_7\,.\end{aligned} \tag{17}$$

Thus, Eq. (17) can be reduced to the following complex octonion form:

$$\mathbb{O} = \mathbb{O}_0 e_0 + \sum_{A=1}^{7} \mathbb{O}_A e_A\,, \tag{18}$$

where $\mathbb{O}_0 \rightarrow (\mathcal{O}_0 + i\mathcal{O}'_0)$, $\mathbb{O}_A \rightarrow (\mathcal{O}_A + i\mathcal{O}'_A)$ $(\forall\, A = 1, 2, ..., 7)$ are complex numbers of octonions and i denotes the complex unit $(i^2 = -1)$. The complex octonions have similar algebraic properties as the octonions, while they differ by having 16 components and an additional complex unit (or extra hypercomplex unit) i.

3. Matrix formulation of octonion wave equations

Let us consider the wave function Ψ in the form of 8-component octonion algebra $\mathcal{O}$ by a complex extension of the quaternions as

$$\begin{aligned}\Psi &= \sum_{\mu=0}^{3} e_\mu \psi_\mu + \sum_{\mu=0}^{3} e_{\mu+4}\chi_\mu \\ &= (e_0\psi_0 + e_1\psi_1 + e_2\psi_2 + e_3\psi_3) + (e_4\chi_0 + e_5\chi_1 + e_6\chi_2 + e_7\chi_3) \ ,\end{aligned} \tag{19}$$

where the components $\psi_\mu(\vec{r},t)$ and $\chi_\mu(\vec{r},t)$ $(\mu = 0,1,2,3)$ are scalar (complex in general) functions of spatial coordinates and time. The octonion wave function of Eq. (19) can be written in a simpler and compact form:

$$\Psi = \left(\psi_0 + e_j\vec{\psi}\right) + (e_k\chi_0 + e_l\vec{\chi}) \ . \qquad (\forall\, j = 1,2,3;\ k = 4;\ l = 5,6,7) \tag{20}$$

Here the wave function $\vec{\psi} \to (\psi_1, \psi_2, \psi_3)$ and $\vec{\chi} \to (\chi_1, \chi_2, \chi_3)$. Since octonions are neither commutative nor associative, one has to be very careful to multiply the octonion either from left or from right in terms of regularity conditions [38]. In order to explain the 8 different octonion wave functions, we can multiply from the left by octonion basis $e_0, e_1, e_2, ...e_7$, and then we get:

$$\begin{aligned}
e_0\,(\Psi) &= e_0\psi_0 + e_1\psi_1 + e_2\psi_2 + e_3\psi_3 + e_4\chi_0 + e_5\chi_1 + e_6\chi_2 + e_7\chi_3\,; \\
e_1\,(\Psi) &= e_1\psi_0 - e_0\psi_1 + e_3\psi_2 - e_2\psi_3 + e_7\chi_0 - e_6\chi_1 + e_5\chi_2 - e_4\chi_3\,; \\
e_2\,(\Psi) &= e_2\psi_0 - e_3\psi_1 - e_0\psi_2 + e_1\psi_3 + e_6\chi_0 + e_7\chi_1 - e_4\chi_2 - e_5\chi_3\,; \\
e_3\,(\Psi) &= e_3\psi_0 + e_2\psi_1 - e_1\psi_2 - e_0\psi_3 - e_5\chi_0 + e_4\chi_1 + e_7\chi_2 - e_6\chi_3\,; \\
e_4\,(\Psi) &= e_4\psi_0 - e_7\psi_1 - e_6\psi_2 + e_5\psi_3 - e_0\chi_0 - e_3\chi_1 + e_2\chi_2 + e_1\chi_3\,; \\
e_5\,(\Psi) &= e_5\psi_0 + e_6\psi_1 - e_7\psi_2 - e_4\psi_3 + e_3\chi_0 - e_0\chi_1 - e_1\chi_2 + e_2\chi_3\,; \\
e_6\,(\Psi) &= e_6\psi_0 - e_5\psi_1 + e_4\psi_2 - e_7\psi_3 - e_2\chi_0 + e_1\chi_1 - e_0\chi_2 + e_3\chi_3\,; \\
e_7\,(\Psi) &= e_7\psi_0 + e_4\psi_1 + e_5\psi_2 + e_6\psi_3 - e_1\chi_0 - e_2\chi_1 - e_3\chi_2 - e_0\chi_3\,.
\end{aligned} \tag{21}$$

Similarly, we can multiply from the right by octonion basis $(e_0, e_1, e_2, ...e_7)$ to the octonion wave function Ψ (19), and we get:

$$\begin{aligned}
(\Psi)\,e_0 &= e_0\psi_0 + e_1\psi_1 + e_2\psi_2 + e_3\psi_3 + e_4\chi_0 + e_5\chi_1 + e_6\chi_2 + e_7\chi_3\,; \\
(\Psi)\,e_1 &= e_1\psi_0 - e_0\psi_1 - e_3\psi_2 + e_2\psi_3 - e_7\chi_0 + e_6\chi_1 - e_5\chi_2 + e_4\chi_3\,; \\
(\Psi)\,e_2 &= e_2\psi_0 + e_3\psi_1 - e_0\psi_2 - e_1\psi_3 - e_6\chi_0 - e_7\chi_1 + e_4\chi_2 + e_5\chi_3\,; \\
(\Psi)\,e_3 &= e_3\psi_0 - e_2\psi_1 + e_1\psi_2 - e_0\psi_3 + e_5\chi_0 - e_4\chi_1 - e_7\chi_2 + e_6\chi_3\,; \\
(\Psi)\,e_4 &= e_4\psi_0 + e_7\psi_1 + e_6\psi_2 - e_5\psi_3 - e_0\chi_0 + e_3\chi_1 - e_2\chi_2 - e_1\chi_3\,; \\
(\Psi)\,e_5 &= e_5\psi_0 - e_6\psi_1 + e_7\psi_2 + e_4\psi_3 - e_3\chi_0 - e_0\chi_1 + e_1\chi_2 - e_2\chi_3\,; \\
(\Psi)\,e_6 &= e_6\psi_0 + e_5\psi_1 - e_4\psi_2 + e_7\psi_3 + e_2\chi_0 - e_1\chi_1 - e_0\chi_2 - e_3\chi_3\,; \\
(\Psi)\,e_7 &= e_7\psi_0 - e_4\psi_1 - e_5\psi_2 - e_6\psi_3 + e_1\chi_0 + e_2\chi_1 + e_3\chi_2 - e_0\chi_3\,.
\end{aligned} \tag{22}$$

