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Extension of the class of exactly solvable nonstationary quantum mechanics problems

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Abstract: A modified factorization method for a charged particle in a time-dependent magnetic field has been developed by virtue of motion quantum integrals. Application of this method to the considered system helps to find exact solutions for a range of nonstationary potentials.

Key words: Factorization method, quantum integrals of motion

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1. Introduction

The classical factorization method (CFM) developed by Schrödinger [1–3] allows one to determine eigenfunctions (EF) and the eigenvalues (EV) for stationary problems. A detailed review of the factorization method can be found in Shi-Hai Dong's book [4].

Mielnik [5] developed a modified method of factorization (MMF) that allows the construction of a class of one-dimensional stationary potentials with a harmonic oscillator spectrum. Those potentials mentioned above are different from those of the harmonic oscillator. Thus, using the MMF one can extend the class of potentials for which exact solutions exist.

To solve the problem for nonstationary systems we need to determine the wave function ψ satisfying the wave equation $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$, where $\hat{H}$ is the Hamiltonian of the considered problem. However, the wave function in the nonstationary problem is not EF of the $\hat{H}$ Hamiltonian. Therefore, it is impossible to generalize directly MMF to the nonstationary case.

In [6], CFM was developed for the nonstationary harmonic oscillator. The purpose of this article is to show an example of the charged particle in a time-dependent magnetic field, that not only the CFM but also MMF can be developed for nonstationary systems, using a method of quantum integrals of motion [7].

Application of MMF to the considered system allows one to find exact solutions for a wide range of nonstationary potentials.

One can determine the stationary potentials and wave functions passing to the limit as $t \to 0$ in the corresponding expressions. Such stationary potentials have the same spectrum, and it seems impossible to distinguish experimentally such systems from each other. However, due to differences in the wave functions of these isospectral systems, any external perturbation allows one to distinguish one system from another. Therefore, even in the case of a stationary magnetic field MMF is of practical interest.

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2. Nonstationary MMF

2.1. Nonstationary potentials

Consider a charged particle in a homogeneous magnetic field $\vec{H}(t)$ directed along the z-axis. In this case, the motion along the $z-$axis is trivial and can be neglected. We will consider the motion in the xy plane. Let us choose the vector potential in the form

$$\vec{A}(\vec{r},t) = \left[-H(t)\ y/2, H(t)\,x/2, 0\right]. \tag{1}$$

In view of the foregoing the Hamiltonian of the system is

$$\hat{H} = \frac{1}{2m}\left[\left(\hat{p}_x + m\omega(t)\,y\right)^2 + \left(\hat{p}_y - m\omega(t)\,x\right)^2\right]. \tag{2}$$

Here x, y are usual canonical coordinates, $\hat{p}_x, \hat{p}_y$ the corresponding momenta,

$\omega(t) = eH(t)/2mc$ the time-dependent frequency, c the velocity of light, m the particle mass, and e its charge.

Let us introduce the operators:

$$\hat{\tilde{A}}^-(t) = f(t)\ \left[\varepsilon\ (t)\ (\hat{p}_x + i\hat{p}_y) - im\dot{\varepsilon}\ \ (t)\ (\delta\ (y) - i\beta\ (x))\right]\Big/\sqrt{2\hbar m}, \tag{3}$$

$$\hat{\tilde{B}}^-(t) = f(t)\ \left[\varepsilon\ (t)\ (\hat{p}_y + i\hat{p}_x) - im\dot{\varepsilon}\ \ (t)\ (\beta\ (x) - i\delta\ (y))\right]\Big/\sqrt{2\hbar m}, \tag{4}$$

where $\beta(x)$, $\delta(y)$ are arbitrary functions of the coordinates, and let us require that

$$\hat{A}^-\hat{A}^+ = \hat{\tilde{A}}^-\hat{\tilde{A}}^+, \tag{5}$$

$$\hat{B}^-\hat{B}^+ = \hat{\tilde{B}}^-\hat{\tilde{B}}^+. \tag{6}$$

In [8] it is shown that $\hat{A}^-$, $\hat{B}^-$ are annihilation operators, and $\hat{A}^+$, $\hat{B}^+$ are creation operators for the system with Hamiltonian (2),

$$\hat{A}^-(t) = f(t)\ \left[\varepsilon(t)\ (\hat{p}_x + i\hat{p}_y) - im\dot{\varepsilon}(t)\ (y - ix)\right]\Big/(2\hbar m)^{1/2}, \tag{7}$$

$$\hat{B}^-(t) = f(t)\ \left[\varepsilon(t)\ (\hat{p}_y + i\hat{p}_x) - im\dot{\varepsilon}(t)\ (x - iy)\right]\Big/(2\hbar m)^{1/2}, \tag{8}$$

$$[\hat{A}^-, \hat{A}^+] = [\hat{B}^-, \hat{B}^+] = 1, [\hat{A}^-, \hat{B}^-] = [\hat{A}^-, \hat{B}^+] = 0. \tag{9}$$

Here $\varepsilon(t)$ is a definite solution of the equation for a classical oscillator

$$\ddot{\varepsilon} + \omega^2(t)\ \varepsilon = 0, \tag{10}$$

$$f(t) = \frac{1}{\sqrt{2}}\exp\left(i\int_0^t \omega(\tau)\ d\tau\right), \tag{11}$$

$$\dot{\varepsilon}\varepsilon^{*}-\dot{\varepsilon}^{*}\varepsilon=2i. \tag{12}$$

Hereinafter the point over any function denotes the time derivative, and the prime is the derivative on coordinate.

It is known [8] that $\hat{A}^{\pm},\hat{B}^{\pm}$ are integrals of motion, i.e.

$$\left[i\hbar\frac{\partial}{\partial t}-\hat{H},\,\hat{A}^{\pm}\right]=0,\left[i\hbar\frac{\partial}{\partial t}-\hat{H},\,\hat{B}^{\pm}\right]=0. \tag{13}$$

Let us use equations (3), (4), (7), and (8) in the formulas (5) and (6), and then sum up the results term by term. As a result, we obtain an equation in which some of the terms depend only on x, and the other terms only on y. Hence, instead of a single equation we get two, namely

$$m\mid\dot{\varepsilon}\mid^{2}\mid\beta\mid^{2}-\dot{\varepsilon}\varepsilon^{*}\beta\hat{p}_{x}-\varepsilon\dot{\varepsilon}^{*}\hat{p}_{x}\beta^{*}=m\mid\dot{\varepsilon}\mid^{2}x^{2}-\dot{\varepsilon}\varepsilon^{*}x\hat{p}_{x}-\varepsilon\dot{\varepsilon}^{*}\hat{p}_{x}x \tag{14}$$

The second equation has the same form, but $x\rightarrow y,\ \hat{p}_{x}\rightarrow\hat{p}_{y}$.

Taking into account f. (0.411.1) [9], it is easy to verify that the solution of the equation (13) is

$$\beta=x+i\frac{\Phi\ (x)}{\dot{\varepsilon}\varepsilon^{*}},Im\,\Phi\ (x)=0, \tag{15}$$

where

$$\Phi\left(x\right)=C^{-1/2}\left[\gamma+\int\limits_{0}^{\sqrt{C}x}dz\,\exp\left(-z^{2}\right)\right]^{-1}\exp\left(-Cx^{2}\right),C=\frac{m}{\hbar\mid\varepsilon\mid^{2}},\gamma\in R. \tag{16}$$

Then

$$\delta=y+i\frac{F\left(y\right)}{\dot{\varepsilon}\varepsilon^{*}},Im\,F\left(y\right)=0,\Phi\left(x\right)\rightarrow F\ (y)\ \text{ if } x\rightarrow y,\gamma\rightarrow\gamma_{F},\gamma_{F}\in R. \tag{17}$$

Using f. (15) and f. (17) it is easy to prove that

$$\left[\hat{\hat{A}}^{-},\hat{\hat{A}}^{+}\right]=\left[\hat{\hat{B}}^{-},\hat{\hat{B}}^{+}\right]=\frac{1+\Phi'}{2}+\frac{1+F'}{2}. \tag{18}$$