Thus, the octonion left-handed multiplication (21) is not equal to the right-handed multiplication (22). The basic differences between these 2 multiplication are:

$$\begin{aligned}
&[e_0,\,\Psi]=0\,;\\
&\{e_j,\,\Psi\}=2(e_j\psi_0-\psi_j)\,;\qquad(\forall\, j=1,2,3)\\
&\{e_j,\,\Psi\}=2(e_j\psi_0-\chi_{j-4})\,.\qquad(\forall\, j=4,5,6,7)
\end{aligned}\tag{23}$$

On the other hand, the idea of 8×8 matrix representation [39, 40] of octonions can be an extension of 2×2 matrices of split octonions. The equivalent matrix visualization in terms of octonionic 8×8 matrix representation may be expressed as

$$\mathcal{O}=\,\mathcal{O}_0e_0+\sum_{A=1}^{7}\mathcal{O}_Ae_A\;=\begin{pmatrix}
\mathcal{O}_0 & -\mathcal{O}_1 & -\mathcal{O}_2 & -\mathcal{O}_3 & -\mathcal{O}_4 & -\mathcal{O}_5 & -\mathcal{O}_6 & -\mathcal{O}_7\\
\mathcal{O}_1 & \mathcal{O}_0 & -\mathcal{O}_3 & \mathcal{O}_2 & -\mathcal{O}_7 & \mathcal{O}_6 & -\mathcal{O}_5 & \mathcal{O}_4\\
\mathcal{O}_2 & \mathcal{O}_3 & \mathcal{O}_0 & -\mathcal{O}_1 & -\mathcal{O}_6 & -\mathcal{O}_7 & \mathcal{O}_4 & \mathcal{O}_5\\
\mathcal{O}_3 & -\mathcal{O}_2 & \mathcal{O}_1 & \mathcal{O}_0 & \mathcal{O}_5 & -\mathcal{O}_4 & -\mathcal{O}_7 & \mathcal{O}_6\\
\mathcal{O}_4 & \mathcal{O}_7 & \mathcal{O}_6 & -\mathcal{O}_5 & \mathcal{O}_0 & \mathcal{O}_3 & -\mathcal{O}_2 & -\mathcal{O}_1\\
\mathcal{O}_5 & -\mathcal{O}_6 & \mathcal{O}_7 & \mathcal{O}_4 & -\mathcal{O}_3 & \mathcal{O}_0 & \mathcal{O}_1 & -\mathcal{O}_2\\
\mathcal{O}_6 & \mathcal{O}_5 & -\mathcal{O}_4 & \mathcal{O}_7 & \mathcal{O}_2 & -\mathcal{O}_1 & \mathcal{O}_0 & -\mathcal{O}_3\\
\mathcal{O}_7 & -\mathcal{O}_4 & -\mathcal{O}_5 & -\mathcal{O}_6 & \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_3 & \mathcal{O}_0
\end{pmatrix}.\tag{24}$$

In the case of quaternion representation, i.e. $\mathbb{H}=\,H_0e_0+\sum_{A=1}^{3}H_Ae_A$ in terms of an 8×8 matrix, it can be easily expressed from Eq. (24) if $\mathcal{O}_j\,=\,0\,,\,(\forall\, j\,=\,4,5,6,7)$. Keeping in view the properties of octonions [10, 11, 12, 13] and its 8-dimensional connection, we may write the wave function Ψ (19) in terms of 8×8 matrix representation as

$$\Psi=\;\sum_{\mu=0}^{3}e_\mu\psi_\mu+\sum_{\mu=0}^{3}e_{\mu+4}\chi_\mu=\begin{pmatrix}
\psi_0 & -\psi_1 & -\psi_2 & -\psi_3 & -\chi_0 & -\chi_1 & -\chi_2 & -\chi_3\\
\psi_1 & \psi_0 & -\psi_3 & \psi_2 & -\chi_3 & \chi_2 & -\chi_1 & \chi_0\\
\psi_2 & \psi_3 & \psi_0 & -\psi_1 & -\chi_2 & -\chi_3 & \chi_0 & \chi_1\\
\psi_3 & -\psi_2 & \psi_1 & \psi_0 & \chi_1 & -\chi_0 & -\chi_3 & \chi_2\\
\chi_0 & \chi_3 & \chi_2 & -\chi_1 & \psi_0 & \psi_3 & -\psi_2 & -\psi_1\\
\chi_1 & -\chi_2 & \chi_3 & \chi_0 & -\psi_3 & \psi_0 & \psi_1 & -\psi_2\\
\chi_2 & \chi_1 & -\chi_0 & \chi_3 & \psi_2 & -\psi_1 & \psi_0 & -\psi_3\\
\chi_3 & -\chi_0 & -\chi_1 & -\chi_2 & \psi_1 & \psi_2 & \psi_3 & \psi_0
\end{pmatrix},\tag{25}$$

and we may now introduce the octonion differential operator $\mathscr{D}$ as

$$\mathscr{D}=\,\partial_0e_0+\sum_{A=1}^{7}\partial_Ae_A\;=\begin{pmatrix}
\partial_0 & -\partial_1 & -\partial_2 & -\partial_3 & -\partial_4 & -\partial_5 & -\partial_6 & -\partial_7\\
\partial_1 & \partial_0 & -\partial_3 & \partial_2 & -\partial_7 & \partial_6 & -\partial_5 & \partial_4\\
\partial_2 & \partial_3 & \partial_0 & -\partial_1 & -\partial_6 & -\partial_7 & \partial_4 & \partial_5\\
\partial_3 & -\partial_2 & \partial_1 & \partial_0 & \partial_5 & -\partial_4 & -\partial_7 & \partial_6\\
\partial_4 & \partial_7 & \partial_6 & -\partial_5 & \partial_0 & \partial_3 & -\partial_2 & -\partial_1\\
\partial_5 & -\partial_6 & \partial_7 & \partial_4 & -\partial_3 & \partial_0 & \partial_1 & -\partial_2\\
\partial_6 & \partial_5 & -\partial_4 & \partial_7 & \partial_2 & -\partial_1 & \partial_0 & -\partial_3\\
\partial_7 & -\partial_4 & -\partial_5 & -\partial_6 & \partial_1 & \partial_2 & \partial_3 & \partial_0
\end{pmatrix}.\tag{26}$$

Here we may consider the 8-dimensional space as the combination of 2 (external and internal) 4-dimensional spaces. In order to write the octonion wave equations [41, 42, 43], we operate the octonion differential operator $\mathscr{D}$ (26) to the octonion wave function Ψ (25), i.e.

$$\mathscr{D}\,\Psi=\mathcal{G}\longmapsto e_0\mathcal{G}_0+\sum_{\mu=1}^{7}e_\mu\mathcal{G}_\mu,\tag{27}$$

where $\mathcal{G}$ is again an octonion 8-dimensional representation:

$$\mathcal{G}=\begin{pmatrix}\mathcal{G}_0 & -\mathcal{G}_1 & -\mathcal{G}_2 & -\mathcal{G}_3 & -\mathcal{G}_4 & -\mathcal{G}_5 & -\mathcal{G}_6 & -\mathcal{G}_7\\ \mathcal{G}_1 & \mathcal{G}_0 & -\mathcal{G}_3 & \mathcal{G}_2 & -\mathcal{G}_7 & \mathcal{G}_6 & -\mathcal{G}_5 & \mathcal{G}_4\\ \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_0 & -\mathcal{G}_1 & -\mathcal{G}_6 & -\mathcal{G}_7 & \mathcal{G}_4 & \mathcal{G}_5\\ \mathcal{G}_3 & -\mathcal{G}_2 & \mathcal{G}_1 & \mathcal{G}_0 & \mathcal{G}_5 & -\mathcal{G}_4 & -\mathcal{G}_7 & \mathcal{G}_6\\ \mathcal{G}_4 & \mathcal{G}_7 & \mathcal{G}_6 & -\mathcal{G}_5 & \mathcal{G}_0 & \mathcal{G}_3 & -\mathcal{G}_2 & -\mathcal{G}_1\\ \mathcal{G}_5 & -\mathcal{G}_6 & \mathcal{G}_7 & \mathcal{G}_4 & -\mathcal{G}_3 & \mathcal{G}_0 & \mathcal{G}_1 & -\mathcal{G}_2\\ \mathcal{G}_6 & \mathcal{G}_5 & -\mathcal{G}_4 & \mathcal{G}_7 & \mathcal{G}_2 & -\mathcal{G}_1 & \mathcal{G}_0 & -\mathcal{G}_3\\ \mathcal{G}_7 & -\mathcal{G}_4 & -\mathcal{G}_5 & -\mathcal{G}_6 & \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_0\end{pmatrix}. \tag{28}$$

The value of $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_7$ may be written in the following manner:

$$\begin{aligned}
\mathcal{G}_0 &= \partial_0\psi_0 - \partial_1\psi_1 - \partial_2\psi_2 - \partial_3\psi_3 - \partial_4\chi_0 - \partial_5\chi_1 - \partial_6\chi_2 - \partial_7\chi_3;\\
\mathcal{G}_1 &= \partial_0\psi_1 + \partial_1\psi_0 + \partial_2\psi_3 - \partial_3\psi_2 + \partial_6\chi_1 - \partial_5\chi_2 - \partial_7\chi_0 + \partial_4\chi_3;\\
\mathcal{G}_2 &= \partial_0\psi_2 + \partial_2\psi_0 + \partial_3\psi_1 - \partial_1\psi_3 + \partial_4\chi_2 - \partial_6\chi_0 - \partial_7\chi_1 + \partial_5\chi_3;\\
\mathcal{G}_3 &= \partial_0\psi_3 + \partial_3\psi_0 + \partial_1\psi_2 - \partial_2\psi_1 + \partial_6\chi_3 - \partial_7\chi_2 + \partial_5\chi_0 - \partial_4\chi_1;\\
\mathcal{G}_4 &= \partial_0\chi_0 + \partial_4\psi_0 + \partial_3\chi_1 - \partial_5\psi_3 - \partial_2\chi_2 + \partial_6\psi_2 - \partial_1\chi_3 + \partial_7\psi_1;\\
\mathcal{G}_5 &= \partial_0\chi_1 + \partial_5\psi_0 + \partial_1\chi_2 - \partial_6\psi_1 + \partial_7\psi_2 - \partial_2\chi_3 - \partial_3\chi_0 + \partial_4\psi 3;\\
\mathcal{G}_6 &= \partial_0\chi_2 + \partial_6\psi_0 - \partial_1\chi_1 + \partial_5\psi_1 + \partial_2\chi_0 - \partial_4\psi_2 - \partial_3\chi_3 + \partial_7\psi_3;\\
\mathcal{G}_7 &= \partial_0\chi_3 + \partial_7\psi_0 + \partial_1\chi_0 - \partial_4\psi_1 + \partial_2\chi_1 - \partial_5\psi_2 - \partial_6\psi_3 + \partial_3 3\chi_2.
\end{aligned} \tag{29}$$

These equations represent the octonion wave equations in terms of 8 components. Thus, we may now interpret these octonion wave equations as the classical wave (field) equations of physical variables. One-dimensional octonion representation is identical to 8-dimensional spaces over the field of real numbers. It is isomorphic to 4-dimensional space representation over the field of complex variables, which is equivalent to 2-dimensional space representation over quaternion field variables. Similarly, 1-dimensional quaternion space is isomorphic to 4-dimensional space over the field of real numbers, which is identical to 2-dimensional space over the field of complex numbers.