Let us calculate $\hat{\hat{A}}^{+}\hat{\hat{A}}^{-}$, knowing f. (18) and f. (5):

$$\hat{\hat{A}}^{+}\hat{\hat{A}}^{-}=\hat{\hat{K}}-\frac{1}{2}, \tag{19}$$

where we introduced the notation

$$\hat{\hat{K}}=\hat{K}-\frac{\Phi'+F'}{2},\hat{K}=\hat{A}^{+}\hat{A}^{-}+\frac{1}{2}. \tag{20}$$

Similarly,

$$\hat{\hat{B}}^{+}\hat{\hat{B}}^{-}=\hat{\hat{L}}-\frac{1}{2},\hat{\hat{L}}=\hat{L}-\frac{\Phi'+F'}{2},\hat{L}=\hat{B}^{+}\hat{B}^{-}+\frac{1}{2}. \tag{21}$$

With f. (13) it is easy to see that the operators $\hat{K}$ and $\hat{L}$ are integrals of motion for the system (2):

$$\left[i\hbar\frac{\partial}{\partial t}-\hat{H},\,\hat{K}\right]=0,\left[i\hbar\frac{\partial}{\partial t}-\hat{H},\,\hat{L}\right]=0, \tag{22}$$

and $[\hat{K},\hat{L}]=0$.

Let us find out what is the Hamiltonian for a nonstationary system with respect to which $\hat{\tilde{K}}$ and $\hat{\tilde{L}}$ are the integrals of motion, i.e. where $\hat{\tilde{H}}$ satisfies the following conditions:

$$\left[i\hbar\frac{\partial}{\partial t} - \hat{\tilde{H}},\, \hat{\tilde{K}} \right] = 0, \left[i\hbar\frac{\partial}{\partial t} - \hat{\tilde{H}},\, \hat{\tilde{L}} \right] = 0. \tag{23}$$

We assume that

$$\hat{\tilde{H}} = \hat{H} + V\ (x) + W\,(y) \tag{24}$$

and $V\ (x)$, $W\,(y)$ are arbitrary functions of x and y. Substitute f. (24), $\hat{\tilde{K}}$ from f. (20) and $\hat{\tilde{L}}$ from f. (21) in f. (23). Let us take into account f. (22). We will obtain

$$i\hbar\frac{\partial\Phi'}{\partial t} + i\hbar\frac{\partial F'}{\partial t} + \left[V,\ \hat{K}\right] + \left[W,\ \hat{K}\right] - \frac{1}{2}\left[\hat{H}\,,\ \Phi'\right] - \frac{1}{2}\left[\hat{H},\, F'\right] = 0 \tag{25}$$

Then instead of $\hat{H}$ we will use the expression (2), and instead of $\hat{K}$- f. (20), in which we will replace $\hat{A}^{\pm}$ according to f. (7). After all these substitutions, from the terms in f. (25) we select those that are operators. For equation (25) to take place, it is necessary that the sum of these terms shall be zero. From this, one can determine the type of V and W:

$$V = -\frac{\hbar}{|\,\varepsilon\,|^2}\Phi', W = -\frac{\hbar}{|\,\varepsilon\,|^2}F' \tag{26}$$

Direct verification shows that $V,\ W$ given by f. (26) are solutions of equation (25), if we use f. (1.3.3.8) and (1.3.3.1) [10], and consider time-depending constants γ and γ_F entering into $\Phi\ (x)$ and $F\ (y)$ as follows:

$$\dot{\gamma} = \alpha\gamma, \dot{\gamma}_F = \alpha\gamma_F. \tag{27}$$

Here

$$\alpha = -\frac{d}{dt}\ln\left(\omega_0\,|\varepsilon|\right)^2, \omega_0 = \omega\,(t = 0) \tag{28}$$