4. GDM equations for dyons

The octonion division algebra is nonassociative and noncommutative. Keeping in view the important properties of octonions [25, 26, 27, 28] and their 8×8 matrix connection, we may write the usual coordinates $(\partial_x, \partial_y, \partial_z, i\partial_t)$ in internal space and the other coordinates $(\partial_X, \partial_Y, \partial_Z, i\partial_T)$ in octonionic external space with each signature $(+,+,+,-)$. In this section, we have used two 4-dimensional spaces, namely internal and external spaces, for the existence of both magnetic monopoles and dyons in higher-dimensional formalism. Since octonions can expresses the 8-dimensional space, the octonion differential operator $\mathscr{D}$ (26) can be written in two 4-dimensional spaces as

$$\mathscr{D} = (\partial_x, \partial_y, \partial_z, -i\partial_t, \partial_X, \partial_Y, \partial_Z, -i\partial_T) = \left(\vec{\nabla}, -i\partial_t, \vec{\nabla}', -i\partial_T\right). \tag{30}$$

In terms of octonionic basis elements, the octonion differential operator $\mathscr{D}$ may be expressed as

$$\mathscr{D} = \left(-i\frac{\partial}{\partial T} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right)e_0 + e_7\left(e_1\frac{\partial}{\partial X} + e_2\frac{\partial}{\partial Y} + e_3\frac{\partial}{\partial Z} - i\frac{\partial}{\partial t}\right). \tag{31}$$

Here the octonion unit e_7 playing the role of an imaginary quantity is no longer invariant scalar as it does not commute with other quaternion units e_1, e_2, e_3 and we have used the octonion product, i.e. $(e_4 e_7 = e_1)$, $(e_5 e_7 = e_2)$, $(e_6 e_7 = e_3)$, etc. We have used the unit value of coefficients along with natural units $(c = \hbar = 1)$ throughout the text. Accordingly, the 8×8 matrix representation of Eq. (31) is

$$\mathscr{D} = \begin{pmatrix} -i\partial_T & -\partial_x & -\partial_y & -\partial_z & -\partial_X & -\partial_Y & -\partial_Z & -i\partial_t \\ \partial_x & -i\partial_T & -\partial_z & \partial_y & -i\partial_t & \partial_Z & -\partial_Y & \partial_X \\ \partial_y & \partial_z & -i\partial_T & -\partial_x & -\partial_Z & -i\partial_t & \partial_X & \partial_Y \\ \partial_z & -\partial_y & \partial_x & -i\partial_T & \partial_Y & -\partial_X & -i\partial_t & \partial_Z \\ \partial_X & i\partial_t & \partial_Z & -\partial_Y & -i\partial_T & \partial_z & -\partial_y & -\partial_x \\ \partial_Y & -\partial_Z & i\partial_t & \partial_X & -\partial_z & -i\partial_T & \partial_x & -\partial_y \\ \partial_Z & \partial_Y & -\partial_X & i\partial_t & \partial_y & -\partial_x & -i\partial_T & -\partial_z \\ i\partial_t & -\partial_X & -\partial_Y & -\partial_Z & \partial_x & \partial_y & \partial_z & -i\partial_T \end{pmatrix}. \tag{32}$$

Octonion conjugate (i.e. $e_0 \to e_0,\ e_j \to -e_j\ ,\ \forall\, j = 1, 2, ..., 7$) of Eq. (31) can be written as

$$\bar{\mathscr{D}} = \left(-i\frac{\partial}{\partial T} - e_1\frac{\partial}{\partial x} - e_2\frac{\partial}{\partial y} - e_3\frac{\partial}{\partial z}\right) e_0 - e_7\left(e_1\frac{\partial}{\partial X} + e_2\frac{\partial}{\partial Y} + e_3\frac{\partial}{\partial Z} - i\frac{\partial}{\partial t}\right), \tag{33}$$

and in terms of 8×8 matrix representation, we get

$$\bar{\mathscr{D}} = \begin{pmatrix} -i\partial_T & \partial_x & \partial_y & \partial_z & \partial_X & \partial_Y & \partial_Z & i\partial_t \\ -\partial_x & -i\partial_T & \partial_z & -\partial_y & i\partial_t & -\partial_Z & \partial_Y & -\partial_X \\ -\partial_y & -\partial_z & -i\partial_T & \partial_x & \partial_Z & i\partial_t & -\partial_X & -\partial_Y \\ -\partial_z & \partial_y & -\partial_x & -i\partial_T & -\partial_Y & \partial_X & i\partial_t & -\partial_Z \\ -\partial_X & -i\partial_t & -\partial_Z & \partial_Y & -i\partial_T & -\partial_z & \partial_y & \partial_x \\ -\partial_Y & \partial_Z & -i\partial_t & -\partial_X & \partial_z & -i\partial_T & -\partial_x & \partial_y \\ -\partial_Z & -\partial_Y & \partial_X & -i\partial_t & -\partial_y & \partial_x & -i\partial_T & \partial_z \\ -i\partial_t & \partial_X & \partial_Y & \partial_Z & -\partial_x & -\partial_y & -\partial_z & -i\partial_T \end{pmatrix}. \tag{34}$$

The product of $\mathscr{D}\,\bar{\mathscr{D}}$ may be written in the following manner:

$$\begin{aligned} \mathscr{D}\,\bar{\mathscr{D}} &= \left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right\} + \left\{\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} - \frac{\partial^2}{\partial T^2}\right\} \\ &= \left\{\nabla^2 - \frac{\partial^2}{\partial t^2}\right\} + \left\{\nabla'^2 - \frac{\partial^2}{\partial T^2}\right\} \\ &= \Box + \Box' = \boxdot \equiv \bar{\mathscr{D}}\,\mathscr{D}. \end{aligned} \tag{35}$$

Here $\Box \to \left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)$ is the D'Alembertian operator in internal space-time coordinates while $\Box' \to \left(\nabla'^2 - \frac{\partial^2}{\partial T^2}\right)$ is for external space-time coordinates, and $\boxdot$ is an 8-dimensional structure having both internal and external space-time coordinates.

On the other hand, a dyon [33] is a hypothetical particle with both electric and magnetic charges. A dyon with zero electric charge is usually referred to as a magnetic monopole. Many grand unified theories predict the existence of both magnetic monopoles and dyons. In order to consider the generalized electromagnetic fields of dyons, we may write the various quantum equations from the octonion 8×8 matrix formulation. Thus, the octonion potential $\mathscr{V} \to (\mathscr{V}_0, \mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \mathscr{V}_4, \mathscr{V}_5, \mathscr{V}_6, \mathscr{V}_7)$ in 8-dimensional formalism as the combinations of two 4-dimensional spaces is defined as

$$\mathscr{V} = \mathscr{V}_0 e_0 + \sum_{A=1}^{7} \mathscr{V}_A e_A = \begin{pmatrix} \mathscr{V}_0 & -\mathscr{V}_1 & -\mathscr{V}_2 & -\mathscr{V}_3 & -\mathscr{V}_4 & -\mathscr{V}_5 & -\mathscr{V}_6 & -\mathscr{V}_7 \\ \mathscr{V}_1 & \mathscr{V}_0 & -\mathscr{V}_3 & \mathscr{V}_2 & -\mathscr{V}_7 & \mathscr{V}_6 & -\mathscr{V}_5 & \mathscr{V}_4 \\ \mathscr{V}_2 & \mathscr{V}_3 & \mathscr{V}_0 & -\mathscr{V}_1 & -\mathscr{V}_6 & -\mathscr{V}_7 & \mathscr{V}_4 & \mathscr{V}_5 \\ \mathscr{V}_3 & -\mathscr{V}_2 & \mathscr{V}_1 & \mathscr{V}_0 & \mathscr{V}_5 & -\mathscr{V}_4 & -\mathscr{V}_7 & \mathscr{V}_6 \\ \mathscr{V}_4 & \mathscr{V}_7 & \mathscr{V}_6 & -\mathscr{V}_5 & \mathscr{V}_0 & \mathscr{V}_3 & -\mathscr{V}_2 & -\mathscr{V}_1 \\ \mathscr{V}_5 & -\mathscr{V}_6 & \mathscr{V}_7 & \mathscr{V}_4 & -\mathscr{V}_3 & \mathscr{V}_0 & \mathscr{V}_1 & -\mathscr{V}_2 \\ \mathscr{V}_6 & \mathscr{V}_5 & -\mathscr{V}_4 & \mathscr{V}_7 & \mathscr{V}_2 & -\mathscr{V}_1 & \mathscr{V}_0 & -\mathscr{V}_3 \\ \mathscr{V}_7 & -\mathscr{V}_4 & -\mathscr{V}_5 & -\mathscr{V}_6 & \mathscr{V}_1 & \mathscr{V}_2 & \mathscr{V}_3 & \mathscr{V}_0 \end{pmatrix}. \tag{36}$$

We may now identify the components of generalized potential as

$$(\mathscr{V}_0, \mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \mathscr{V}_4, \mathscr{V}_5, \mathscr{V}_6, \mathscr{V}_7) \Longrightarrow (-\varphi, \mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_z, -i\mathcal{B}_x, -i\mathcal{B}_y, -i\mathcal{B}_z, -i\phi), \quad (i = \sqrt{-1}) \tag{37}$$

where $(\phi, \overrightarrow{\mathcal{A}}) \equiv \{\mathcal{A}^\mu\}$ and $(\varphi, \overrightarrow{\mathcal{B}}) \equiv \{\mathcal{B}^\mu\}$ are respectively described as the components of electric $\{\mathcal{A}_\mu\}$ and magnetic $\{\mathcal{B}_\mu\}$ 4-potentials of dyons. Thus, from Eq. (36), we get

$$\mathscr{V} = \begin{pmatrix} -\varphi & -\mathcal{A}_x & -\mathcal{A}_y & -\mathcal{A}_z & i\mathcal{B}_x & i\mathcal{B}_y & i\mathcal{B}_z & i\phi \\ \mathcal{A}_x & -\varphi & -\mathcal{A}_z & \mathcal{A}_y & i\phi & -i\mathcal{B}_z & i\mathcal{B}_y & -i\mathcal{B}_x \\ \mathcal{A}_y & \mathcal{A}_z & -\varphi & -\mathcal{A}_x & i\mathcal{B}_z & i\phi & -i\mathcal{B}_x & -i\mathcal{B}_y \\ \mathcal{A}_z & -\mathcal{A}_y & \mathcal{A}_x & -\varphi & i\mathcal{B}_y & i\mathcal{B}_x & i\phi & -i\mathcal{B}_z \\ -i\mathcal{B}_x & -i\phi & -i\mathcal{B}_z & -i\mathcal{B}_y & -\varphi & \mathcal{A}_z & -\mathcal{A}_y & -\mathcal{A}_x \\ -i\mathcal{B}_y & i\mathcal{B}_z & -i\phi & -i\mathcal{B}_x & -\mathcal{A}_z & -\varphi & \mathcal{A}_x & -\mathcal{A}_y \\ -i\mathcal{B}_z & -i\mathcal{B}_y & i\mathcal{B}_x & -i\phi & \mathcal{A}_y & -\mathcal{A}_x & -\varphi & -\mathcal{A}_z \\ -i\phi & i\mathcal{B}_x & i\mathcal{B}_y & i\mathcal{B}_z & \mathcal{A}_x & \mathcal{A}_y & \mathcal{A}_z & -\varphi \end{pmatrix}. \tag{38}$$