Finally, for the time-dependent system with Hamiltonian

$$\hat{\tilde{H}} = \hat{H} - \frac{\hbar}{|\varepsilon|^2}\left(\Phi' + F'\right) \tag{29}$$

we have the expressions

$$\Phi\,(x,\,t) = C^{-1/2}\exp\,\left(-Cx^2\right)\left[\left(\omega_0\,|\varepsilon|^2\right)^{-1}\tilde{\gamma} + \int\limits_0^{\sqrt{C}x} dz\,\exp\left(-z^2\right)\right]^{-1}, \tilde{\gamma} \in R \tag{30}$$

$$F\,(y,\,t) = C^{-1/2}\exp\,\left(-Cy^2\right)\left[\left(\omega_0\,|\varepsilon|^2\right)^{-1}\tilde{\gamma}_F + \int\limits_0^{\sqrt{C}y} dz\,\exp\left(-z^2\right)\right]^{-1}, \tilde{\gamma}_F \in R. \tag{31}$$

A family of nonstationary Hamiltonians $\hat{\tilde{H}}$ for which exact solutions of $\tilde{\psi}$ exist can be constructed knowing the expressions for $\Phi\,(x,\,t)$, $F\,(y,\,t)$, starting from the Hamiltonian $\hat{H}$ of the charged particle in a time-dependent magnetic field, and using the formula (29).

2.2. Exact solutions

Let us obtain these exact solutions. As noted above, operators $\hat{\tilde{K}}$ and $\hat{\tilde{L}}$ are integrals of motion. Consequently, the operators $\left(i\hbar\frac{\partial}{\partial t}-\hat{\tilde{H}}\right)$, $\hat{\tilde{K}}$ and $\hat{\tilde{L}}$ have a common EF system. This implies that the wave function of the time-dependent system can be found by solving the problem for EV of $\hat{\tilde{K}}$ and $\hat{\tilde{L}}$ operators:

$$\hat{\tilde{K}}\tilde{\psi}=\tilde{k}\tilde{\psi},\hat{\tilde{L}}\tilde{\psi}=\tilde{l}\tilde{\psi} \tag{32}$$

Taking into consideration (19), (5), (9), (20), we have

$$\hat{\tilde{K}}\hat{\tilde{A}}^{+}=\hat{\tilde{A}}^{+}\left(\hat{K}+1\right) \tag{33}$$

In [11] EF $\psi_{nd}\ (t)$ of $\hat{K}$ operator has been defined, which satisfies the known relations $\hat{A}^{-}\psi_{nd}=\sqrt{n}\psi_{n-1,d}$, $\hat{A}^{+}\psi_{nd}=\sqrt{n+1}\psi_{n+1,d}$, $n,d=1,2,3...$

Let us take it into account in the expression (20) for $\hat{K}$, premultiplying both sides of (33) on $\psi_{n-1,d}\,(t)$. We obtain

$$\hat{\tilde{K}}\tilde{\psi}_{nd}(t)=k_{n}\tilde{\psi}_{nd}\,(t)\,,k_{n}=n+\frac{1}{2}(n=1,2,3....,d=0,1.2,3,...), \tag{34}$$

where

$$\tilde{\psi}_{nd}\,(t)\sim\hat{\tilde{A}}^{+}\psi_{n-1,\,d}\,(t) \tag{35}$$

will be EF of $\hat{\tilde{K}}$ operator. Thus, the EV $\tilde{k}$ and EF $\tilde{\psi}$ of the operator $\hat{\tilde{K}}$ are obtained automatically from the EV k and EF ψ of the operator $\hat{K}$.