Now multiplying $\mathscr{D}$ (32) to octonion potential $\mathscr{V}$ (38) for the octonionic field equations, we get

$$\mathscr{D}\mathscr{V} = \mathcal{F} \longmapsto \mathcal{F}_0 e_0 + \sum_{A=1}^{7} \mathcal{F}_A e_A \cong \begin{pmatrix} \mathcal{F}_0 & -\mathcal{F}_1 & -\mathcal{F}_2 & -\mathcal{F}_3 & -\mathcal{F}_4 & -\mathcal{F}_5 & -\mathcal{F}_6 & -\mathcal{F}_7 \\ \mathcal{F}_1 & \mathcal{F}_0 & -\mathcal{F}_3 & \mathcal{F}_2 & -\mathcal{F}_7 & \mathcal{F}_6 & -\mathcal{F}_5 & \mathcal{F}_4 \\ \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_0 & -\mathcal{F}_1 & -\mathcal{F}_6 & -\mathcal{F}_7 & \mathcal{F}_4 & \mathcal{F}_5 \\ \mathcal{F}_3 & -\mathcal{F}_2 & \mathcal{F}_1 & \mathcal{F}_0 & \mathcal{F}_5 & -\mathcal{F}_4 & -\mathcal{F}_7 & \mathcal{F}_6 \\ \mathcal{F}_4 & \mathcal{F}_7 & \mathcal{F}_6 & -\mathcal{F}_5 & \mathcal{F}_0 & \mathcal{F}_3 & -\mathcal{F}_2 & -\mathcal{F}_1 \\ \mathcal{F}_5 & -\mathcal{F}_6 & \mathcal{F}_7 & \mathcal{F}_4 & -\mathcal{F}_3 & \mathcal{F}_0 & \mathcal{F}_1 & -\mathcal{F}_2 \\ \mathcal{F}_6 & \mathcal{F}_5 & -\mathcal{F}_4 & \mathcal{F}_7 & \mathcal{F}_2 & -\mathcal{F}_1 & \mathcal{F}_0 & -\mathcal{F}_3 \\ \mathcal{F}_7 & -\mathcal{F}_4 & -\mathcal{F}_5 & -\mathcal{F}_6 & \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_0 \end{pmatrix}, \tag{39}$$

where $\mathcal{F}$ is an octonion that reproduces the generalized complex-electromagnetic fields of dyons. Thus, the components of generalized complex-electromagnetic fields $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_7)$ may be expressed as

$$\begin{aligned}
\mathcal{F}_0 &= \left[-\frac{\partial \mathcal{A}_x}{\partial x} - \frac{\partial \mathcal{A}_y}{\partial y} - \frac{\partial \mathcal{A}_z}{\partial z} - \frac{\partial \phi}{\partial t}\right] + i\left[\frac{\partial \mathcal{B}_x}{\partial X} + \frac{\partial \mathcal{B}_y}{\partial Y} + \frac{\partial \mathcal{B}_z}{\partial Z} + \frac{\partial \varphi}{\partial T}\right] \implies 0\ ;\\
\mathcal{F}_1 &= \left[-\frac{\partial \varphi}{\partial x} + \frac{\partial \mathcal{A}_z}{\partial y} - \frac{\partial \mathcal{A}_y}{\partial z} - \frac{\partial \mathcal{B}_x}{\partial t}\right] + i\left[-\frac{\partial \phi}{\partial X} - \frac{\partial \mathcal{B}_z}{\partial Y} + \frac{\partial \mathcal{B}_y}{\partial Z} - \frac{\partial \mathcal{A}_x}{\partial T}\right] \implies \mathcal{H}_x + i\mathscr{E}_X\ ;\\
\mathcal{F}_2 &= \left[-\frac{\partial \varphi}{\partial y} + \frac{\partial \mathcal{A}_x}{\partial z} - \frac{\partial \mathcal{A}_z}{\partial x} - \frac{\partial \mathcal{B}_y}{\partial t}\right] + i\left[-\frac{\partial \phi}{\partial Y} - \frac{\partial \mathcal{B}_x}{\partial Z} + \frac{\partial \mathcal{B}_z}{\partial X} - \frac{\partial \mathcal{A}_y}{\partial T}\right] \implies \mathcal{H}_y + i\mathscr{E}_Y\ ;\\
\mathcal{F}_3 &= \left[-\frac{\partial \varphi}{\partial z} + \frac{\partial \mathcal{A}_y}{\partial x} - \frac{\partial \mathcal{A}_x}{\partial y} - \frac{\partial \mathcal{B}_z}{\partial t}\right] + i\left[-\frac{\partial \phi}{\partial Z} - \frac{\partial \mathcal{B}_y}{\partial X} + \frac{\partial \mathcal{B}_x}{\partial Y} - \frac{\partial \mathcal{A}_z}{\partial T}\right] \implies \mathcal{H}_z + i\mathscr{E}_Z\ ;\\
\mathcal{F}_4 &= \left[-\frac{\partial \varphi}{\partial X} + \frac{\partial \mathcal{A}_z}{\partial Y} - \frac{\partial \mathcal{A}_y}{\partial Z} - \frac{\partial \mathcal{B}_x}{\partial T}\right] - i\left[-\frac{\partial \phi}{\partial x} - \frac{\partial \mathcal{B}_z}{\partial y} + \frac{\partial \mathcal{B}_y}{\partial z} - \frac{\partial \mathcal{A}_x}{\partial t}\right] \implies \mathscr{H}_X - i\mathcal{E}_x\ ;\\
\mathcal{F}_5 &= \left[-\frac{\partial \varphi}{\partial Y} + \frac{\partial \mathcal{A}_x}{\partial Z} - \frac{\partial \mathcal{A}_z}{\partial X} - \frac{\partial \mathcal{B}_y}{\partial T}\right] - i\left[-\frac{\partial \phi}{\partial y} - \frac{\partial \mathcal{B}_x}{\partial z} + \frac{\partial \mathcal{B}_z}{\partial x} - \frac{\partial \mathcal{A}_y}{\partial t}\right] \implies \mathscr{H}_Y - i\mathcal{E}_y\ ;\\
\mathcal{F}_6 &= \left[-\frac{\partial \varphi}{\partial Z} + \frac{\partial \mathcal{A}_y}{\partial X} - \frac{\partial \mathcal{A}_x}{\partial Y} - \frac{\partial \mathcal{B}_z}{\partial T}\right] - i\left[-\frac{\partial \phi}{\partial z} - \frac{\partial \mathcal{B}_y}{\partial x} + \frac{\partial \mathcal{B}_x}{\partial y} - \frac{\partial \mathcal{A}_z}{\partial t}\right] \implies \mathscr{H}_Z - i\mathcal{E}_z\ ;\\
\mathcal{F}_7 &= \left[-\frac{\partial \mathcal{A}_x}{\partial X} - \frac{\partial \mathcal{A}_y}{\partial Y} - \frac{\partial \mathcal{A}_z}{\partial Z} - \frac{\partial \phi}{\partial T}\right] - i\left[\frac{\partial \mathcal{B}_x}{\partial x} + \frac{\partial \mathcal{B}_y}{\partial y} + \frac{\partial \mathcal{B}_z}{\partial z} + \frac{\partial \varphi}{\partial t}\right] \implies 0\ .
\end{aligned} \tag{40}$$