Similarly, but in view of (21), (6), $\hat{B}^{-}\psi_{nd}=\sqrt{d}\psi_{n,d-1}$, $\hat{B}^{+}\psi_{nd}=\sqrt{d+1}\psi_{n,,d+1}$ we obtain

$$\hat{\tilde{L}}\hat{\tilde{B}}^{+}=\hat{\tilde{B}}^{+}\left(\hat{L}+1\right),\tilde{\psi}_{nd}\,(t)\sim\hat{\tilde{B}}^{+}\psi_{n,\,d-1}\,(t)\,(n=0,\,1,\,2,...,\;\;d=1,\,2,\,3,...) \tag{36}$$

- of EF operator $\hat{\tilde{L}}$ with EV $\tilde{l}=l$:

$$\tilde{L}\tilde{\psi}_{nd}(t)=l_{d}\tilde{\psi}_{nd}\,(t)\,. \tag{37}$$

From the expressions (35) and (36) and the commutation relation $[\hat{\tilde{A}}^{+},\hat{\tilde{B}}^{+}]=0$ it is obvious that the EF of the problem with Hamiltonian $\hat{\tilde{H}}$ will be

$$\tilde{\psi}_{nd}\,(t)=C_{nd}\hat{\tilde{A}}^{+}\hat{\tilde{B}}^{+}\psi_{n-1,\,d-1}\,(t)\,,n,\,m=1,\,2,\,3,...$$

$$\tilde{\psi}_{n0}\,(t)=C_{no}\hat{\tilde{A}}^{+}\psi_{n-1,\,0}\,(t)\,,\hat{\tilde{\psi}}_{0d}\,(t)=C_{0d}\hat{\tilde{B}}^{+}\psi_{0,\,d-1}\,(t)\,. \tag{38}$$

It is easy to verify that all $\tilde{\psi}_{nd}\,(t)$ are orthogonal to each other:

$$\left(\tilde{\psi}_{n'd'}\,(t)\,,\tilde{\psi}_{nd}(t)\right)=0\text{ at }n\neq n',d\neq d', \tag{39}$$

$\tilde{\psi}_{0d}\,(t)$ is orthogonal to all $\tilde{\psi}_{nd}\,(t)$, if $\hat{\tilde{A}}^{-}\tilde{\psi}_{0d}=0$. Similarly, $\tilde{\psi}_{n0}\,(t)$ is orthogonal to all $\tilde{\psi}_{nd}\,(t)$, if $\hat{\tilde{B}}^{-}\tilde{\psi}_{n0}=0$.

However, formula (38) does not cover all EF of the Hamiltonian $\hat{\tilde{H}}$. Indeed, one can construct another EF $\tilde{\psi}_{00}$ based on its orthogonality to all other EF. For this purpose let us set $n = d = 0$ in formula (39), which leads to the condition

$$\hat{\tilde{A}}^{-}\tilde{\psi}_{00} = 0, \hat{\tilde{B}}^{-}\tilde{\psi}_{00} = 0, \tag{40}$$

from which in view of the expressions (3), (4), (14) and the expression for C from (16), we have

$$\tilde{\psi}_{00} = const \cdot \exp\left(\frac{i\dot{\varepsilon}m\ \left(x^2 + y^2\right)}{2\hbar\varepsilon}\right) \cdot \exp\left(-C\int\limits_0^x \Phi\left(\tilde{x}, t\right)d\tilde{x} - C\int\limits_0^y F\ \left(\tilde{y},\ t\right)\ d\tilde{y}\right). \tag{41}$$

$\tilde{\psi}_{00}$ is another, as seen from (34) and (37), the EF of $\hat{\tilde{H}}$ corresponding to the EV $l_0 = k_0 = 1/2$.

If in expressions (38), (41), (26), (30), (31) for the wave functions and the potential considered nonstationary problem put $\varepsilon = \omega_0^{-1/2}\exp\left(i\omega_0 t\right)$ and tend t to zero (in this case $\omega\left(t\right) \to \omega_0$), then we obtain the corresponding expressions of the stationary problem.

3. Conclusion

MMF is developed for a nonstationary system, namely for a charged particle in a time-dependent magnetic field. Based on the results in this paper and assigning specific dependence of the magnetic field on time, one can find the exact solutions for a variety of anharmonic potentials.

The use of motion quantum integrals makes the procedure of the solution for the considered nonstationary task similar to that for a stationary task.

In the case of a stationary task, MMF generates a family of isospectral stationary potentials (see also (5)). If we impose an external perturbation, then it is possible to distinguish experimentally isospectral systems mentioned above from each other. For example, this finding can be used in the study of band structure.

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