In Eq. (40) above, we have used the Lorenz gauge conditions respectively for the dynamics of electric and magnetic charges of dyons in internal and external space-time coordinates:

$$\begin{aligned}
\frac{\partial \mathcal{A}_x}{\partial x} + \frac{\partial \mathcal{A}_y}{\partial y} + \frac{\partial \mathcal{A}_z}{\partial z} + \frac{\partial \phi}{\partial t} &= 0\,, \quad \frac{\partial \mathcal{A}_x}{\partial X} + \frac{\partial \mathcal{A}_y}{\partial Y} + \frac{\partial \mathcal{A}_z}{\partial Z} + \frac{\partial \phi}{\partial T} = 0;\\
\frac{\partial \mathcal{B}_x}{\partial x} + \frac{\partial \mathcal{B}_y}{\partial y} + \frac{\partial \mathcal{B}_z}{\partial z} + \frac{\partial \varphi}{\partial t} &= 0\,, \quad \frac{\partial \mathcal{B}_x}{\partial X} + \frac{\partial \mathcal{B}_y}{\partial Y} + \frac{\partial \mathcal{B}_z}{\partial Z} + \frac{\partial \varphi}{\partial T} = 0;
\end{aligned} \tag{41}$$

and $\overrightarrow{\mathcal{E}}$, $\overrightarrow{\mathcal{H}}$ are the generalized electric and magnetic fields of dyons described in internal space-time coordinates, while $\overrightarrow{\mathscr{E}}$ and $\overrightarrow{\mathscr{H}}$ are the generalized electric and magnetic fields in the case of external space-time coordinates, i.e.

$$\begin{aligned}
\overrightarrow{\mathcal{E}}\{x,y,z,t\} &= -\overrightarrow{\nabla}\phi - \frac{\partial \overrightarrow{\mathcal{A}}}{\partial t} - \overrightarrow{\nabla} \times \overrightarrow{\mathcal{B}}\,, \qquad \overrightarrow{\mathcal{H}}\{x,y,z,t\} = -\overrightarrow{\nabla}\varphi - \frac{\partial \overrightarrow{\mathcal{B}}}{\partial t} + \overrightarrow{\nabla} \times \overrightarrow{\mathcal{A}};\\
\overrightarrow{\mathscr{E}}\{X,Y,Z,T\} &= -\overrightarrow{\nabla'}\phi - \frac{\partial \overrightarrow{\mathcal{A}}}{\partial T} - \overrightarrow{\nabla'} \times \overrightarrow{\mathcal{B}}\,, \quad \overrightarrow{\mathscr{H}}\{X,Y,Z,T\} = -\overrightarrow{\nabla'}\varphi - \frac{\partial \overrightarrow{\mathcal{B}}}{\partial T} + \overrightarrow{\nabla'} \times \overrightarrow{\mathcal{A}}.
\end{aligned} \tag{42}$$

The generalized octonion field equation can be written in the following compact notation:

$$\bar{\mathscr{D}}\,\mathcal{F} = \mathcal{J} \longmapsto \mathcal{J}_0 e_0 + \sum_{A=1}^{7} \mathcal{J}_A e_A \cong \begin{pmatrix}
\mathcal{J}_0 & -\mathcal{J}_1 & -\mathcal{J}_2 & -\mathcal{J}_3 & -\mathcal{J}_4 & -\mathcal{J}_5 & -\mathcal{J}_6 & -\mathcal{J}_7\\
\mathcal{J}_1 & \mathcal{J}_0 & -\mathcal{J}_3 & \mathcal{J}_2 & -\mathcal{J}_7 & \mathcal{J}_6 & -\mathcal{J}_5 & \mathcal{J}_4\\
\mathcal{J}_2 & \mathcal{J}_3 & \mathcal{J}_0 & -\mathcal{J}_1 & -\mathcal{J}_6 & -\mathcal{J}_7 & \mathcal{J}_4 & \mathcal{J}_5\\
\mathcal{J}_3 & -\mathcal{J}_2 & \mathcal{J}_1 & \mathcal{J}_0 & \mathcal{J}_5 & -\mathcal{J}_4 & -\mathcal{J}_7 & \mathcal{J}_6\\
\mathcal{J}_4 & \mathcal{J}_7 & \mathcal{J}_6 & -\mathcal{J}_5 & \mathcal{J}_0 & \mathcal{J}_3 & -\mathcal{J}_2 & -\mathcal{J}_1\\
\mathcal{J}_5 & -\mathcal{J}_6 & \mathcal{J}_7 & \mathcal{J}_4 & -\mathcal{J}_3 & \mathcal{J}_0 & \mathcal{J}_1 & -\mathcal{J}_2\\
\mathcal{J}_6 & \mathcal{J}_5 & -\mathcal{J}_4 & \mathcal{J}_7 & \mathcal{J}_2 & -\mathcal{J}_1 & \mathcal{J}_0 & -\mathcal{J}_3\\
\mathcal{J}_7 & -\mathcal{J}_4 & -\mathcal{J}_5 & -\mathcal{J}_6 & \mathcal{J}_1 & \mathcal{J}_2 & \mathcal{J}_3 & \mathcal{J}_0
\end{pmatrix}, \tag{43}$$

where $\mathcal{J}$ is also an octonion described by the octonion form of generalized current,

$$(\mathcal{J}_0, \overrightarrow{\mathcal{J}}) = \{\mathcal{J}_\mu\} \Longrightarrow (\rho, \overrightarrow{\mathcal{J}}) = \{\mathcal{J}_\mu\}, \text{ and } (\varrho, \overrightarrow{\mathcal{K}}) = \{\mathcal{K}_\mu\} \tag{44}$$

are respectively taken as the 4 currents associated with electric charge and magnetic monopole of dyons. In order to obtain both symmetry and duality invariant Maxwell equations, Dirac proposed the magnetic monopole [44, 45]. Thus, from Eq. (43), we can obtain the octonionic GDM equations of dyons respectively in internal and external space-time coordinates as follows:

$$\begin{aligned}
(\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{H}}) &= \varrho\,, \quad (\overrightarrow{\nabla'} \cdot \overrightarrow{\mathscr{H}}) = \varrho\,; \\
(\overrightarrow{\nabla} \times \overrightarrow{\mathcal{H}}) &= \frac{\partial \mathcal{E}}{\partial t} + \mathcal{J}\,, \quad (\overrightarrow{\nabla'} \times \overrightarrow{\mathscr{H}}) = \frac{\partial \mathscr{E}}{\partial T} + \mathcal{J}\,; \\
(\overrightarrow{\nabla} \times \overrightarrow{\mathcal{E}}) &= -\frac{\partial \mathcal{H}}{\partial t} - \mathcal{K}\,, \quad (\overrightarrow{\nabla'} \times \overrightarrow{\mathscr{E}}) = -\frac{\partial \mathscr{H}}{\partial T} - \mathcal{K}\,; \\
(\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{E}}) &= \rho\,, \quad (\overrightarrow{\nabla'} \cdot \overrightarrow{\mathscr{E}}) = \rho\,.
\end{aligned} \tag{45}$$

Eq. (45) reduces to the usual transform as

$$X \to x, \quad Y \to y, \quad Z \to z, \quad T \to t \Longrightarrow \overrightarrow{\nabla'} \to \overrightarrow{\nabla} \text{ and } \Box' \to \Box\,. \tag{46}$$

Thus, we get

$$\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{H}} &= \varrho\,; \\
\overrightarrow{\nabla} \times \overrightarrow{\mathcal{H}} &= \frac{\partial \mathcal{E}}{\partial t} + \mathcal{J}\,; \\
\overrightarrow{\nabla} \times \overrightarrow{\mathcal{E}} &= -\frac{\partial \mathcal{H}}{\partial t} - \mathcal{K}\,; \\
\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{E}} &= \rho\,;
\end{aligned} \tag{47}$$

which are the GDM equations of generalized fields of dyons (monopole). Eqs. (45) and (47) are fully symmetric Dirac–Maxwell equations and allow for the possibility of magnetic charges and currents, analogous to electric charges and currents. Like the quaternion formulation of generalized electromagnetic fields of dyons, octonionic matrix formulation is compact and simpler.

5. Conclusion

In this paper, we have expressed the octonion matrix representation, which is 8×8 matrix representation since the octonions (Cayley numbers) are the largest division algebra with 8 dimensions, double the number of the quaternions. The octonion wave equation has been discussed from the octonion differential operator in terms of 8×8 matrix representation given by Eq. (32). It consists of an 8-dimensional space as the combination of two 4-dimensional spaces associated with quaternions. We have used two 4-dimensional spaces, namely internal and external spaces, for the existence of both magnetic monopoles and dyons in higher-dimensional formalism. We have obtained the octonion valued potential for dyons in 8×8 matrix formulation, which is the combination of 4-dimensional external space followed by 4-dimensional internal space. The electric 4-potential A^μ and magnetic 4-potential B^μ of dyons are respectively associated with the electric and magnetic charges in external and internal spaces. The octonion is the best way for the theory of high-dimensional representation. As such, we have considered the electric 4-potential in usual 4-dimensional internal space while the magnetic 4-potential has been considered in external 4-dimensional space. The magnetic charge in internal space plays the role of electric charge in external space or vice versa. Eq. (40) represents the generalized electromagnetic field equations of dyons in terms of the components of two 4-potentials. Eq. (45) is the compact form of GDM equations of dyons (presence of magnetic monopole) in the case of 8 (i.e. two 4-)dimensional formalism. As such, that is the beauty of the octonion, instead of writing the 8 different differential Maxwell-type equations in internal and external space-time coordinates for higher-dimensional theory. The GDM equations are fully symmetric Maxwell equations and allow for the possibility of magnetic charges and currents, analogous to electric charges and currents. We have obtained a compact GDM equation, Eq. (47), which is understood to be manifestly covariant, dual invariant, simpler, and consistent. It reproduced the dynamics of electric (magnetic) energy in the absence of magnetic (electric) charge of dyons or vice versa.

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